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On the minimum diameter spanning tree problem

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Abstract

We point out a relation between the minimum diameter spanning tree of a graph and its absolute 1-center. We use this relation to solve the diameter problem and an extension of it efficiently.

Keywords: Algorithms; Combinatorial problems; Minimum diameter spanning tree; Absolute 1-center

1. Introduction

Let $G = (V, E)$ be an undirected graph, where V is the set of nodes and E is the set of edges. Also let $|V| = n$ and $|E| = m$. Suppose that each edge $e \in E$ is associated with a positive weight (length) d_e. A spanning tree of G is a connected subgraph $T = (V, E_T)$ without cycles. The *diameter* of T, $D(T)$, is defined as the longest of the shortest paths in T among all the pairs of nodes in V. The *minimum diameter spanning tree* (MDST) problem is to find a spanning tree of G of minimum diameter.

Ho, Lee, Chang and Wong [5] consider the case where the graph G is a complete Euclidean graph, induced by a set S of n points in the Euclidean plane. They call this special case the *geometric* MDST problem. They prove that in this geometric problem there is an optimal tree which is either monopolar or dipolar. A spanning tree is *monopolar* if there exists a point in S called a *monopole* such that each of the remaining points is connected to it; and it is said to be *dipolar* if there exist a pair of points in S called a *dipole* such that all the remaining points are directly connected to exactly one of the two points in the dipole. Based on the latter property, Ho et al. [5] develop an $O(n^3)$ algorithm to find a spanning tree of minimum diameter of a Euclidean graph. They also mention that the above results extend to any graph whose edge lengths satisfy the triangle inequality.

In this note we consider the general case where the edge lengths do not necessarily satisfy the triangle inequality. We then observe that the MDST problem is identical to the well studied absolute 1-center problem introduced by Hakimi in 1964 [3]. As such, one can apply existing algorithms and solve the MDST problem on a general graph in $O(mn + n^2 \log n)$ time.

To facilitate the discussion suppose that each edge of the given graph $G = (V, E)$ is rectifiable. Thus, we refer also to interior points on an edge by their distances (along the edge) from the two nodes of the edge. We let $A(G)$ denote the continuum set of points on the edges of G. The edge lengths induce a distance function on $A(G)$. For any pair of points x and y in $A(G)$ let $d_G(x, y)$ denote the length of a shortest path in $A(G)$ connecting x and y. For each x in $A(G)$ let $T(x)$ denote a shortest path tree connecting x to all

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the nodes in V. $T(x)$ can be found by augmenting x to the node set V, and computing the shortest paths from x to all nodes in V in $O(m + n\log n)$ time by the algorithm in [2].

For each x in $A(G)$ define

$$F(x) = \max_{v\in V} d_G(x, v).$$

The *absolute 1-center problem* (A1CP) on G is to minimize the function $F(x)$. A point x^* in $A(G)$ is an *absolute 1-center* of G if the function F attains its minimum at x^*. There are several efficient algorithms to locate an absolute 1-center. See for example, [9,6,8,1]. The most efficient known algorithm is due to Kariv and Hakimi [8]. It can be implemented in $O(mn + n^2\log n)$ time, if one uses the procedure of Fredman and Tarjan [2] to compute the distances between all pairs of nodes in V.

2. Equivalence of A1CP and MDST

We now observe that an absolute 1-center of a graph defines a spanning tree of minimum diameter.

Theorem 1. *Let x^* be an absolute 1-center of G and let $T(x^*)$ be a shortest path tree connecting x^* to all nodes in V. Then $T(x^*)$ is a minimum diameter spanning tree of G.*

Proof. Let T be an arbitrary spanning tree of G. Let $A(T)$ be the metric space consisting of the continuum set of points on the edges of T, with the respective distance function, $d_T(x, y)$, induced by the edges of T. Also let $y^*(T)$ be the absolute 1-center of T, i.e., $y^*(T)$ is a minimizer of the function $F_T(x) = \max_{v\in V}\{d_T(x, v)\}$ over $A(T)$.

