Generalizations of Hoffman's Existence Theorem for Circulations*

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Hoffman's Existence Theorem for circulations gives a necessary and sufficient condition for the existence of a feasible circulation in a directed network with upper and lower bounds on the flow along each of the arcs. This paper presents new existence theorems for more general types of flows in directed networks: flows with gains, two-commodity flows, and flows with set constraints.

I. INTRODUCTION

The classic network circulation model deals with the flow of a single kind of commodity through a network in which the flow in every arc is conserved and its value is restricted by lower and upper bounds. Numerous applications of this model exist in the literature; here we mention only the models of production systems developed by Dorsey, Hodgson, and Ratliff [1], and Zangwill [14, 15]. Many authors have recognized that the classic model is often inadequate in practical situations and have extended it to include more general cases, three of which will be considered here:

(1) *Models of flow with gains*, where the commodity shipped along an arc undergoes transformation (see [8], [11], [13] and their references).

(2) *Multicommodity models*, in which several kinds of commodities are shipped simultaneously through the same arc. ([10] contains a survey of such models.)

(3) *Models with (disjoint) set constraints*, where bounds are imposed on the sum of flows traversing (disjoint) sets of arcs.

Each of these extensions apply to common situations in production systems, arising from deterioration of stock, the production of several products simultaneously at the same facility, constraints on total production in different production periods and so forth.

Hoffman's classic existence theorem for feasible circulations states a necessary and sufficient condition for the existence of a single commodity circulation which satisfies both the flow conservation conditions in the nodes and arcs of the network and the

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bounds imposed on the flow through the arcs [7]. Hoffman also extended his theorem to the case where bounds are imposed on the net flow through the nodes.

Existence theorems for circulations are of interest for a variety of reasons. First, they are useful tools in discovering and proving other theorems [2-4, 11]. Second, their proofs suggest algorithms for feasible (or minimum cost) circulations [11, 12]. The theorems themselves supply optimality conditions and convergence proofs for such algorithms [6]. Finally, in special cases, a feasibility theorem may serve to prove that no feasible circulation exists. This is demonstrated by the examples which follow the theorems of this paper.

In the next section I present notation and definitions. In Secs. III, IV, and V I state and prove generalizations of Hoffman's theorem for directed networks with gains, two-commodities, and disjoint set constraints, respectively.

II. NOTATION AND TERMINOLOGY

A *directed network* (N, A) consists of a finite set of *nodes* N and a set of arcs $A \subseteq N \times N$. A *circulation* is an assignment of *flow* values x_{ij} to the arcs $(i,j) \in A$ such that the following condition holds:

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = 0 \qquad i \in N. \tag{1}$$

Let d_{ij} and k_{ij} be given sets of lower and upper bounds, respectively, imposed on the flow through the arcs $(i,j) \in A$. A circulation is *feasible* if

$$d_{ij} \leqslant x_{ij} \leqslant k_{ij} \qquad (i,j) \in A. \tag{2}$$

Hoffman's existence theorem for circulations [7, 12] states that: A necessary and sufficient condition for a feasible solution to (1) and (2) to exist is that for every set $S \subseteq N$

$$\sum_{\substack{(i,j)\in A\\ i\in S\\ j\in N-S}} k_{ij} \geqslant \sum_{\substack{(j,i)\in A\\ i\in S\\ j\in N-S}} d_{ji}.$$

A *path* in (N, A) is a sequence $(\alpha_1, \ldots, \alpha_n)$ of $n (n \geqslant 1)$ distinct arcs having, for $m = 1, \ldots, n$, arc $\alpha_m \in A$ and either $\alpha_m = (i_m, i_{m+1})$ or $\alpha_m = (i_{m+1}, i_m)$. This path is a *cycle* if $i_1 = i_{n+1}$. Arc α_m in this path has *positive orientation* if $\alpha_m = (i_m, i_{m+1})$ and *negative orientation* if $\alpha_m = (i_{m+1}, i_m)$. In a *directed cycle* all the arcs have common orientation. A *subgraph* (M, B) of (N, A) has $\phi \neq M \subseteq N$, $B \subseteq M \times M$, and $B \subseteq A$. A subgraph (M, B) of (N, A) is *connected* if (M, B) contains a path from each $i \in M$ to each $j \in M$ having $j \neq i$. A subgraph is called a *tree* if it is connected and has no cycles. A set of node-disjoint trees is called a *forest*, and each of these trees is a *component* of the forest. A tree (M, B) having $M = N$ is called a *spanning tree*.

