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# Equilibrium strategies and the value of information in a two line queueing system with threshold jockeying

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**Abstract**

We consider memoryless two-line system with threshold jockeying. Upon arrival each customer decides whether to purchase the information about which line is shorter, or randomly selects one of the lines. Since the decision of a customer is affected by the decision of the others, we are interested in Nash-equilibrium policies. Indeed, we show explicitly how to find these policies. We are also interested in the externalities imposed by an informed customer on the others. We derive an explicit expression for these externalities in the case that jockeying takes place as soon as the lines differ by three. Some of the results may seem to be counterintuitive. For example, when the threshold is three, the value of information may increase with the portion of informed customers.

## 1 INTRODUCTION

Queueing systems with with two waiting lines in parallel have been analyzed extensively. In one extreme lies the instantaneous jockeying model in which whenever the difference between the lengths of the two lines becomes greater than one, the customer in the back of the longer line jockeys to the back of the other line. In particular, in this model arrivals always join the shorter line. As the total number of customers in the system coincides with the number in the corresponding one-line system, the analysis of this model is tractable. Apparently the first to consider this model was Koenigsberg [10]. See also Maekawa [11] for the Laplace transform of the waiting time and of some of its moments and Haviv and Ritov [8] and Zhao and Grassmann [15] for generalizations of this model.

On the other extreme lies the non-jockeying model where arrivals still join the shorter line but are do not jockey later to the other line even if it is much shorter. Contrary to the first model, this model turned out to be intractable. Starting with Haight [6] and Kingman [9] and up to recently by Zhao and Grassmann [16] and Adan, Wessels and Zijm [2], expressions for the stationary distribution were developed. All are relatively hard to work with as they involve complex variables analysis or infinite summation of series expansions, calling the practitioner to use truncations and approximations. Also, this problem was reduced by Fayolle and Iasnogorodski [4] to a Riemann–Hilbert boundary problem. Moreover, finite expressions for the waiting time moments or for its distribution are not available.

Starting with Gertsbakh [5] attention was directed to an intermediate model with *threshold* jockeying. Again, customers join the shorter line, but jockeying occurs only when the difference between the lengths of the two lines reaches some threshold value, say $N$. Note that the two models described above are the two extreme cases, $N = 2$ and $N = \infty$, respectively. For finite values of $N$ these models are more tractable than the case $N = \infty$ since the stationary distribution follows a matrix geometric pattern where the size of the rate matrix is at most $N \times N$. See Zhao and Grassmann [17] and Adan, Wessels and Zijm [2,3] for details and see Wang and Hlynka [14] for a special analysis for $N = 3$. In particular, the latter reference describes some real life applications, like toll booth queues, where threshold jockeying takes place. Moreover, as shown by Adan, van Houtum and van der Wal [1], results from queues with threshold jockeying can serve as bounds on the corresponding results in systems without jockeying. Once the stationary distribution is known, it is possible to compute the expected number of customers in the system and the expected waiting time. To the best of our knowledge neither the distribution of the waiting time nor its Laplace transform appear in the literature. [1]

The question we address is different from the traditional ones of stationary distributions and waiting times. It is concerned with the issue of the value of information. Suppose that upon arrival a customer does not

[1]Although this is not our main purpose, we actually fill this gap here.

know which queue is shorter and hence he joins each queue with probability 0.5. Also, suppose he has the option of acquiring the information on which queue is shorter and if he exercises this option, he indeed joins the shorter queue. This option does not come free: we assume that the cost has some value which for convenience is measured in units of time. Thus, an arrival who is self-optimization oriented compares the cost associated with acquiring the information with the expected gain from joining the shorter line rather than doing so only with probability 0.5. Of course, in the case of instantaneous jockeying ($N = 2$) the information has no value. But this is not the case in general. We point out that in our model, informed and uninformed customers behave in the same way with respect to jockeying.

A crucial observation for this model is that the value of the information for an individual, and therefore also his decision of whether to acquire it, depend on what is done by the others. Thus, the relevant solution concept is that of a Nash-equilibrium strategy. By limiting ourselves to symmetric strategies, a Nash-equilibrium strategy will be characterized by a parameter $p$, $0 \leq p \leq 1$, where $p$ is the probability of purchasing the information. Of course, $p = 0$ ($p = 1$, resp.), namely nobody (everybody, resp.) acquires the information is possible. If $p = 0$ or if $p = 1$ the strategy is called *pure*. Otherwise, it is called *mixed*.

In order to find a Nash-equilibrium strategy we first have to derive the value of information for an individual arrival, namely his expected reduction in expected waiting time due to information acquiring, when the rest of the customers are using strategy $p$. Call this value $g(p)$. The value of information has to be compared with the cost of information (measured in units of time) which is denoted here by $C$. A probability $p$ strictly between zero and one specifies a Nash-equilibrium strategy if and only if $g(p) = C$. Also, if $g(0) \leq C$ then $p = 0$ is a Nash-equilibrium strategy and if $g(1) \geq C$ than $p = 1$ is a Nash-equilibrium strategy. It may happen that more then one Nash-equilibrium strategy exists. For example, in a numerical example we report later, when $N = 3$, $g(0) \leq g(p) \leq g(1)$ for all $p$, $0 \leq p \leq 1$. Thus, even if $g(p) = C$ for some $p$, $0 < p < 1$, resulting in a mixed Nash-equilibrium strategy, the pure strategies $p = 0$ and $p = 1$ are also Nash-equilibria.

We will claim that the actual information value for a customer depends only on the state of the system upon arrival. If the customers to arrive later on are informed or not is not of relevance to him. Hence, all that matters for the information value for an individual customer are the probabilities that he sees states where the information is useful and the gains corresponding to these states (which are not functions of $p$). For example, in the case where $N = 3$ these are the states in which the difference between the two queue lengths is exactly one.

One could argue intuitively that the value of information is a decreasing function of $p$ as follows. The larger the portion of informed customers, the better the utilization of the two servers: a server is less likely to be idle while customers are waiting. Customers clear the system faster, the expected waiting time is smaller, and hence the expected difference in waiting time due to information acquiring is also smaller. Moreover, when the portion of informed customers is larger, then, for a fixed number of customers in the system, the lengths of the two lines tend to be more even, so that the value of information is smaller.

