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Approximation Algorithms for a Capacitated Network Design Problem[1]

Refael Hassin,[2] R. Ravi,[3] and F. Sibel Salman[4]

Abstract. We study a capacitated network design problem with applications in local access network design. Given a network, the problem is to route flow from several sources to a sink and to install capacity on the edges to support the flow at minimum cost. Capacity can be purchased only in multiples of a fixed quantity. All the flow from a source must be routed in a single path to the sink. This NP-hard problem generalizes the Steiner tree problem and also more effectively models the applications traditionally formulated as capacitated tree problems. We present an approximation algorithm with performance ratio $(\rho_{ST} + 2)$ where ρ_{ST} is the performance ratio of any approximation algorithm for the minimum Steiner tree problem. When all sources have unit demand, the ratio improves to $(\rho_{ST} + 1)$ and, in particular, to 2 when all nodes in the graph are sources.

Key Words. Network design, Approximation algorithms, Routing flow, Capacity installation.

1. Introduction. We consider the problem of capacity installation and the routing of traffic from a set of source nodes to a single sink node in a network. We model applications in which capacity can be purchased in multiples of a fixed quantity. In telecommunication network design this corresponds to installing transmission facilities such as fiber-optic cables on the edges of a centralized network, and in transportation networks this applies to assigning vehicles of fixed capacity to routes from several destinations to a single hub node. A typical telecommunication network consists of a backbone network and several local access networks. Each local access network collects traffic from user nodes at a centralized gateway node which is part of the backbone network. The backbone provides access via high-speed connections between the gateway nodes. The performance measures for the effectiveness of a network design include cost, reliability and quality of service. While reliability is a major criterion for a backbone network, cost and quality of service play more important roles in the design of local access networks. The problem we study models the design of a cost-effective local access network.

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[2] Department of Statistics and Operations Research, Tel Aviv University, Tel Aviv 69978, Israel. hassin@post.tau.ac.il.
[3] GSIA, Carnegie Mellon University, Pittsburgh, PA 15213, USA. ravi@cmu.edu.
[4] Krannert School of Management, Purdue University, West Lafayette, IN 47907, USA. salmanf@mgmt.purdue.edu.

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The *capacitated network design problem*, which we refer to as CND in short, can be stated as follows. We are given an underlying undirected graph $G = (V, E)$. A subset S of nodes is specified as sources of traffic and a single sink node t is specified as the destination. Each source node $s_i \in S$ has a positive integer-valued demand dem_i, all of which must be routed to t via a single path, that is, flow cannot be bifurcated. A transmission facility with capacity U is available for purchasing and installation on the edges of the graph. Purchasing and installing k lines of the transmission facility on edge e costs kc_e and provides kU capacity, where c_e is a positive real number, and U, k are positive integers. The problem is to find a minimum cost installation of facilities that provides sufficient capacity to route all of the demand simultaneously. The problem requires choosing a path from each source to the sink node and finding the number of facilities to be installed on each edge such that all the demand is routed without exceeding edge capacities. Traffic originating in different sources may share the capacity on the installed cables and the capacity installed on an edge has to be at least as much as the *total* traffic routed through this edge in both directions.

Based on the telecommunication application, we refer to the transmission facility installed on the edges as the "cable", even though it could represent other modes of transmission. In telecommunications, typically a set of cable types with differing capacities is available to the network designer. The minimum cost capacity installation and routing problem has been studied in the literature as the network loading problem in its most general form as a multicommodity flow problem with multiple facility types. For a survey on exact solution methods in this area, the reader is referred to the chapter on multicommodity capacitated network design by Gendron, Crainic and Frangioni in [SS], and the chapter on network design by Balakrishnan, Magnanti and Mirchandani in [DMM]. An exact solution approach has been proposed in [SRH] for the single-sink multifacility problem. In spite of the recent computational progress, the size of the instances that can be solved to optimality in reasonable time contains no more than 50 nodes and 100 edges, whereas real-life instances may contain hundreds of nodes and edges. The CND corresponds to the single-sink single-facility case.

The capacitated network design problem is NP-hard since it generalizes the Steiner tree problem which has been shown to be NP-hard [GJ]. Given a graph with a subset of vertices distinguished as the set of terminals, and costs on the edges, the *Steiner tree problem* is to find a minimum-cost subtree spanning the set of terminals. In the CND problem, when cable capacity is larger than the total demand in the network, one copy of the cable will be installed on every edge that carries flow. This special case of our problem is equivalent to a Steiner tree problem, where the set of terminals to be connected is the set of the source nodes and the sink node.