Following Handler [4] we conclude that $y^*(T)$ is unique and furthermore, that $D(T)$, the diameter of T, satisfies

$$D(T) = 2F_T(y^*(T)).$$

Let $T(x^*)$ be a shortest path tree connecting x^*, the absolute 1-center of G, to all nodes in V. From the optimality of x^* it follows that x^* is the midpoint of every diameter of $T(x^*)$. Then,

$$D(T(x^*)) = 2\max_{v\in V} d_{T(x^*)}(x^*, v) \quad (1)$$

$$= 2\max_{v\in V} d_G(x^*, v) \quad (2)$$

$$\leqslant 2\max_{v\in V} d_G(y^*(T), v) \quad (3)$$

$$\leqslant 2\max_{v\in V} d_T(y^*(T), v) \quad (4)$$

$$= D(T). \quad (5)$$

Eqs. (1) and (5) follow from Handler's result. Eq. (2) holds since, by the definition of $T(x^*)$, $d_G(x^*, v) = d_{T(x^*)}(x^*, v)$ for every $v \in V$. Inequality (3) holds since x^* is the absolute 1-center of G. Inequality (4) holds since for any pair of points $x, y \in A(G)$, $d_G(x, y) \leqslant d_T(x, y)$. We conclude therefore that $D(T(x^*))$ is the minimum diameter of any spanning tree of G. $\square$

3. Applications

The theorem and the above discussion imply that a minimum diameter spanning tree of a general graph G can be found in $O(mn + n^2\log n)$ time. If G is complete (e.g., a Euclidean graph) this bound reduces to $O(n^3)$. Note that the result of Ho et al. [5] about the existence of an optimal minimum diameter spanning tree of a Euclidean graph, which is either monopolar or dipolar follows directly from the theorem. Indeed, if the edge lengths satisfy the triangle inequality, an optimal tree is monopolar if x^* is a node, and it is dipolar otherwise (the dipole consists of the two endpoints of the edge containing x^*).

The equivalence between the 1-center problem and the MDST problem clearly holds also for the multicenter problem. Let $X_p = \{x_1, \ldots, x_p\}$ be a set of p points in $A(G)$. Define

$$H(X_p) = \max_{v\in V} \min\{d_G(x_i, v) \mid i = 1, \ldots, p\}.$$

X_p^* is called an *absolute p-center* of G if the function H attains its minimum at X_p^*. From the above theorem it follows that the absolute p-center problem is equivalent to the following *minimum diameter p-forest* problem:

Find a subgraph $G' = (V, E_{G'})$ without cycles and with at most p connected components (subtrees), such that the maximum of the diameters of its connected components is minimized. As indicated by Kariv and Hakimi [8] the above problem is NP-hard when p is part of the input. However, the problem is polynomially solvable for a fixed value of p. The best known complexity bound, $O(m^p n^{p-1} \log^3 n)$ for $p \geqslant 2$, is given in [10].

Ihler, Reich and Widmayer [7] generalized the results of [5] to the geometric *MDST problem with classes* defined as follows: Given the graph $G = (V, E)$ and a partition $V_1, \ldots, V_c$ of V into c *classes*, compute a tree of minimum diameter that contains at least one vertex of each class. They show how to compute an optimal solution in $O(n^3)$ time also for this problem. We provide now an outline of how Theorem 1 can be used also for this problem and its generalization to general graphs and p-forest. Consequently, these problems can also be solved within the same time that we stated above.

We show how to imitate the algorithm for the absolute 1-center problem. This algorithm first solves the all pair shortest path problem on G, and then treats the edges one by one. While considering x an edge $(i, j) \in E$ where $i \in V_r$ and $j \in V_q$ (possibly $r = q$), the graph is treated as if the classes V_l ($l \neq r, q$) are represented by a single vertex. The distance of this vertex from i (j) is the minimum distance of a vertex in V_l from i (respectively, j). The vertices in V_r and V_q, apart from i and j are ignored. The best location on (i, j) can be found now as in the 1-center algorithm, with some minor changes.

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