We claim that the above intuitive argument maybe true but it is not complete. Actually, there need not be any direct relation between the expected waiting time and the reduction in expected waiting time gained when purchasing information. As a matter of fact, we will show numerically that the opposite is true for $N = 3$ where actually the value of information increases with $p$. In particular, for this model the information for an individual customer is more valuable when all the others acquire it than when none of them does. However, when $N \geq 4$, we show numerically that the value of information decreases with $p$ as expected by our original intuition.

We explain now, through an example, why when $N = 3$ it makes sense that the value of information increases with $p$. Suppose an arrival faces state $(0, 1)$, namely he comes to a system in which there is one customer in service and where one server is idle. If he is informed, the next state will be $(1, 1)$. Otherwise, it might be $(0, 2)$. This implies that if this customer is informed, then the next to arrive is more likely to arrive to state $(0, 1)$. [2] This is a state in which information is valuable. Hence, the fact that a customer is informed tends, in this case, to increase the value of information for others.

Another question which we address here is the question of the *externalities* associated with purchasing the information. Clearly, the larger the portion of arrivals who acquire the information the less is the overall waiting time. This is so since the probability that a server is idle while a customer is waiting for service decreases. However, when $C > 0$, we cannot say without further examination that customers must be encouraged to purchase more information. The reason is that the reduction in waiting time of those who buy information may be on account of others who will have to wait more. Hence, an interesting qualitative question is whether the effect (externality) of one's action due to the acquisition of information on *others* is positive or negative. If it is negative, then for some possible values of $C$ self optimizing individuals (who behave as prescribe by the equilibrium strategy) will buy more information than it is socially desired. (See, Hassin [7]) for an

[2] Note that transition from state $(1, 1)$ to state $(0, 1)$ takes place as soon as one of the two busy server completes service. For transition form state $(0, 2)$ into state $(0, 1)$ it is required that the single busy server completes his service.

analysis for a similar question for a single server model. There, it is shown that under suitable conditions, social welfare will be increased if information about the queue length will be suppressed from the customers.) For $N = 3$, we derive closed form expressions for the positive and the negative parts of the externality. We show numerically that the total externality is a monotone increasing function of $p$. For small values of the traffic intensity the externality as function of $p$ is positive. For large values, the externality is negative for small values of $p$ and positive for large values of $p$. Thus, depending on $C$, the externality at a Nash equilibrium strategy may be positive or negative. Treating the issue of externalities for an arbitrary value of $N$ seems to be a much harder task.

The paper is organized as follows. In Section 2 we show how to compute the stationary distribution of the system for any value of $N$ and for any value of $p$. In Section 3 we derive the expected sojourn times for customers in various positions in the queue. In Section 4, the results of the previous two sections are combined and the value of information is derived. This leads to the Nash-equilibrium strategies. Section 5 is devoted to the computation of the externality for $N = 3$. We conclude in Section 6 with some remarks.

## 2 THE STATIONARY DISTRIBUTION

In this section we show how to compute the stationary distribution for the memoryless two parallel line model when jockeying occurs as soon as the difference between the two queues reaches the value of $N$ for some $N$, $3 \leq N < \infty$. In order to simplify notation, we omit the parameter $N$ from now on in this section. We use matrix-geometric computation (see Neuts, 1981). The technique requires a decomposition of the state space. The decomposition is not unique and the one we have selected is the most convenient for our purpose.

Let $\lambda$ be the rate of arrival and let $1/\mu$ be the expected service requirement. Scale $\lambda$ and $\mu$ such that $\lambda + 2\mu = 1$. Let $\pi_{i,j}$ be the stationary probability that $i$ customers are in one line (no matter which) and $j$ customers are in the other line. These numbers include customers in service. Without loss of generality, assume that $i \leq j$ For $i \geq 0$, let $L(i)$ be the set of $N$ states $(i, j)$ for $i \leq j \leq i + N - 1$. Order these $N$ states by $(i, i), (i, i+1), \cdots, (i, i+N-1)$. Let $\underline{\pi}_i$ be the row-vector of the stationary probabilities of the states in $L(i)$ ordered as above. For $i \geq 1$, transition from a state in $L(i)$ can take place only to states in $L(i-1)$, $L(i)$ or $L(i+1)$. Thus, for some easy to find matrices $Q_0$,$Q_1$ and $Q_2$ in $R^{N \times N}$, for $i \geq 1$,

$$\underline{\pi}_i Q_0 + \underline{\pi}_{i+1} Q_1 + \underline{\pi}_{i+2} Q_2 = \underline{0} \quad . \tag{1}$$

Specifically, let $\lambda_1 = \lambda(1+p)/2$ and let $\lambda_2 = \lambda(1-p)/2$. Then,

$$Q_0(ij) = \begin{cases} \lambda_1 & i = 2,\ldots,N-1 \ , \ j = i-1 \\ \lambda & i = N \ , \ j = N-1 \\ 0 & otherwise \end{cases}$$

or

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \lambda & 0 \end{pmatrix}$$

$$Q_1(ij) = \begin{cases} -1 & j = i \ , \ i = 1,\ldots,N \\ \lambda & i = 1 \ , \ j = 2 \\ \mu & i = 2,\ldots,N-1 \ , \ j = i-1 \\ \lambda_2 & i = 2,\ldots,N-1 \ , \ j = i+1 \\ 2\mu & i = N \ , \ j = N-1 \\ 0 & otherwise \end{cases}$$

or

$$Q_1 = \begin{pmatrix} -1 & \lambda & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \mu & -1 & \lambda_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \mu & -1 & \lambda_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu & -1 & \lambda_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \mu & -1 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2\mu & -1 \end{pmatrix}$$

and

$$Q_2(ij) = \begin{cases} 2\mu & i = 1 \ , \ j = 2 \\ \mu & i = 2,\ldots,N-1 \ , \ j = i+1 \ , \\ 0 & otherwise \end{cases}$$