The NP-hardness of CND implies that the existence of an algorithm that finds an optimal solution in polynomial time is very unlikely. Therefore, we focus on obtaining provable near-optimal solutions in polynomial time. An algorithm is said to be a factor α approximation algorithm for a minimization problem, if it produces a feasible solution with objective function value at most α times the optimal objective function value for all instances of the problem, and its running time is bounded by a fixed polynomial in the instance size. The factor α is interchangeably called the approximation (or worst-case, or performance) factor (or ratio). Clearly, the closer α is to 1, the better the approximation algorithm. For CND, a constant factor approximation was obtained earlier by Salman et al. in [SCR$^+$]. The algorithm in [SCR$^+$] is based on an algorithm of Mansour and

Peleg [MP] which approximates the multicommodity and the single cable type problem in an n-node graph with an $O(\log n)$ performance ratio. While Mansour and Peleg used a spanner to route traffic, Salman et al. [SCR$^+$] showed that routing through a Light Approximate Shortest Path Tree (defined in [KRY]) gives an approximation factor of 7 for the single-sink problem. When all the nodes in the input network except the sink node are source nodes, the approximation factor in [SCR$^+$] reduces to $(2\sqrt{2}+2)$. Other constant factor approximations for CND also follow from the work of Andrews and Zhang [AZ] who gave an $O(k^2)$-approximation for the single-sink problem with k cable types, and the work of Garg et al. [GKK$^+$] who gave an improved $O(k)$-approximation for the same problem, but the resulting constant factors are rather high.

In this paper we present an approximation algorithm with a better approximation factor by making use of the relation of the CND problem to the Steiner tree problem. The algorithm utilizes a Steiner tree of approximately minimum cost, that is, a tree of cost at most ρ_{ST} times the optimal cost for routing smaller amounts of demand. When the demand accumulates to a value close to the cable capacity, the algorithm sends the aggregated demand by a shortest path to the sink through cables dedicated specifically for the accumulated demand.

We show that this algorithm has a worst-case ratio $(\rho_{ST}+2)$, where ρ_{ST} is the performance ratio of the Steiner tree approximation. When every source has unit demand, we give a slightly more intricate version of the algorithm and obtain an improved worst-case ratio $(\rho_{ST}+1)$. When all the nodes in the input network except the sink node are source nodes, we first find a minimum spanning tree instead of a Steiner tree. As a result, the approximation ratio reduces to 3 with arbitrary integer demands, and to 2 with unit demands.

Although the Steiner tree problem is NP-hard even with Euclidean or rectilinear costs [GJ], a Steiner tree with approximately minimum cost can be constructed in polynomial time. A minimum cost tree spanning all the terminals has cost at most twice the cost of an optimal Steiner tree [TM]. Furthermore, approximation algorithms with improved worst-case ratios have been developed in a series of papers [Z], [BR], [KZ], [PS], [HP], [RZ] over the last two decades. As a result, for graphs with arbitrary costs, the worst-case ratio was gradually decreased from 2 to 1.55. Robins and Zelikovsky [RZ] gave the currently best known approximation ratio of 1.55. On the other hand, it has also been shown that the Steiner tree problem cannot be approximated within a factor of $1+\varepsilon$ for sufficiently small $\varepsilon > 0$, unless $P = NP$ [BP], [CT].

With the above-mentioned results on the approximation of the Steiner tree problem, the algorithm we propose in this paper improves the approximation ratio of the CND problem to 3.55. We note that any improvements in the approximation of the Steiner tree problem will be reflected in the approximation ratio of our algorithm.

The problem we study is also related to the *capacitated MST problem* (*c-MST*) [P], [AG], [G], [KB1], [KB2], [CL], [S]: given an undirected edge-weighted graph with a root node, the unit demand at every other node and a positive integer U, the problem is to find a minimum spanning tree such that every subtree of the root node has a total demand of at most U. This problem has been cited by Kershenbaum and Boorstyn [KB1] as well as later by Gavish [G] to model the local access network design problem when traffic is routed from a set of sources to a sink node under a limited capacity of links. In c-MST, network links have a fixed capacity U, whereas in our single cable problem capacity can be purchased in multiples of U and the multiplicity of cables on each link is a decision variable. While a tree is required as a solution in c-MST, in most applications

the actual requirement is to send the demand of each source node via a single path to the sink node, which is known as the non-bifurcating requirement for the demands. Our single cable problem enforces the non-bifurcating requirement without requiring that the solution be a tree. As a result, a solution to our problem may contain cycles and the total traffic going through a node may exceed U, as opposed to a solution of c-MST. We can contrast our approximation results with the best known approximation results for c-MST. Altinkemer and Gavish [AG] gave a 4-approximation for the non-uniform demand case and a 3-approximation for the uniform demand case of c-MST. In the non-uniform demand case, our $(\rho_{ST} + 2)$-approximation has a performance ratio less than 4, and handles the Steiner version that does not require all non-sink nodes to be source nodes (i.e., other Steiner nodes in the graph are allowed). When all non-sink nodes are sources the performance ratio improves to 3, and to 2 for the corresponding uniform demand case.