For sets $B \subseteq N \times N$, $S \subseteq N$ and $T \subseteq N$, a function f_{ij} $(i, j) \in A$ and a scalar x, we adopt the following abbreviations:

$$f(B) = \sum_{(i,j)\in B\cap A} f_{ij},$$

$$(S,T) = \{(i,j)\in A : i\in S, j\in T\}$$

$$N - S = \{i\in N : i\notin S\}, \text{ and}$$

$$(x)^+ = \max\{0, x\}.$$

For example, the condition of Hoffman's theorem can be expressed as

$$k(S, N-S) \geqslant d(N-S, S) \text{ for every } S\subseteq N.$$

III. FLOW WITH GAINS

In this model a nonzero multiplier M_{ij} a lower bound d_{ij} and an upper bound k_{ij} are associated with each arc. A *feasible circulation* is a circulation which satisfies the following conditions:

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} M_{ji}x_{ji} = 0 \qquad i\in N \tag{3}$$

$$d_{ij} \leqslant x_{ij} \leqslant k_{ij} \qquad (i,j)\in A. \tag{4}$$

Theorem 1. A necessary and sufficient condition for a feasible solution to (3), (4) to exist, is that for every tree $(T,B)\subseteq(N,A)$ and any set of real numbers t_i $i\in N$, such that $t_i = 0$ for $i\in N - T$, and $t_i = t_jM_{ij}$ for $(i,j)\in B$ the following condition holds:

$$\sum_{(i,j)\in A} (t_i - M_{ij}t_j)^+k_{ij} \geqslant \sum_{(i,j)\in A} (M_{ij}t_j - t_i)^+\, d_{ij}. \tag{5}$$

Jewell provides an inclusive description of the patterns of flow changes which preserve feasibility in networks with gains [8]. It is essential to realize that, in contrast with flow changes in networks without gains, it is possible to change x_{mn} and preserve a feasible circulation by changing flows in arcs that do not necessarily include a cycle containing (m,n).

Proof: (a) *Necessity.* Suppose x_i units of flow enter the tree at node i, travel along the unique path of arcs of the tree (T,B) to node j, and then leave the tree. When traversing arc (m,n) from m to n, the flow is multiplied by M_{mn} and when traversing it in the opposite direction, it is multiplied by $-1/M_{mm}$ so that (3) holds. By the definition of t_i, the amount of flow *leaving* the tree is $x_j = x_it_i/t_j$ (i.e., $x_jt_j = x_it_i$). Suppose that the flow *into* node $i\in T$ is x_i; then

$$\sum_{i\in T} x_it_i = 0.$$

Let x_{ij} be a circulation, then

$$\sum_{(i,j)\in A} (M_{ij}t_j - t_i)x_{ij} = 0.$$

This equality becomes clear from the following five cases:

Case 1. If $(i,j)\in(T, N-T)$, then $t_j = 0$ and $-x_{ij}$ is the flow into T through (i,j).

Case 2. If $(i,j)\in(N-t, T)$, then $t_i = 0$ and $M_{ij}x_{ij}$ is the flow into T through (i,j).

Case 3. If $(i,j)\in A-B$ and $i\in T, j\in T$, then $M_{ij}x_{ij}$ units of flow enter T from (i,j) through node j and $-x_{ij}$ units through node i.

Case 4. If $(i,j)\in(N-T,N-T)$ then $t_i = t_j = 0$ and no flow enters (or leaves) T through (i,j).

Case 5. If $(i,j)\in B$ then $M_{ij}t_j - t_i = 0$. No flow enters T through (i,j).

Replacing flows with a positive coefficient by their lower bounds and flows with a negative coefficient by their upper bound yields eq. (5). Therefore, the condition is necessary.