or

$$Q_2 = \begin{pmatrix} 0 & 2\mu & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Consequently, there exists a rate matrix (which is a function of $p$) $R \in R^{N\times N}$ such that for $i \geq 0$, $\underline{\pi}_{i+1} = \underline{\pi}_i R$. See Neuts [13, pp. 80–83]. Actually $R$ is the minimal matrix which satisfies the matrix equation $X^2Q_2 + X(I + Q_1) + Q_0 = X$. Moreover, $R = \lim_{k\to\infty} X(k)$ where $X(0)$ is the zero $N \times N$ matrix and where for $k \geq 0$, $X(k+1) = X(k)^2Q_2 + X(k)(I+Q_1) + Q_0$. Zhao and Grassmann [17] proved that for $p = 1$ the expression for the stationary distribution simplifies even further as the rate matrix can be replaced by a rate scalar, $\rho^2$. Specifically, for $i \geq 1$ each entry in $\underline{\pi}_{i+1}$ equals $\rho^2$ times the corresponding entry in $\underline{\pi}_i$. Unfortunately, this is not necessarily true for any value of $p$. However, by utilizing the spectral representation of the rate matrix $R$, an alternative expression for the stationary probabilities is available. Specifically, let $\omega_1, \omega_2, \ldots, \omega_N$ be the eigenvalues of $R$. Some of them may coincide but we assume that the matrix is not defective, namely a basis for $R^N$ formed by eigenvectors of $R$ exists. Hence, there exist $N$ rank-one (projection) matrices $E_1, E_2, \ldots, E_N$ with the properties that $E_iE_j = 0$ if $i \neq j$ and $E_iE_i = E_i$ for $1 \leq i, j \leq N$. Moreover,

$$R = \Sigma_{i=1}^{N} \omega_i E_i$$

is the spectral representation of $R$. Hence,

$$R^k = \Sigma_{i=1}^{N} \omega_i^k E_i \quad , \quad k \geq 1.$$

It is routine to find the spectral representation of $R$ as many numerical packages contain codes for executing the required computation. It was shown by Neuts [13] that when the stationary distribution exists then all eigenvalues of $R$ are in the unit disk. Moreover, since one row of $Q_0$ is zero, the same is the case with the matrix $R$ and hence at least one of the eigenvalues of $R$, say $w_1$, is zero. Finally, the spectral radius of $R$ is $\rho^2$. We omit the proof for this observation.

The above indicates how to compute $\underline{\pi}_i$ for $i \geq 1$ once $\underline{\pi}_0$ is known. Thus, we turn our attention now to the computation of $\underline{\pi}_0$. Note that in Equation (1) we state the balance equations for all the states in $L(i)$ for $i \geq 1$. There are $N$ balance equations corresponding to the $N$ states in $L(0)$. One of this equations is redundant. One can write down the other $N-1$ equations. Specifically, whenever an entry from $\underline{\pi}_1$ is needed, the corresponding entry from $\underline{\pi}_0 R$ can be taken and by treating $\pi_{0,0}$ as constant and solving for $\pi_{0,1}, \pi_{0,2}, \ldots, \pi_{0,N-1}$, one can get all entries in $\underline{\pi}_0$ expressed in terms of $\pi_{0,0}$. In particular, one gets a row vector $\underline{v} \in R^{1\times N}$ with $\underline{c}_1 = 1$, such that $\underline{\pi}_0 = \pi_{0,0}\underline{v}$. Finally, using the fact that the sum of the probabilities is one, one can solve for $\pi_{0,0}$. Specifically, $\pi_{0,0} = [\underline{v}(I-R)^{-1}\underline{1}]^{-1}$ where $\underline{1} \in R^{N\times 1}$ is a vector all its entries are one. Thus, we got a complete explicit expression for the stationary distribution.

## 3 THE EXPECTED SOJOURN TIME

The following observation, that holds for any value $N$ as the jockeying threshold, is fundamental for our analysis.

*At any instance, the (future) sojourn time of a customer depends on the total number of customers in the system and how many of them are in front of him in his own line. However, given these two numbers, the sojourn time does not depend on the split of the customers between the two lines.*

In accordance with the above observation, for $k \geq 0$ and for $i \geq 0$, let $M_{k,i}$ be the expected (future) sojourn time of a customer who sees $k$ customers in front of him (in his line) and a total of $i$ customers behind him in his line and from position $\max\{k-N+3, 0\}$ and up in the other line. Note that under the model's assumption if a customer sees $k$ customers in front of him in his line for some $k$ with $k \geq N-2$, then all positions in the other line from position one and up to position $k-N+2$ are occupied. Clearly, $M_{k,i} = (k+1)/\mu$ for $0 \leq k \leq N-2$ and $i \geq 0$.

Before solving for $M_{k,i}$ we point out that our interest is in computing the expected value of the information as a function of $p$:

$$g(p) \equiv \sum_{k=0}^{\infty}\sum_{i=1}^{N-2} \pi_{k,k+i}\left[\frac{M_{k+i,N-i-2} + M_{k,N-2+i}}{2} - M_{k,N-2+i}\right] \quad . \tag{2}$$

Here we show how to compute the conditional expected values and their generating functions. The function $g(p)$ itself is given in the next section.

If one assumes that $M_{-1,i} = 0$ then the following partial difference equation is satisfied by the $M_{k,i}$ (assuming $\lambda + 2\mu = 1$),

$$M_{k,i} = 1 + \lambda M_{k,i+1} + \mu M_{k,i-1} + \mu M_{k-1,i+1} \quad , \quad k \geq 0 \;\; , \;\; i \geq 1 \qquad (3)$$

[3] Also, if one assumes that for $k \leq N-2$, $M_{k-N+1,2N-3} = (k+1)/\mu$, then the boundary conditions are

$$M_{k,0} = 1 + \lambda M_{k,1} + \mu M_{k-1,1} + \mu M_{k-N+1,2N-3} \;\; , \;\; k \geq 0, \qquad (4)$$

$$M_{k,i} = \frac{k+1}{\mu} \quad , \quad 0 \leq k \leq N-2 \;\; , \;\; i \geq 0 \qquad (5)$$

[4] and finally,

$$\lim_{i \to \infty} M_{k,i} \leq \frac{k+1}{\mu} \quad , \quad k \geq 0. \qquad (6)$$

Note that equations (3)-(6) and hence their solution(s) are independent of $p$. However, the stationary probabilities, and thus also the unconditional expected sojourn time and the value of information, are functions of $p$. Also, note that the difference Equation (3) but with different boundary conditions appears in Maekawa [11, Equation (23)].