In the next two sections we present the algorithms for the unit demand case and the more general integer demand case and analyze their worst-case performances. We conclude with an extension of the local access design problem.

2. Unit Demands. In this section we consider the case when every source node has unit demand and U is a positive integer. If $U = 1$, an optimal solution can be computed easily in polynomial time. In this case, sending the demand of each source to the sink through a path with minimum total cost (a shortest path) and installing one dedicated copy of the cable for each source on its shortest path is an optimal solution. When $U < 1$, for each source using again a shortest path, but installing $\lceil 1/U \rceil$ copies of the cable on the path, gives a 2-approximation. When U is large enough so that one copy of the cable will be sufficient whenever an edge contains positive flow (for example, for any instance where U is at least the number of sources this will hold), a minimum cost Steiner tree connecting the sources and the sink is an optimal solution. Therefore, the interesting case of the problem with regard to approximation arises when $1 < U < |S|$, and typically this will be the valid case for telecommunications applications with fiber-optic cables. For this case we propose an approximation algorithm that constructs a solution by accumulating the demand of value U at selected hub nodes and by sending the aggregated demand at the hub nodes via shortest paths to the sink. The algorithm utilizes a Steiner tree to route the demand from the sources to the hub nodes.

The algorithm can be outlined as follows. Construct a Steiner tree T with terminal set $S \cup \{t\}$ and cost c_e on each edge e in polynomial time such that the cost of the tree is at most ρ_{ST} times the optimal Steiner tree cost. Find a path from each node to the sink in G with minimum total cost by solving shortest path problems in polynomial time. In each iteration, identify a subtree of T such that the total remaining demand in the subtree equals the cable capacity U. In this subtree select a node with the smallest cost of reaching the sink as the hub node (ties can be broken arbitrarily). Route the demand of each source in the subtree to the hub node using the unique paths in T. Send the aggregated demand at the hub to the sink via a minimum cost path in the input graph G by installing a dedicated cable on the path. Repeat these steps until no more such subtrees can be identified. Calculate the flow (in both directions) on each edge of T. If the total flow on an edge exceeds U, cancel flow in opposite directions by reassigning the source nodes to the hub nodes.

Let P_v be a minimum cost of a path from node v to the sink t in G with cost $c(v, t)$. In the Steiner tree T we designate the sink node t as the *root* of the tree and define the *level* of a node as the number of edges on its path to the sink node t. Any node adjacent to v whose level is one larger than the level of v is called a *child* of v. For each node v, T_v denotes the subtree of T rooted at v. At a given iteration of the algorithm (we drop the iteration number for simplicity of notation), R is the set of unprocessed source nodes and $D(T_v)$ is the total unprocessed demand in T_v. That is, $D(T_v) = \sum_{s_i \in R \cap T_v} dem_i = |R \cap T_v|$ since $dem_i = 1$ for all i.

The Algorithm UNIFORM, given below, takes T as input, and outputs (1) for each source, a route through which all of its demand is sent to the sink in G, and (2) the number of cables that are installed on the edges of the input network to support the flow of traffic going through them.

Algorithm UNIFORM

Initialize: $R := S$
Main step:
  Pick a node v such that $D(T_v) \geq U$ and the level of v is maximum.
  If no such node exists (that is, $D(T_t) < U$) or $v = t$, then go to the final step.
  Find a node, say w, in $R \cap T_v$ such that $c(w, t)$ is minimum.
  Designate w as a "hub" node and set $C = \{w\}$.
  Collect $U - 1$ additional source nodes into C (*described below*).
  Assign these sources to w.
  Route the demand of each source in C to the hub node w via the unique paths in T.
  Route demand aggregated at w via a minimum cost path to the sink in G, namely P_w.
  Install one copy of the cable on P_w, in addition to any existing cables.
  Remove C from R.
  If R is not empty, repeat the main step.
  If R is empty, go to the final step.
Final step:
  If R is not empty, then
    Designate t as the hub node for sources in R.
    Route all the demand in R to t via the unique paths in T.
  Send demand of each source to its assigned hub node in T.
  Calculate the resulting flow in both directions on each edge of T.
  For all edges e of T
    If the sum of flow on e in both directions exceeds U,
      Cancel maximal amount of flow in opposite directions.
      Reallocate the sources whose flow has been cancelled between two source collections.
      Reselect the hub nodes for the two collections (*described below*).
      Update flows.
  Install one copy of cable on the edges of T which have positive flow, in addition to any existing cables.

The following procedure is used to collect unprocessed source nodes from T_v in the set C at the main step of the algorithm. The set C contains only the hub node w when the procedure is called. At the end of the procedure, C contains U nodes. The algorithm searches subtrees in a depth-first manner, starting with the subtree that contains the hub node, to avoid partially assigning subtrees unless necessary.