(b) *Sufficiency.* Start with $x_{ij} = 0$ for all $(i,j)\in A$. Choose any arc with infeasible flow and change this flow to a feasible one so that no new infeasibilities are created and circulation is maintained. If no feasible circulation exists, this process will stop with an infeasible flow value y_{mn}, which cannot be made feasible without creating other infeasibilities. Assume $y_{mn} < d_{mn}$ (a similar proof holds for $y_{mn} > k_{mn}$). Then the solution to the following auxiliary problem is bounded (in fact, zero):

$$\text{Maximize } x_{mn}$$

subject to

$$\sum_{(j,i)\in A} M_{ji}x_{ji} - \sum_{(i,j)\in A} x_{ij} = 0 \qquad i\in N,$$

$$x_{ij} \leqslant 0 \qquad (i,j)\in A: y_{ij} \geqslant k_{ij},$$

$$-x_{ij} \leqslant 0 \qquad (i,j)\in A: y_{ij} \leqslant d_{ij}.$$

Therefore, the following dual system of equations has a feasible solution:

$$M_{ij}u_j - u_i + a_{ij}w_{ij} - k_{ij}v_{ij} \begin{cases} = 0 & (i,j)\neq(m,n) \\ > 0 & (i,j) = (m,n), \end{cases}$$

$$w_{ij} \geqslant 0, v_{ij} \geqslant 0 \qquad (i,j)\in A,$$

where $a_{ij}[b_{ij}]$ equals one if $y_{ij} \geqslant k_{ij} [y_{ij} \leqslant d_{ij}]$ and zero otherwise.

Let $S = \{(i,j)\in A: d_{ij} < y_{ij} < k_{ij}\}$; then for every $(i,j)\in S$, $a_{ij} = b_{ij} = 0$ and thus

$u_i = M_{ij}u_j$. Suppose that S contains a cycle. If this is a directed cycle then the product of the multipliers of its arcs is equal to one. Else, the product of the multipliers of its positively oriented arcs is equal to the product of the multipliers of its negatively oriented arcs. For example, in a cycle with arcs (1, 2), (2, 3), (1, 3), $u_1 = M_{12}u_2 = M_{12}M_{23}u_3 = (M_{12}M_{23}/M_{13})u_1$, and thus $M_{12}M_{23} = M_{13}$. We can change the flow, y_{ij}, along this cycle until some arc leaves S, i.e., the flow through this arc becomes equal to one of its bounds, and repeat this procedure until (N, S) is a forest.

Note that

$$
\begin{aligned}
(y_{mn} - d_{mn})(M_{mn}u_n - u_m) &< 0 & & \\
(y_{ij} - d_{ij})(M_{ij}u_j - u_i) &\leqslant 0 & & (i,j) \in A \text{ and } y_{ij} \leqslant d_{ij} \\
(y_{ij} - k_{ij})(M_{ij}u_j - u_i) &\leqslant 0 & & (i,j) \in A \text{ and } y_{ij} \geqslant k_{ij} \\
(M_{ij}u_j - u_i) &= 0 & & (i,j) \in S.
\end{aligned}
$$

Summation yields:

$$\sum_{(i,j)\in A} y_{ij}(M_{ij}u_j - u_i) < \sum_{(i,j)\in A} (M_{ij}u_j - u_i)^+ d_{ij} - \sum_{(i,j)\in A} (u_i - M_{ij}u_j)^+ k_{ij}.$$

As we have shown in part (a) of this proof, the left hand side of the inequality equals zero, hence we have proved that eq. (5) does not hold for the forest (N, S) with $t_i = u_i$.

To complete the proof, we show that it is possible to formulate the auxiliary problem so that $u_i = 0$ for all nodes except for one component, which is incident with (m, n).

Consider any component of (N, S) not incident with (m, n), and check whether the deletion of the flow conservation equations of its nodes changes the solution of the auxiliary problem. If it does, there must exist an arc $(i, j) \in A - S$ between the component and its complement, such that the deletion of its constraint from the auxiliary problem does not affect the solution. In this case we can change a_{ij} and b_{ij} to zero and assume $(i, j) \in A$. (Note that (N, S) remains a forest.) Otherwise the flow conservation equations for the nodes of the component may be deleted, and u_i for these nodes can be restricted to zero.