We are now ready to solve (3)-(6). Actually, we next find the generating function of the solution. First, for $i \geq 0$, and for a complex number $t$ with $|t| < 1$, let $G_i(t) = \Sigma_{k=0}^{\infty} M_{k,i} t^k$. Second, fix a value of $i$ with $i \geq 1$. Then, by multiplying the $k-th$ equation in (3) with $t^k$, summing them up and remembering that $M_{-1,i} = 0$, one gets that

$$(\lambda + t\mu)G_{i+1}(t) - G_i(t) + \mu G_{i-1}(t) = \frac{-1}{1-t} \quad , \quad i \geq 1 \;\; . \qquad (7)$$

Note that for a given value of $t$, the difference Equation (7) is ordinary and of the second order. A particular (formal) solution is:

$$G_i(t) \equiv \frac{1}{\mu(1-t)^2} \, , \quad i \geq 0 \;\; .$$

Looking at the homogeneous part we see that both sequences $(f_-(t))^i$ and $(f_+(t))^i$, $0 \leq i < \infty$, with

$$f_-(t) = \frac{1 - \sqrt{1 - 4\mu(\lambda + \mu t)}}{2(\lambda + \mu t)}$$

and with

$$f_+(t) = \frac{1 + \sqrt{1 - 4\mu(\lambda + \mu t)}}{2(\lambda + \mu t)}$$

solve it. Hence,

[3]Note that $M_{0,1} = 1/\mu$, hence Equation (3) holds also for $k = 0$ only if one assigns $M_{-1,0}$ to have a value of zero.

[4]The boundary condition (4) is straightforward for the case that $k - N + 1 \geq 0$. For $0 \leq k \leq N-2$ (and hence $M_{k,0} = (k+1)/\mu$) only the abovementioned definition of $M_{k-N+1,2N-3}$ as $(k+1)/\mu$ validates (4) also for case that $0 \leq k \leq N-2$. Finally, note that the fact that above we assigned $M_{0,-1} = 0$ should not disturb us as long as we remember the different values when we treat the equations.

$$G_i(t) = \frac{1}{\mu(1-t)^2} + C_-(t)(f_-(t))^i + C_+(t)(f_+(t))^i$$

for some functions $C_-(t)$ and $C_+(t)$. Since we assumed $\lambda + 2\mu = 1$ then $\mu \leq .5$ and hence $|f_+(t)| \geq 1$ for any $t$ with $|t| \leq 1$ (with equality if and only if $t = 1$). This coupled with (2) implies that $C_+(t) = 0$. Likewise, $|f_-(t)| \leq 1$ for any $t$ with $|t| \leq 1$ (with equality if and only if $t = 1$). Hence, all that is left is to find the value of $C_-(t)$. This will be done by utilizing (4). Specifically, multiplying each of the equations in (4) by $t^k$ and summing up from $k = 0$ to infinity, leads to an identity (with respect to the variable $t$) which involves $G_0(t)$, $G_1(t)$ and $G_{2N-3}(t)$. Using the form $\frac{1}{\mu(1-t)^2} + C_-(t)(f_-(t))^i$ for $G_i(t)$ when $i$ gets the values 0, 1 and $2N-3$ in the identity, leads to an equation for the single unknown $C_-(t)$ whose value in turn is found to be

$$C_-(t) = \frac{(N-1)t^{N-1}}{(1-t)[\mu t^{N-1}(f_-(t))^{2N-3} + (\lambda + \mu t) f_-(t) - 1]}$$

# 4 THE VALUE OF INFORMATION AND NASH EQUILIBRIUM STRATEGIES

In order to compute the value of the information (see the definition of $g(p)$ in Equation (2)), there is a need to compute first the value of some of the transforms $G_i(t)$ when $t$ runs over the non-zero eigenvalues of $R$. Specifically, the value of information is one half of

$$\sum_{k=0}^{\infty} \pi_{k,k+1}(M_{k+1,N-3} - M_{k,N-1}) + \sum_{k=0}^{\infty} \pi_{k,k+2}(M_{k+2,N-4} - M_{k,N}) + \\ \cdots + \sum_{k=0}^{\infty} \pi_{k,k+N-2}(M_{k+N-2,0} - M_{k,2N-4}) \tag{8}$$

Let $I$ be the set of non-zero eigenvalues of $R$ and let $(x)_i$ be the i-th entry of the vector $x$. Then the value of the above sum equals

$$\pi_{0,1}(M_{1,N-3} - M_{0,N-1}) + \sum_{i \in I}(\underline{\pi}_0 E_i)_2 \sum_{k=1}^{\infty} w_i^k (M_{k+1,N-3} - M_{k,N-1}) +$$
$$\pi_{0,2}(M_{2,N-4} - M_{0,N}) + \sum_{i \in I}(\underline{\pi}_0 E_i)_3 \sum_{k=1}^{\infty} w_i^k (M_{k+2,N-4} - M_{k,N}) +$$
$$\pi_{0,N-2}(M_{N-2,0} - M_{0,2N-4}) +$$
$$\sum_{i \in I}(\underline{\pi}_0 E_i)_{N-1} \sum_{k=1}^{\infty} w_i^k (M_{k+N-2,0} - M_{k,2N-4})$$

which equals

$$\frac{\pi_{0,1}}{\mu} + \sum_{i \in I} (\underline{\pi}_0 E_i)_2 \{ \frac{1}{w_i} [G_{N-3}(w_i) - \frac{1}{\mu} - \frac{2w_i}{\mu}] - [G_{N-1}(w_i) - \frac{1}{\mu}] \} +$$

$$2\frac{\pi_{0,2}}{\mu} + \sum_{i \in I} (\underline{\pi}_0 E_i)_3 \{ \frac{1}{w_i^2} [G_{N-4}(w_i) - \frac{1}{\mu} + \frac{2w_i}{\mu} + \frac{3w_i^2}{\mu}] -$$

$$[G_N(w_i) - \frac{1}{\mu}] \} + \cdots + (N-2)\frac{\pi_{0,N-2}}{\mu} +$$

$$\sum_{i \in I} (\underline{\pi}_0 E_i)_{N-1} \{ \frac{1}{w_i^{N-2}} [G_0(w_i) - \frac{1}{\mu} - \frac{2w_i}{\mu} - \cdots - \frac{(N-1)w_i^{N-2}}{\mu}] -$$

$$[G_{2N-4}(w_i) - \frac{1}{\mu}] \}$$

We evaluated the function $g(p)$ for selected values of $N$ and $\rho = \lambda/2\mu$. [5] To put all values in the same time scale we fixed $\mu = 0.5$. From the graphs one can see that $g(p)$ is monotone increasing when $N = 3$ and monotone decreasing for larger values of $N$. Thus, in order to analyze the corresponding Nash equilibria we split the discussion into the two different cases. We start with the case $N = 3$. Here, the more acquire the information, the higher is the value of information for an individual. Hence, if $C > g(1)$ then nobody will acquire the information and this is the unique Nash-equilibrium strategy. If $g(p^\star) = C$ for some $0 \leq p^\star \leq 1$, then $p^\star$ prescribes a Nash-equilibrium strategy. However, it is easy to see that in this case $p = 1$ and $p = 0$ are also Nash-equilibrium strategies. We point out that $p = 0$ and $p = 1$ are also *evolutionarily stable strategies* (ESS). [6] Finally, if $C < g(0)$ then everybody will acquire the information. This is a strategy which dominates all the other strategies. In particular, it is the unique Nash-quilibrium strategy. It is interesting to observe from Figure 1 that for a fixed $p$, $g(p)$ is monotone with respect to $\rho$ where $\rho = \lambda/2\mu$.