Procedure to collect source nodes of T_v

Add v to C, if $v \in R$.
Let $v_1, \ldots, v_k$ be the children of v.
If $w \neq v$, then
    Let v_p be the child of v such that the hub node w is in T_{v_p}.
    Add $T_{v_p} \cap R$ to C.
While $|C| < U$,
    Pick an unprocessed child of v, say v_i.
    If $D(T_{v_i}) + |C| \leq U$, then
    Add $T_{v_i} \cap R$ to C.
    Else (T_{v_i} is collected partially),
      Scan T_{v_i} depth-first and add sources in $R \cap T_{v_i}$ to C one at a time
      until $|C| = U$.
Return C.

The procedure to cancel flow on an edge e of T is called when the total flow exceeds U on edge e. We denote the set of sources sending flow on e in the direction towards t by S_{out}. We will show that these sources, which have a total flow of $f_{\text{out}} = |S_{\text{out}}|$, send their flow to a common hub node, which we denote by w_{out}. Let C_{out} be the set of all sources assigned to w_{out}. Similarly, let S_{in} be the sources sending flow on e in the opposite direction that have total flow $f_{\text{in}} = |S_{\text{in}}|$, and are assigned to the hub w_{in}. Let C_{in} be the set of all sources assigned to w_{in}.

Procedure to cancel flow on an edge e

If $f_{\text{in}} \leq f_{\text{out}}$, then cancel f_{in} by swapping f_{in} nodes between C_{in} and C_{out} as follows:
    Remove S_{in} from C_{in}.
    Remove a subset of S_{out} of size f_{in} from C_{out} and add it to C_{in}.
    Add S_{in} to C_{out}.
    Find a node w in C_{out} such that $c(w, t)$ is minimum.
    Designate w as the new "hub" node of C_{out}.
    Route demand of each source in C_{out} to w.
    Find a node w in C_{in} such that $c(w, t)$ is minimum.
    Designate w as the new "hub" node of C_{in}.
    Route demand of each source in C_{in} to w.
If $f_{\text{in}} > f_{\text{out}}$, then cancel f_{out} as follows:
    Remove S_{out} from C_{out}.
    Remove a subset of S_{in} of size f_{out} from C_{in} and add it to C_{out}.
    Add S_{out} to C_{in}.
    Find a node w in C_{in} such that $c(w, t)$ is minimum.

Designate w as the new "hub" node of C_{in}.
Route demand of each source in C_{in} to w.
Find a node w in C_{out} such that $c(w, t)$ is minimum.
Designate w as the new "hub" node of C_{out}.
Route demand of each source in C_{out} to w.

It can be easily verified that Algorithm UNIFORM has a polynomial time complexity. We proceed to analyze the worst-case performance of the solutions output by the algorithm.

LEMMA 2.1. *Algorithm UNIFORM assigns sources to hubs such that when the demand of each source is routed to its hub node simultaneously, the total flow on an edge of the tree T is at most the cable capacity U.*

PROOF. We pick an arbitrary edge e of T, and let v be the incident node on e with the higher level (see Figure 1). Consider the solution output by the algorithm. In this solution the total flow on e equals the sum of the flow coming out of T_v and the flow going into T_v. Our proof is based on these two claims:

CLAIM 1. *The total flow going out of T_v is at most $U - 1$.*

CLAIM 2. *The total flow coming into T_v is at most $U - 1$.*

To prove Claims 1 and 2, we consider two cases based on how the sources in T_v are assigned to hub nodes by the algorithm. A *partially assigned subtree* has at least two sources assigned to distinct hub nodes.

Suppose T_v is partially assigned (see Figure 2). There exists outflow from T_v if at least one source in T_v is assigned to a hub node out of T_v. When the algorithm assigns sources in T_v to a hub node outside T_v for the first time, a subtree $T_{\bar{v}}$ with $\bar{v}$ at a smaller level than v is being processed by the algorithm. Since $\bar{v}$ is a node with maximum level such

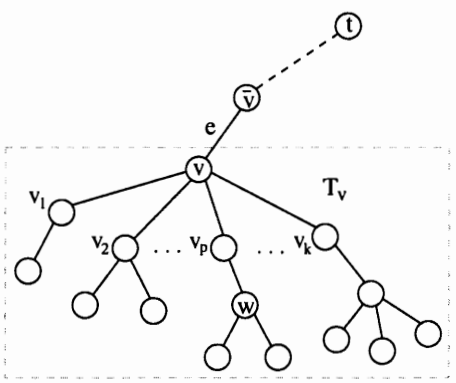

Fig. 1. A node v of T, its children and its subtree.