Repeating this procedure results in a forest with either one or two components with $u_i \neq 0$ incident with (m, n). If two components remain, the flow conservation equations of only one component are needed to block the increase in y_{mn} and the dual variables u_i of the nodes of the second component may be restricted to zero.

Example. Consider a network with three nodes and the following capacities: $k_{12} = 1$, $d_{12} = 0$; $k_{23} = 1$, $d_{23} = 1$; $k_{31} = 1$, $d_{31} = 0$. Let $M_{12} = 2$ and $M_{23} = M_{31} = 1$. Clearly no feasible circulation exists. The condition of the theorem does not hold for the tree which consists of the arcs $(1, 2)$, $(3, 1)$ and for $t_1 = t_3 = 1$, $t_2 = 1/2$:

$$\sum_{(i,j)\in A} (t_i - M_{ij}t_j)^+ k_{ij} = 0 < 1/2 = \sum_{(i,j)\in A} (M_{ij}t_j - t_i)^+ d_{ij}.$$

IV. TWO-COMMODITY CIRCULATIONS

In this model, a circulation consists of two kinds of flows, namely x_{ij} and x'_{ij}, $(i,j) \in A$. A feasible circulation satisfies the following conditions:

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = 0 \qquad i \in N \tag{6}$$

$$\sum_{(i,j)\in A} x'_{ij} - \sum_{(j,i)\in A} x'_{ji} = 0 \qquad i \in N \tag{7}$$

$$k_{ij} \geqslant x_{ij} \geqslant d_{ij} \qquad (i,j) \in A \tag{8}$$

$$k'_{ij} \geqslant x'_{ij} \geqslant d'_{ij} \qquad (i,j) \in A \tag{9}$$

$$k''_{ij} \geqslant x_{ij} + x'_{ij} \geqslant d''_{ij} \qquad (i,j) \in A \tag{10}$$

We assume without any loss of generality that for each arc $(i,j) \in A$;

$$k + k' \geqslant k'' \geqslant k + d', \; k' + d \geqslant d'' \geqslant d + d'. \tag{11}$$

Let S and S' be two sets such that $t_i(t'_i)$ elements of $S(S')$ correspond to node i. Define the following sets:

$B'' = \{t''_{ij}$ elements correspond to any arc $(i,j) \in A$ such that $t''_{ij} = \min[t_i - t_j, t'_i - t'_j] > 0\}$,

$B' = \{t'_{ij}$ elements correspond to any arc $(i,j) \in A$ such that $t'_{ij} = t'_i - t'_j - \max[0, t_i - t_j] > 0\}$,

$B = \{t_{ij}$ elements correspond to any arc $(i,j) \in A$ such that $t_{ij} = t_i - t_j - \max[0, t'_i - t'_j] > 0\}$,

C'', C', and C are defined as B'', B', and B, respectively, for $(j,i) \in A$.

This is illustrated by the network in Figure 1, based on an example of Jewell [9]. The values (t, t') for every node of the network are given in the figure. Note that the nodes denoted by w are the same and the same holds for β_1. The relevant sets are:

$B'' = \{(\alpha_i, \beta_i) \quad i = 1, \ldots, m\}$,

$B' = \{M - 1$ times $(w, v)\}$,

$B = \phi$,

$C'' = \{(\beta_1, \alpha_m)\}$,

$C' = M - 1$ times $(\beta_1, \alpha_m), M - i$ times (β_i, W), and i times $(v, \alpha_i)\, i = 1, \ldots, m\}$ and $C = \{(\beta_{i+1}, \alpha_i)\, i = 1, \ldots, M - 1$ and $(w, v)\}$.