Suppose now that $N \geq 4$. If $g(0) \leq C$ (resp, $g(1) \geq C$), then there is a unique Nash equilibrium at $p = 0$ (resp., $p = 1$), it is an ESS and it is an optimal strategy for each of the customers regardless what the others do. If $g(p^\star) = C$ for some $p^\star \in (0,1)$ then $p^\star$ prescribes the unique Nash equilibrium which is also an ESS. However, it is not a strategy which dominates all the other strategies.

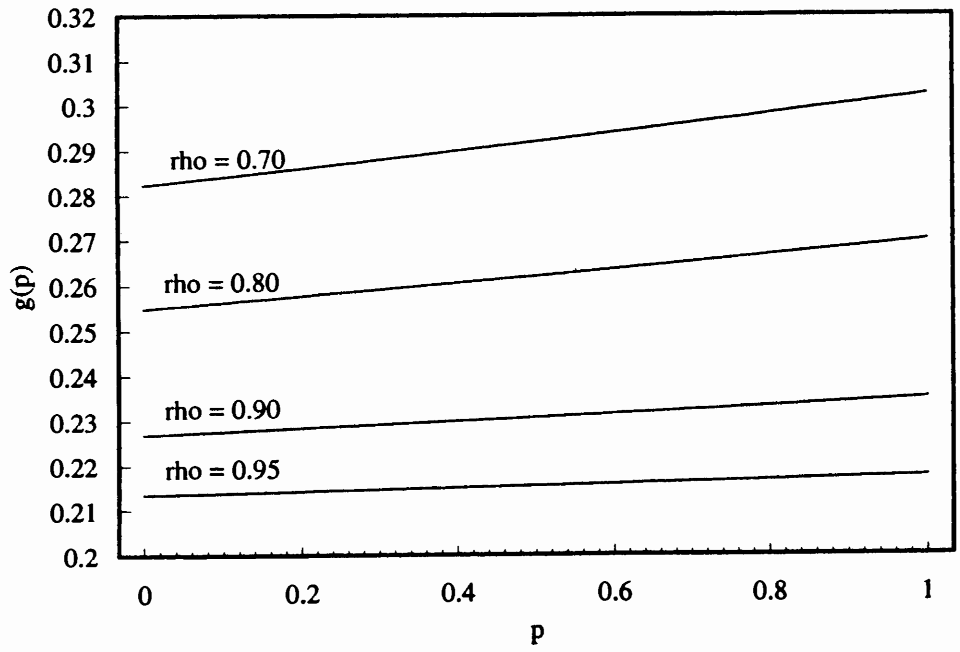


Figure 1: The Value of Information - N=3

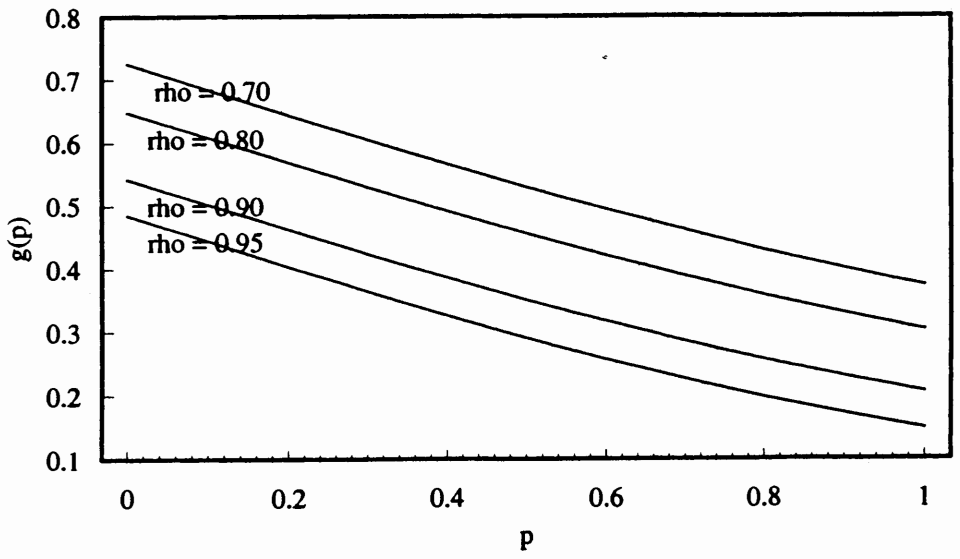


Figure 2: The Value of Information - N=5

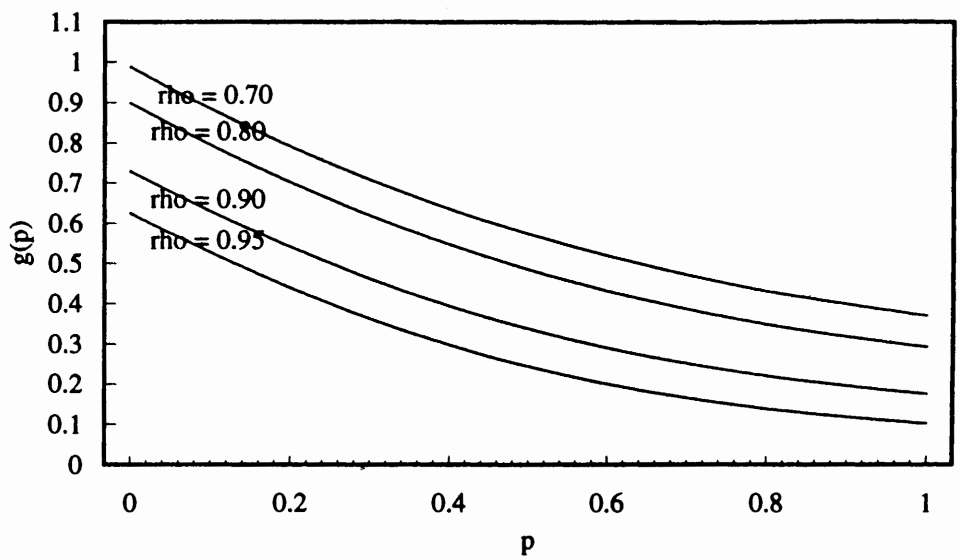


Figure 3: The Value of Information - N=7

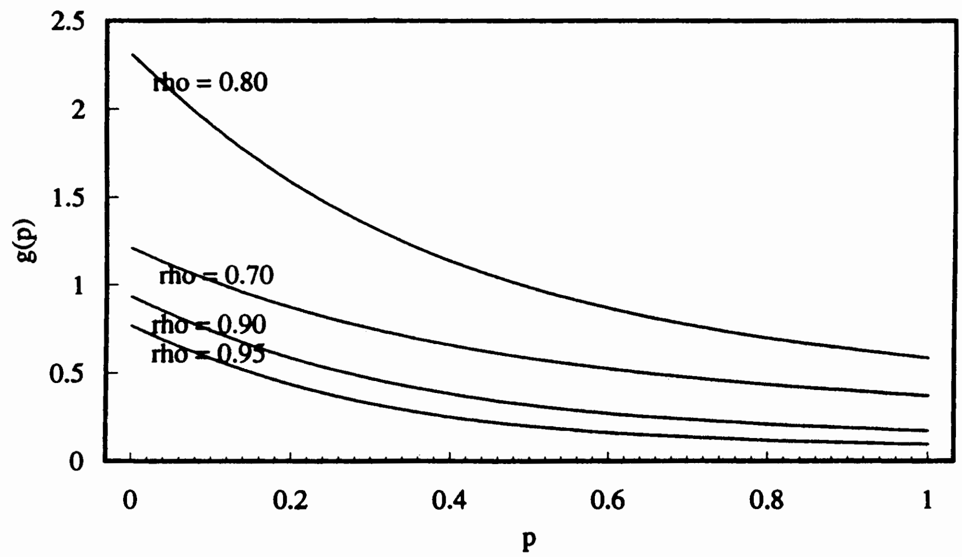


Figure 4: The Value of Information - N=10

# 5 THE EXTERNALITIES OF BUYING INFORMATION WHEN $N = 3$.

The action of purchasing information by an individual on which line is shorter, has an effect on the waiting times of others. This effect is the externality associated with the action. In this section we analyze the externalities of purchasing information for $N = 3$. In particular, we derive a closed form expression for these externalities. Note that it is not at all clear whether the externalities are positive or negative. On one hand, the information may help an individual in overtaking others and hence causing them an additional delay (i.e., negative externalities) but on the other hand, it may lead to a better utilization of the two servers and thus customers which otherwise might have waited for this individual, would not have to do so (i.e., positive externalities). Note that we define the externality of one's action on the rest of the society as the expected difference between the expected costs *to the rest of the society* when this action is not taken and when it is taken. The effect on the individual whose action is under consideration is not considered as part of the externality.

We start with an observation (which holds for $N = 3$):

*Unless the state is* $(1, 0)$, *the action of acquiring the information by an arrival either affects nobody else or it only affects the next arrival.*

We next argue that this is indeed the case. First, it is easy to see that those who are already in the system are not affected by the possibility that a new to arrive acquires the information on the queue size. Second, suppose that upon arrival the state is $(i, i+1)$ for some $i \geq 1$. Note that this is the only case where information has any value. If the next event is a departure, then the next state and the relative positions of the customers are the same regardless of whether the current arrival joins the shorter or the longer queue. If the next event is an arrival, then no matter if the new to arrive acquires the information or not, state $(i+1, i+2)$ will be reached. The only possible difference may be that the new arrival and the current arrival will interchange places depending if the current arrival joins the shorter or the longer queue. Those who arrive afterwards will not be affected. □