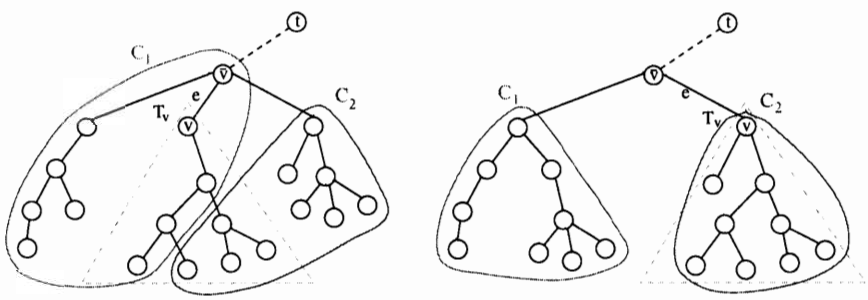

Fig. 2. Examples of partially and completely assigned subtrees.

that $D(T_{\bar{v}}) \geq U$, we can conclude that the total unprocessed demand in T_v is strictly less than U. Therefore, the total outflow from T_v will be at most $U - 1$. Hence, Claim 1 holds in this case.

The reason Claim 2 holds for a partially assigned T_v is as follows. When there exists an inflow into T_v, the flow is accumulated at a hub node in T_v. Since the algorithm accumulates a flow of exactly U at any hub node, a flow of at most $U - 1$ will go into T_v. The algorithm first picks a subtree and a hub node in it, and collects the demand starting with the subtrees of T_v. Therefore, the algorithm will not collect sources out of T_v unless all the sources in T_v have already been collected. This implies that once flow enters T_v, none of the nodes in T_v will become a hub node again. Thus Claims 1 and 2 are proved for the case of partially assigned T_v.

Now let us assume that T_v is not partially assigned. Then all the sources in T_v are assigned to a common hub node. If these sources are routed to a hub node out of the subtree, then the outflow is at most $U - 1$. If the sources are routed to a hub node in the subtree, then the inflow is at most $U - 1$. Inflow or outflow on this edge occurs only once throughout the iterations of the algorithm. Thus, Claims 1 and 2 hold in this case, too.

For any edge of T, the flow in *one* direction does not exceed U, by Claims 1 and 2. If the edge e has flow in both directions such that $f_{\text{in}}(e) + f_{\text{out}}(e) > U$, then at the final step of the algorithm we cancel flow of equal value in opposite directions. As a result, the flow in one direction will be equal to zero, and the flow in the opposite direction will be equal to the difference of $f_{\text{in}}(e)$ and $f_{\text{out}}(e)$; hence, the total flow on e will not exceed U. The cancellation of flow of value $k = |f_{\text{in}}(e) - f_{\text{out}}(e)|$ will lead to the swapping of k sources which sent flow on e in one direction with that of k other sources which sent flow in the opposite direction between the two collections C_{in} and C_{out}. As a result, hub nodes for the updated source sets C_{in} and C_{out} are reselected. However, the new hub for C_{in} will still be a node in T_v because the new additions to C_{in} are nodes in T_v, and similarly the new hub for C_{out} will still be a node outside T_v because the new additions to C_{out} are not in T_v. Although some of the routes for the sources in C_{in} and C_{out} will change, the total flow on any of the edges on their routes will not increase due to the following. Let T_{in} be the subtree composed of the union of paths from each s_i in S_{in} to w_{in}, and similarly let T_{out} be the subtree composed of the union of paths from each s_i in S_{out} to w_{out}. For the edges in the intersection of T_{in} and T_{out}, flow in opposite directions

will be cancelled; hence the total flow on these edges will decrease. For the remaining edges of T_{in} and T_{out}, some of the flow in one direction (of value up to k) will reverse its direction but the total flow will not increase in amount. □

An example to flow cancellation is given in Figure 3. In the figure, T_5 is a partially assigned tree, where sources 1, 2, 3, 4, and 5 of T_5 are assigned to the outside hub node 11