Theorem 2. A necessary and sufficient condition for a feasible solution to exist for (6)-(10) is that for any two sets of nodes S, S', the corresponding sets of arcs B'', B', B, C'', C', C satisfy

$$k''(B'') + k'(B') + k(B) \geqslant d''(C'') + d'(C') + d(C). \tag{12}$$

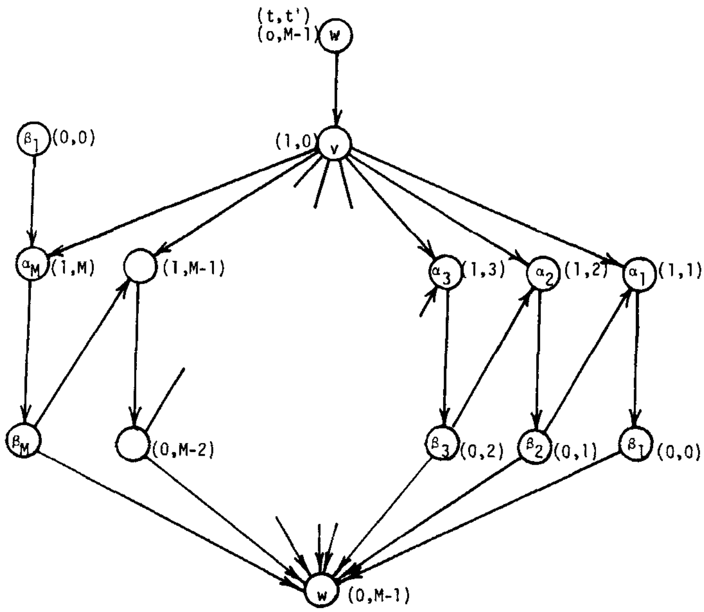

FIG. 1

Proof: (a) Let $T_q = \{i \in N : t_i \geqslant q\} \subseteq N$ and $T'_q = \{i \in N : t'_i \geqslant q\} \subseteq N$. Then for a feasible circulation

$$\begin{aligned} k''(B'') + k'(B') + k(B) &\geqslant x(B'') + x'(B'') + x'(B') + x(B) \\ &= \sum_{q=1,2,\ldots} [x(T_q, N - T_q) + x'(T'_q, N - T'_q)] \\ &= \sum_{q=1,2,\ldots} [x(N - T_q, T_q) + x'(N - T'_q, T'_q)] \\ &= x(C'') + x'(C'') + x'(C') + x(C) \\ &\geqslant d''(C'') + d'(C') + d(C). \end{aligned}$$

Therefore (12) is a necessary condition.

(b) To prove sufficiency, start with $x = x' = 0$. Choose arcs with an infeasible (two commodity) flow and make this flow feasible while maintaining circulation and without creating new infeasibilities.

If no feasible circulation exists, the process will terminate with some circulation y, y' and at least one arc, say (m, n), with an infeasible flow.

Let K'', K', K, D'', D', and D be the sets of arcs with

$$y_{ij} + y'_{ij} \geqslant k''_{ij},\ y'_{ij} \geqslant k'_{ij},\ y_{ij} \geqslant k_{ij},\ y_{ij} + y'_{ij} \leqslant d''_{ij},\ y'_{ij} \leqslant d'_{ij} \text{ and } y_{ij} \leqslant d_{ij},$$

respectively. We shall now consider the different kinds of infeasibilities that may exist in the flow through (m, n) and show that each implies that (12) does not hold.

Suppose $y_{mn} + y'_{mn} < d''_{mn}$. Then the solution to the following auxiliary problem is bounded (in fact, zero):

$$\text{Maximize } x_{mn} + x'_{mn}$$

subject to

$$\begin{aligned}
\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} &= 0 \quad i \in N,\\
\sum_{(i,j)\in A} x'_{ij} - \sum_{(j,i)\in A} x'_{ji} &= 0 \quad i \in N,\\
x_{ij} + x'_{ij} &\leqslant 0 \quad (i,j) \in K'',\\
x'_{ij} &\leqslant 0 \quad (i,j) \in K',\\
x_{ij} &\leqslant 0 \quad (i,j) \in K,\\
-x_{ij} - x'_{ij} &\leqslant 0 \quad (i,j) \in D'',\\
-x'_{ij} &\leqslant 0 \quad (i,j) \in D',\\
-x_{ij} &\leqslant 0 \quad (i,j) \in D.
\end{aligned}$$