Consider again some state $(i, i+1)$ for $i \geq 0$. If the current arrival joins the shorter line and if the next event is an arrival (the latter is with probability $\lambda$), then the new to come will have an expected waiting time of $M_{i+1,1}$ while if the current arrival joins the longer queue, the corresponding value is $M_{i,3}$. Thus, the negative externality that an informed arrival to state $(i, i+1)$ imposes on the next arrival (if the next event is indeed an arrival) is $(0.5M_{i,3} + 0.5M_{i+1,1}) - M_{i+1,1} = 0.5(M_{i,3} - M_{i+1,1})$. Thus, the portion of the externality due to information acquiring and consequently the possibility of changing positions between an informed customer and the one arriving next to him is

$$EX1 \equiv \sum_{i=0}^{\infty} 0.5\lambda(M_{i,3} - M_{i+1,1})(\underline{\pi}_i)_2$$

Hence,

$$EX1 = 0.5\lambda\pi_{0,1}(M_{0,3} - M_{1,1})$$
$$+0.5\lambda \sum_{k=1}^{\infty}(M_{k,3} - M_{k+1,1}) \sum_{i\in I} w_i^k(\pi_0 E_i)_2$$
$$= -0.5\lambda\pi_{0,1}/\mu + 0.5\lambda \sum_{i\in I}(\pi_0 E_i)_2(G_3(w_i) - 1/\mu) -$$
$$0.5\lambda \sum_{i\in I}(\pi_0 E_i)_2 \frac{1}{w_i}(G_1(w_i) - 1/\mu - 2w_i/\mu)$$

The better utilization of the two servers due to the use of information is a source of positive externalities. As remarked above, these effects can be seen only when the arrival faces state $(0,1)$. In other states, acquiring information may affect the relative position of the customers but has no affect on the servers utilization. Thus, we turn our attention to state (0,1). Here, many future arrivals may be affected by waiting less because the server is utilized better when the arrival is informed. We now investigate the difference of expected future *total* waiting time when the arrival joins the longer line to reach state $(0,2)$, or the shorter line to reach state $(1,1)$. This difference comes since it is possible that the same random process of events in the system under the latter case will cause a transition from state $(1,1)$ into state $(0,1)$ while under the former case state $(0,2)$ will stay as is. (In particular, the idle server at $(0,2)$ will be looked at as he serves a fictitious customer.) This will happen with probability $\mu$. In this case there will be one customer more in the system which initiates with $(0,2)$ in comparison with the system which initiates with $(0,1)$ and this will be the case as long as the states in the two systems differ. Thus, we have interest in computing the expected time until the states in the two systems coincide (or coupled) as this is the social gain from the information acquiring. However, from this value we have to subtract $M_{1,0} = 2/\mu$ which is the expected gain of the customer in question. This customer, if he selects the longer line, is placed second in the line at the initial moment (and then state $(0,2)$ is reached), and we do not want to incorporate his gain while computing the externalities. To see why $2/\mu$ is the right value to subtract from the total gain, consider an arrival to state $(0,1)$. Of course, the next state is $(1,1)$ or $(0,2)$ depending if he joins the shorter or the longer line. Note that in the former case he commences service by one of the servers immediately upon arrival. Suppose the next event (for both cases) is service completion at this server (a probability $\mu$ event). Up to that point, the last arrival has spent the same time in both systems. However, had he joined the longer queue, he would have to stay in the system for an additional time whose expected value is $2/\mu$.