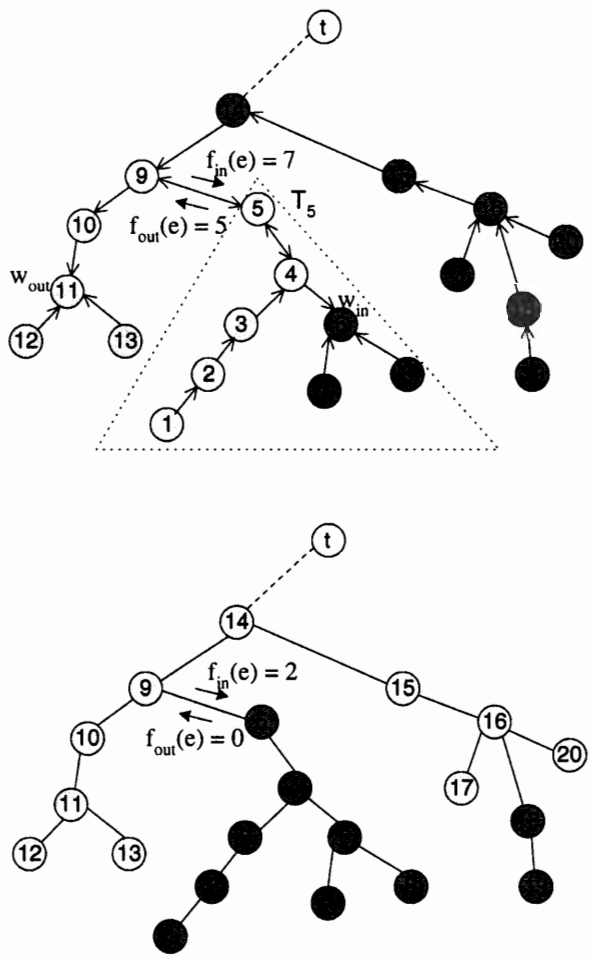

Fig. 3. An example instance with unit demands and $U = 10$ to illustrate the cancellation of flow. (a) Subtree T_5 is partially assigned: nodes 1, 2, 3, 4, and 5 of T_5 are assigned to hub node 11 along with nodes 9, 10, 11, 12 and 13; nodes 6, 7, 8 of T_5 are assigned to hub node 6 along with nodes 14, 15, 16, 17, 18, 19 and 20. The direction of positive flow arising from these assignments are indicated by arrows on the arcs. On edge $e = (5, 9)$ the sum of flow in both directions exceeds U. (b) Flow of value 5 is cancelled on edge e by assigning nodes 1, 2, 3, 4, 5 to a hub node in T_5 and in return, nodes 14, 15, 16, 17, 20 to a hub outside T_5.

along with nodes 9, 10, 11, 12 and 13; sources 6, 7, 8 of T_5 are assigned to the inside hub node 6 along with nodes 14, 15, 16, 17, 18, 19 and 20. Note that $S_{\text{out}} = \{1, 2, 3, 4, 5\}$, $C_{\text{out}} = \{1, 2, 3, 4, 5, 9, 10, 11, 12, 13\}$, $w_{\text{out}} = 11$; $S_{\text{in}} = \{14, 15, 16, 17, 18, 19, 20\}$, $C_{\text{in}} = \{6, 7, 8, 14, 15, 16, 17, 18, 19, 20\}$, $w_{\text{in}} = 6$. On edge $e = (5, 9)$, the sum of flow in both directions exceeds U, where $U = 10$, since $f_{\text{in}}(e) = |S_{\text{in}}| = 7$ and $f_{\text{out}}(e) = |S_{\text{out}}| = 5$. Flow of value 5 is cancelled on edge e by assigning sources in S_{out} to C_{in}, and a subset of S_{in} of size 5, say $\{14, 15, 16, 17, 20\}$, to C_{out}. After the reassignments, the hub nodes are selected from $C_{\text{in}} = \{1, 2, 3, 4, 5, 6, 7, 8, 18, 19\}$ and $C_{\text{out}} = \{9, 10, 11, 12, 13, 14, 15, 16, 17, 20\}$. Note that the new w_{in} will be one of the sources in T_5 and the new w_{out} will be one of the sources out of T_5 since the newly added sources to C_{in} are exclusively in T_5 and those added to C_{out} are exclusively out of T_5.

THEOREM 2.2. *Algorithm UNIFORM is a* $(1 + \rho_{\text{ST}})$*-approximation algorithm for the capacitated network design problem with unit demands.*

PROOF. Let C_{OPT} be the cost of an optimal solution and let C_A be the cost of a solution output by Algorithm UNIFORM. Let C_{ST} denote the cost of the the cables installed on the edges of the Steiner tree T to support flow from sources to the hub nodes. Let C_G be the cost of the cables installed on the edges of G to send aggregated flow from the hub nodes to the sink node with minimum cost. Then C_A can be expressed as the sum of C_{ST} and C_G. We bound C_{ST} and C_G using two lower bounds on C_{OPT}.

The optimal cost of a Steiner tree with terminal set $S \cup \{t\}$ is a lower bound on C_{OPT} because we must connect the nodes in S to t and install at least one copy of the cable on each connecting edge. By Lemma 2.1, at most one copy of the cable is sufficient to accommodate flow on the edges of the Steiner tree T; hence C_{ST} equals the total edge cost of T. Since the algorithm uses a Steiner tree whose cost is at most ρ_{ST} times the cost of an optimal Steiner tree, $C_{\text{ST}} \leq \rho_{\text{ST}} C_{\text{OPT}}$.