Therefore, the following dual system of equations has a feasible solution:

$$u_j - u_i + a''_{ij}w''_{ij} + a_{ij}w_{ij} - b''_{ij}v''_{ij} - b_{ij}v_{ij} = \begin{cases} 0 & (i,j) \in A - \{(m,n)\}\\ 1 & (i,j) = (m,n), \end{cases}$$

$$u'_j - u'_i + a''_{ij}w''_{ij} + a'_{ij}w'_{ij} - b''_{ij}v''_{ij} - b'_{ij}v'_{ij} = \begin{cases} 0 & (i,j) \in A - \{(m,n)\}\\ 1 & (i,j) = (m,n), \end{cases}$$

$$w''_{ij}, w'_{ij}, w_{ij}, v''_{ij}, v'_{ij}, v_{ij} \geqslant 0 \qquad (i,j) \in A,$$

where $a''_{ij}, a'_{ij}, a_{ij}, b''_{ij}, b'_{ij}, b_{ij}$ equals one if $(i,j) \in K'', K', K, D'', D', D$, respectively, and zero otherwise. Since only the differences $(u_j - u_i)$ and $(u'_j - u'_i)$ matter, we can assume $u'_i \geqslant 0$, $u_i \geqslant 0$ for each $i \in N$. By multiplying these numbers by a sufficiently large integer, we obtain an integral solution.

Let $S = \{u_i \text{ times node } i\}$, $S' = \{u'_i \text{ times node } i\}$.

Suppose $(i,j) \in B''$, then $u'_i > u'_j$ and $u_i > u_j$. Hence, both $a''_{ij} + a_{ij}$ and $a''_{ij} + a'_{ij}$ are positive, i.e., $(i,j) \in K'' \cup (K \cap K')$. Since, by (11) $K'' \supseteq (K \cap K')$, then $(i,j) \in K''$. Suppose $(i,j) \in B'$, then $u'_i > u'_j$ and $u'_i - u'_j > u_i - u_j$, i.e., $a''_{ij} + a'_{ij} \geqslant 1$ and $a'_{ij} + b_{ij} \geqslant 1$. The first condition implies $(i,j) \in K'' \cup K'$ and the second implies $(i,j) \in K' \cup D$. Hence $(i,j) \in K' \cup (K'' \cap D)$. By (11) $K'' \cap D \subseteq K'$, thus $(i,j) \in K'$. Similarily, if $(i,j) \in B$, then $(i,j) \in K$ and if $(i,j) \in C'', C', C$ then $(i,j) \in D'', D', D$, respectively. Therefore,

$$\begin{aligned}
k''(B'') + k'(B') + k(B) &\leqslant y'(B'') + y(B'') + y'(B') + y(B)\\
&= y'(C'') + y(C'') + y'(C') + y(C)\\
&\leqslant d''(C'') + d'(C') + d(C). \qquad (13)
\end{aligned}$$

1. Assume $y_{mn} + y'_{mn} < d''_{mn}$, $y_{mn} < k_{mn}$ and $y'_{mn} < k'_{mn}$. By (11) also $y_{mn} + y'_{mn} < k''_{mn}$, hence, $a''_{mn} = a'_{mn} = a_{mn} = 0$, so that $u_m < u_n$ and $u'_m < u'_n$, i.e., $(m,n) \in C''$. With $y_{mn} + y'_{mn} < d''_{mn}$, this implies that the last inequality in (13) is strict and that (12) does not hold.

2. Suppose $y_{mn} + y'_{mn} < d''_{mn}$ but $y'_{mn} \geq k'_{mn}$. Then by (11) also $y_{mn} < d_{mn}$, so that $a''_{mn} = a_{mn} = 0$. This implies $u_m < u_n$, and therefore if $u'_m < u'_n$, then $(m,n) \in C''$ and if $u'_m \geq u'_n$ then $(u_m - u_n) < (u'_m = u'_n)$ and $(m,n) \in C$. In both cases the last inequality in (13) is strict and (12) does not hold.