In order to find the expected time until coupling, we have to embed the problem in a more general one: given two systems which differ by one customer, what is the expected time until coupling given that the two systems are subject to the same random events (arrivals, departures and information acquiring). For $j \geq 3$, let $f_j$ be the expected time until coupling, when the initial total number in one system is $j$ while in the second it is $j+1$. It is not hard to see that for $j \geq 3$ the expected time until coupling is invariant with respect to how these customers split between the two parallel queues in the two systems (as long as the queueing discipline is maintained). Thus, For $j \geq 4$,

$$f_j = 1 + \lambda f_{j+1} + 2\mu f_{j-1} \quad . \tag{9}$$

For three customers or less in total in the less congested system, we need to elaborate. Thus, denote its state by a superscript, while a subscript will refer to the more congested system. For example, $f_{2,1}^{2,0}$ is the expected time until coupling when the initial state of one system is $(2,0)$ while the initial state of the other is $(2,1)$. Note that our ultimate interest is in $f_{2,0}^{1,0}$. We next write six equations which are satisfied by these values.

$$f_3 = 1 + \lambda f_4 + \mu f_{2,1}^{2,0} + \mu f_{2,1}^{1,1}$$

$$f_{2,1}^{2,0} = 1 + \lambda f_3 + \mu f_{1,1}^{1,0}$$

$$f_{2,1}^{1,1} = 1 + \lambda f_3 + \mu f_{1,1}^{1,0} + \mu f_{2,0}^{1,0}$$

$$f_{1,1}^{1,0} = 1 + \lambda_1 f_{2,1}^{1,1} + \lambda_2 f_{2,1}^{2,0} + \mu f_1$$

$$(1-\mu) f_{2,0}^{1,0} = 1 + \lambda_1 f_{2,1}^{1,1} + \lambda_2 f_{2,1}^{2,0} + \mu f_1$$

$$(1-\mu) f_1 = 1 + \lambda_1 f_{1,1}^{1,0} + \lambda_2 f_{2,0}^{1,0}$$

where $f_1 = f_{0,1}^{0,0}$.

Next we outline how to solve the above equations coupled with the difference Equation (9). It is easy to see that the difference equation defined in Equation (9) is solved by

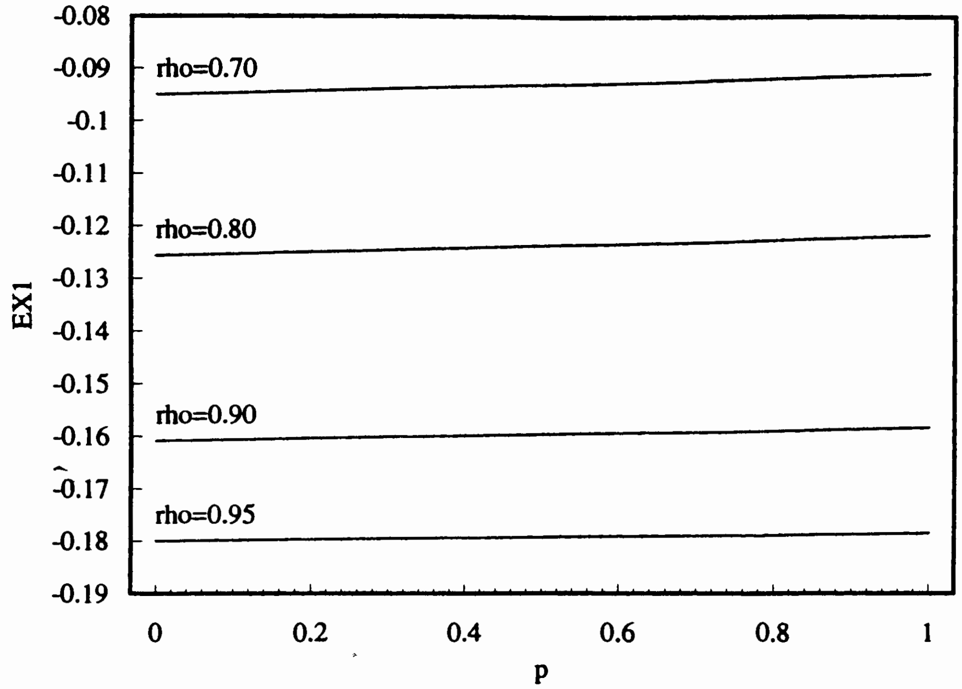


Figure 5: The Value of Externalities - EX1

$$\frac{j}{2\mu - \lambda} + k_1 + k_2\rho^{-j} \quad , \quad j \geq 3$$

for some constants $k_1$ and $k_2$. Clearly, $k_2 = 0$ as otherwise the expected value will be (asymptotically) geometric with a factor larger than 1. This, of course, is not the case since, up to the boundary conditions, the $f_j^{\prime}s$ are like the time until first hit of zero while starting at $j$ in a random walk with a nonzero drift toward zero. Thus, for $j \geq 3$,

$$f_j = \frac{j}{2\mu - \lambda} + k$$

for some constant $k$. Insert these expressions for $f_3$ and for $f_4$ in the six equations defined above and get a system of six linear equations with the six variables $k$,$f_{1,0}^{0,0}$,$f_{1,1}^{0,1}$, $f_{2,0}^{1,0}$, $f_{2,1}^{2,0}$ and $f_{2,1}^{1,1}$. This system can be solved. In particular, the value of $f_{2,0}^{1,0}$ we look for will be found.