A source set C_k collected at iteration k of the main step has demand equal to U and the algorithm installs one copy of the cable on the path from the hub node w_k to the sink t with cost $c(w_k, t)$. The term $\sum_{s_i \in C_k} (c(s_i, t)/U)$ is the cost associated with routing the unit demand of each source s_i of C_k through a minimum cost path and purchasing on each edge of the path $1/U$ of the cable. The algorithm identifies a collection of subtrees with disjoint source sets (but possibly common non-source nodes). Therefore, $\sum_k \sum_{s_i \in C_k} (c(s_i, t)/U)$ is a lower bound on C_{OPT}. Since the algorithm sends the total demand of a set C_k via the source in C_k with the minimum cost of reaching t (that is, the hub node w_k) and $|C_k| = U$, we get

$$c(w_k, t) = \min_{s_i \in C_k} c(s_i, t) \leq \frac{\sum_{s_i \in C_k} c(s_i, t)}{|C_k|} = \sum_{s_i \in C_k} \frac{c(s_i, t)}{U}.$$

Thus, we finally have

$$C_G = \sum_k c(w_k, t) \leq \sum_k \sum_{s_i \in C_k} \frac{c(s_i, t)}{U} \leq C_{\text{OPT}}.$$

Therefore, $C_A = C_{\text{ST}} + C_G \leq (1 + \rho_{\text{ST}}) C_{\text{OPT}}$. □

When every non-sink node in the input graph has a non-zero demand, we do not need to construct a Steiner tree. Instead, we input a minimum cost spanning tree T to Algorithm UNIFORM. In the proof of Theorem 2.2, C_{ST} refers to the cost of the spanning tree T and ρ_{ST} is equal to 1. Therefore, we obtain a performance ratio of 2.

COROLLARY 2.3. *Algorithm UNIFORM is a 2-approximation algorithm for the capacitated network design problem with unit demand at every non-sink node.*

3. Integer Demands. When the demand of a source node can be any positive integral value, it is no longer possible to collect sources with total demand exactly equal to the capacity U. Suppose we are allowed to split the demand of a source into any integral units, each of which can be routed in separate paths to the sink. In that case the algorithm of the previous section can be used by expanding each source s_i with demand dem_i to a set of sources with unit demand connected by zero-length edges in the tree. However, in the more general case all the flow of a source must use the same path to the sink. In this case we modify Algorithm UNIFORM so that we send the demand directly to the sink when it accumulates to an amount between $U/2$ and U. To guarantee that we do not exceed U while collecting the demand, we send all sources with demand at least $U/2$ directly at the beginning of the algorithm.

For a source set C, let $dem(C)$ be the total demand of sources in C. As defined for the uniform demand case, we use $D(C)$ to denote the total *remaining* (unprocessed) demand of C. The modified algorithm, which we call Algorithm NON-UNIFORM, is given below:

Algorithm NON-UNIFORM

Initialize: $R := S$.
Preprocessing (send large demands directly):
  For all sources s_i such that $dem_i \geq U/2$,
    Route the demand to the sink via a minimum cost path in G.
    Install $\lceil dem_i/U \rceil$ additional copies of the cable on the edges of this $s_i - t$ path.
    Remove s_i from R.
Main step:
  Pick a node v such that $D(T_v) \geq U/2$ and the level of v is maximum.
  If no such node exists (that is, $D(T_t) < U/2$) or $v = t$, then go to the final step.
  Find a node, say w, in $R \cap T_v$ such that $c(w, t)$ is minimum.
  Designate w as a "hub" node and set $C = \{w\}$.
  Collect additional source nodes in C (*described below*).
  Assign these sources to w.
  Route the demand of each source in C to the hub node w via the unique paths in T.
  Route demand aggregated at w via a minimum cost path to the sink in G, namely P_w.
  Install one copy of the cable on P_w, in addition to any existing cables.
  Remove C from R.
  If R is not empty, repeat the main step.
  If R is empty, go to the final step.

Final step:

If R is not empty, then route all the demand in R to t via the unique paths in T. Install one copy of the cable on the edges of T which have positive flow, in addition to any existing cables.

The following procedure is used to collect unprocessed source nodes from T_v in the set C at the main step of the algorithm. The set C contains only the hub node w when the procedure is called. At the end of the procedure the demand of C is in the range $[U/2, U]$.

Procedure to collect source nodes of T_v

Add v to C, if $v \in R$.
Let $v_1, \ldots, v_k$ be the children of v.
If $w \neq v$, then
    Let v_p be the child of v such that the hub node w is in T_{v_p}.
    Add $T_{v_p} \cap R$ to C.
While $dem(C) < U/2$,
    Pick an unprocessed child of v, say v_i.
    Add $T_{v_i} \cap R$ to C.
Return C.