3. Suppose $k''_{mn} > y_{mn} + y'_{mn} \geq d''_{mn}$, and $y_{mn} < d_{mn}$. By (11), $y'_{mn} > d'_{mn}$, and the auxiliary problem with the modified primal objective, $\max(x_{mn})$, is bounded. Hence the dual constraint for x'_{mn} equals zero and the dual system is feasible. Since $a''_{mn} = a_{mn} = 0$, then $u_m < u_n$; and since $b'_{mn} = 0, (u_m - u_n) < (u'_m - u'_n)$. Therefore, $(m,n) \in C$. Together with $y_{mn} < d_{mn}$, this implies that (12) does not hold.

4. Suppose $y_{mn} + y'_{mn} \geq k''_{mn}$ and $y_{mn} < d_{mn}$. By (11), $y'_{mn} > k'_{mn}$, and the auxiliary problem with the objective modified to $\max(-x'_{mn})$ is bounded. Hence, the dual constraint for x'_{mn} equals -1, and the dual system is feasible. Since $b''_{mn} = b'_{mn} = 0$, then $u'_m > u'_n$; and since $a_{mn} = 0$, $(u'_m - u'_n) > (u_m - u_n)$ and $(m,n) \in B'$. Hence, $y'_{mn} > k'_{mn}$ implies that the first inequality in (13) is strict and that (12) does not hold.

We conclude that $y_{mn} + y'_{mn} < d''_{mn}$ implies that (12) does not hold. Proofs for the other possible infeasibilities in the flow through (m,n) are obtained by interchanging the roles of the primed and unprimed flows and/or the roles of upper and lower bounds. Thus, we conclude that (12) is a sufficient condition for the existence of a two-commodity circulation.

Example. Consider again Figure 1 with the following bounds:

$$d_{ij} = d''_{ij} = 1 \text{ and } k'_{ij} = 0 \text{ for } (i,j) = (w,v),$$
$$d'_{ij} = d''_{ij} = 1 \text{ for } (i,j) = (\beta_1, \alpha_M).$$

All other lower bounds equal zero and upper bounds equal one. For the pairs (t,t') given in the figure

$$k''(B'') + k'(B') + k(B) = M + 0 + 0 < 1 + (M-1) + 1 = d''(C'') + d'(C) + d(C).$$

Therefore no feasible circulation exists.

V. FLOW WITH SET CONSTRAINTS

In the general network model with set constraints, upper bounds k_S and lower bounds $d_S \leq k_S$ are defined for every $S \subseteq A$. A feasible circulation satisfies the following conditions:

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = 0 \qquad i \in N, \tag{14}$$

$$d_S \leq x(S) \leq k_S \qquad S \subseteq A. \tag{15}$$

A special case of (15) has constraints only on the flow leaving (entering) the nodes, i.e.,

$$d(i,M) \leqslant \sum_{j\in M} x_{ij} \leqslant k(i,M) \qquad i \in N, M \subseteq F(i), \tag{16}$$

where $F(i) = \{j \in N:(i,j) \in A\}$. A simple existence theorem holds when for each $i \in N$ and $S, T \subseteq F(i)$ the following conditions hold:

$$d(i,S) + d(i,T) \leqslant d(i,S \cup T) + d(i,S \cap T) \quad \text{(i.e., } d(n,\cdot) \text{ is } \textit{supermodular}) \tag{17}$$

$$k(i,S) + k(i,T) \geqslant k(i,S \cup T) + k(i,S \cap T) \quad \text{(i.e., } k(i,\cdot) \text{ is } \textit{submodular}), \tag{18}$$

$$k(i,S) - k(i,S - T) \geqslant d(i,T) - d(i,T - S). \tag{19}$$

As the meaning of these conditions and a proof of this theorem are given in [5], here we only state the theroem:

Theorem 3. A necessary and sufficient condition for the existence of a feasible solution to (14) and (16) when for each $i \in N$ and S, $T \subseteq F(i)$, conditions (17)-(19) hold, is that for every $M \subseteq N$,

$$\sum_{i\in M} k(i,N - M) \geqslant \sum_{j\in N-M} d(j,M).$$

Let $S^* = \{S \subseteq A:(d_S,k_S) \neq (-\infty,\infty)\}$. Another important special case of (15) is the one in which the sets $S \subseteq S^*$ are disjoint, hence we have:

Theorem 4. The following condition is necessary and sufficient for the existence of a feasible solution to (14)-(15) when the sets $S \in S^*$ are disjoint: For any set of nodes in which node i is included t_i times, and $t_i - t_j$ equals a common value $t(S)$ for every $(i,j) \in S$,

$$\sum_{S\in S^*} (t_S)^+ k_S \geqslant \sum_{S\in S^*} (-t_S)^+ d_S. \tag{20}$$

Proof: (a) Let $T_q = \{i \in N:t_i \geqslant q\}$, then for a feasible circulation x,

$$\begin{aligned}\sum_{S\in S^*} (t_S)^+ k_S &\geqslant \sum_q x(T_q,N - T_q)\\ &= \sum_q x(N - T_q,T_q)\\ &\geqslant \sum_{S\in S^*} (-t_S)^+ d_S.\end{aligned}$$

Hence (20) is necessary.

(b) Suppose that no feasible circulation exists. Then there is a circulation y_{ij} $(i,j) \in A$ and a set $Q \in S^*$ such that $y(Q)$ is infeasible, (for example $y(Q) < d_Q$), and the solution to the problem stated below is bounded. (A similar proof holds when $y(Q) > k_Q$.)

$$\text{Maximize } x(Q)$$

subject to

$$\sum_{(i,j)\in A} x_{ij} - \sum_{(j,i)\in A} x_{ji} = 0 \qquad i \in N,$$
$$x(S) \leqslant 0 \qquad S \in S^* : y(S) \geqslant k_S,$$
$$-x(S) \leqslant 0 \qquad S \in S^* : y(S) \leqslant d_S.$$

Therefore the following dual system has a feasible solution:

$$u_j - u_i + \sum_{S\in S^*} a_{ijS} w_S - \sum_{S\in S^*} b_{ijS} v_S = \begin{cases} 0 & (i,j) \in A - Q \\ 1 & (i,j) \in Q, \end{cases}$$
$$v_S \geqslant 0, w_S \geqslant 0 \qquad S \in S^*,$$

where $a_{ijS}(b_{ijS})$ is equal to one if $(i, j) \in S$ and $y(S) \geqslant k_S(y(S) \leqslant d_S)$, and zero otherwise. An integral solution of u_i, $i \in N$, is obtained by multiplying the dual constraints by a sufficiently large integer.

Non-negativity can be assumed since only the differences $(u_i - u_j)$ matter.

Suppose $(i, j) \in S \in S^*$. Then $u_S = u_i - u_j > 0$ implies $y(S) \geqslant k_S$, and $u_S = u_i - u_j < 0$ implies $y(S) \geqslant d_S$. Let $t_i = u_i$, $T_q = \{i : t_i \geqslant q\}$. Then

$$\sum_{S\in S^*} (t_S)^+ k_S \leqslant \sum_q y(T_q, N - T_q)$$
$$= \sum_q y(N - T_q, T_q)$$
$$\leqslant \sum_{S\in S^*} (-t_S)^+ d_S.$$

Since $y(Q) < d_Q \leqslant k_Q$, then $a_{ijQ} = 0$ for all $(i, j) \in A$, and $u_Q < 0$. Thus the second inequality is strict and (20) does not hold. Therefore (20) is a sufficient condition.

Example. Consider the triangle with $x_{31} \leqslant 1$, $x_{12} + x_{23} \geqslant 3$. Let $t = (0, 1, 2)$, then $\Sigma_{S\in S^*} (-t_S)^+ d_S = 1 \cdot 3 > 2 \cdot 1 = \Sigma_{S\in S^*} (t_S)^+ k_S$ and no feasible circulation exists.

Necessary and sufficient conditions for the existence of a feasible solution when the constrained sets are not disjoint may be easily obtained by a simple transformation: change any arc which is included in $n > 1$ constrained sets into n arcs in series, and ascribe each to a different set.

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