We can conclude here that the contribution to value of the externalities due to acquiring the information when the state is $(0, 1)$ is

$$EX2 \equiv \pi_{0,1} 0.5\mu[f_{2,0}^{1,0} - \frac{2}{\mu}] \ .$$

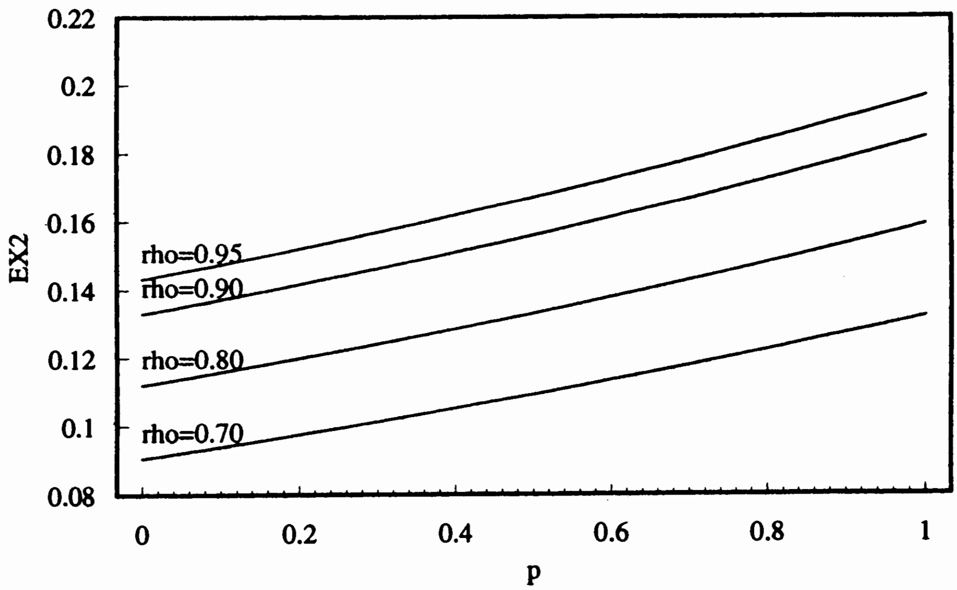


Figure 6: The Value of Externalities - EX2

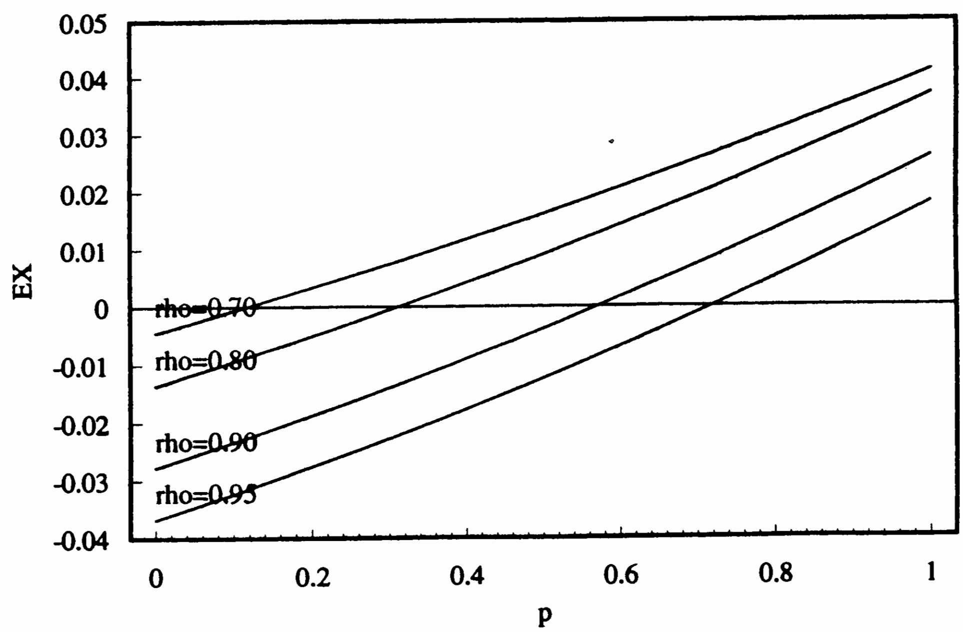


Figure 7: The Value of Externalities - EX

As said above $EX2$ is positive. [7] Also note that $0.5\mu f_{2,0}^{1,0}\pi_{0,1}$ is the expected gain to the entire society when an arrival acquires the information (when the individual acquiring the information is counted as part of the society).

Finally, the externality is $EX1+EX2$ and it is not a priori clear whether this value should be positive or negative. Indeed, consider Figures 5, 6 and 7. Figure 5 contains plots for $EX1$ for various values of $\rho$. As said above it is negative while in this example its absolute value decreases with $p$. Plots for $EX2$ are given in Figure 6. It is positive and monotone increasing in $p$. Finally, in Figure 7 we plot $EX$. For the values of $\rho$ showed there, $EX$ is monotone increasing in $p$ with negative values for small values of $p$ and with positive values for large values of $p$. For smaller values of $\rho$ $EX$ is monotone increasingwith $p$ and is positive for all values of $p$.

# 6 CONCLUDING REMARKS

We introduced the concept of the value of information in a two parallel line system and showed that it leads to Nash-equilibrium strategies in which a portion of the customers purchase the information. We have shown that by the aid of the matrix geometric technique that the value of information and the Nash-equilibrium strategies can be computed for memoryless two parallel line system with a threshold jockeying scheme for any threshold.

We demonstrated numerically that when the threshold value $N$ is three, the value of information for an arrival may be monotone increasing with the portion of the population who acquire it. It particular, it is possible that one will be willing to pay more in order to know which line is shorter when all the others are known to be informed than when they are not informed. This result may seem counterintuitive and we provided therefore also intuitive reasoning for it. As a result, at least one of the pure strategies in always a Nash equilibrium which is also an ESS. It is possible to have an Nash equilibrium based on a mixed strategy but it is never an ESS. However, in this case both pure strategies are ESS. When $N \geq 4$ we have shown numerically that the value of information decreases with the portion of customers who purchase it, a result which conforms with one's intuition. As a result, there is always a unique Nash equilibrium strategy which can be pure or mixed. Moreover, it is an ESS.

We also computed the externalities associated with information acquiring for a threshold value of three. In particular, we were able to decompose the externalities into two sources: The one due to the possibility that an

[7] It can be argued independently that $f_{2,0}^{1,0} \geq 2/\mu$. We omit the details.

informed customer overtakes another customer. The other due to the better utilization of the server when more customers use the information.

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