LEMMA 3.1. *Algorithm NON-UNIFORM assigns sources to hubs such that when the demand of each source is routed to its hub node simultaneously, the total flow on an edge of the tree T is at most the cable capacity U.*

PROOF. The proof is much simpler compared with the unit-demand case because the algorithm does not assign any subtree partially. Consider an edge e of T. Let v be incident on e such that e is not in T_v. Since all the sources in T_v are collected in the same set by the algorithm, the demand of these sources is routed to a hub node either out of the subtree, or in the subtree, but not both. Thus, flow on e exists only in one direction. If the demand of the sources is routed to a hub node out of T_v, then the outflow is at most $U - 1$. If the demand is routed to a hub node in the subtree, then inflow is at most $U - 1$. Thus, for any edge of T, flow does not exceed U. □

LEMMA 3.2. *At each iteration of Algorithm NON-UNIFORM, flow sent from a hub node to the sink in G is at least $U/2$ and at most U.*

PROOF. Due to the subtree selection rule in the algorithm, if a subtree T_v is selected at an iteration, then all the subtrees rooted at its children have remaining demand strictly less than $U/2$. Therefore, when source nodes are collected in C, the first time the total collected demand $dem(C)$ exceeds $U/2$, $dem(C)$ will be at most U. Therefore, the total flow sent from the hub to the sink at this iteration is in the range $[U/2, U]$. □

THEOREM 3.3. *Algorithm NON-UNIFORM is a $(2+\rho_{ST})$-approximation algorithm for the capacitated network design problem with integer demands.*

PROOF. We use the same definitions of C_{OPT}, C_A, C_G and C_{ST} as in the proof of Theorem 2.2.

By Lemma 3.1, at most one copy of the cable is sufficient to accommodate flow on the edges of the Steiner tree T. Therefore, $C_{\text{ST}} \leq \rho_{\text{ST}} C_{\text{OPT}}$.

For a source set C_k collected at iteration k, the algorithm installs one copy of the cable on the minimum cost $w_k - t$ path in G. By Lemma 3.2, at most one copy of the cable is sufficient to accommodate flow on this path and the flow utilizes at least half of the capacity of the cable. Thus, $\sum_{s_i \in C_k} dem_i \geq U/2$. If we were allowed to pay for only the portion of the cable used, then a minimum cost solution would use the minimum cost path from each source to the sink. This solution would have cost $\sum_{s_i \in S} (dem_i/U) \cdot c(s_i, t)$ and this is a lower bound on C_{OPT}. Since source sets collected by the algorithm have disjoint sources, and the demand from a set C_k is sent via the source in C_k that is closest to t (the hub node w_k),

$$\begin{aligned} C_{\text{OPT}} &\geq \sum_k \sum_{s_i \in C_k} \frac{dem_i}{U} c(s_i, t) \geq \sum_k \sum_{s_i \in C_k} \frac{dem_i}{U} \left(\min_{s_i \in C_k} c(s_i, t) \right) \\ &\geq \sum_k \sum_{s_i \in C_k} \tfrac{1}{2} c(w_k, t) = \tfrac{1}{2} C_G. \end{aligned}$$

The last inequality follows since $\sum_{s_i \in C_k} dem_i \geq U/2$ and $\min_{s_i \in C_k} c(s_i, t) = c(w_k, t)$. Therefore, $C_A = C_{\text{ST}} + C_G \leq (2 + \rho_{\text{ST}}) C_{\text{OPT}}$. □

When every non-sink node in the input graph has a non-zero demand, we do not need to construct a Steiner tree. Instead, we input a minimum cost spanning tree T to Algorithm NON-UNIFORM. In the proof of Theorem 3.3, C_{ST} refers to the cost of the spanning tree T and ρ_{ST} is equal to 1. Therefore, we obtain a performance ratio of 3.

COROLLARY 3.4. *Algorithm NON-UNIFORM is a 3-approximation algorithm for the capacitated network design problem with positive integer demand at every non-sink node.*

4. Extensions. Our methods apply to the following extension of the local access network design problem. Instead of specifying a single sink node, any node v in the graph can be used as a node that sinks U units of demand at a cost of f_v. A node is allowed to sink more than U units of demand by paying $\lceil dem/U \rceil \cdot f_v$ cost to sink dem units of flow. The problem is to open a sufficient number of sinks and route all the demands to these sinks at minimum cable plus sink opening costs.

To model this extension, we extend the metric in two steps: (1) create a new sink node t with edges to every vertex v of cost f_v; (2) take the metric completion of this augmented network. Notice that the second step may decrease some of the costs on the edges incident on the new sink t (e.g., if $f_i + c(j, i) < f_j$, then the cost of the edge (j, t) can be reduced from f_j to $f_i + c(j, i)$), or between any pair of original nodes (e.g., if $c(i, j) > f_i + f_j$, then we may replace the former by the latter). Bearing this in mind, it is not hard to see that any solution in the new graph to the single cable problem with t as the sink and with the modified costs can be converted to a solution to the original

problem of the same cost. Thus, our algorithms in the previous sections apply to give the same performance guarantees.

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