Reprinted from JOURNAL OF THE OPERATIONS RESEARCH SOCIETY OF AMERICA
Vol. 3, No. 2, May, 1955
Printed in U.S.A.

USE OF LINEAR PROGRAMMING IN CAPITAL BUDGETING

PAUL GUNTHER
General Electric Company, Schenectady, New York

(Received March 23, 1955)

LINEAR programming has found wide application in solving such diverse problems as determining the optimum utilization of machines, most efficient scheduling procedures, most economical routing of transportation, best system of salary scales, etc. An example has recently come to our attention which illustrates the value of this new technique, and the practical type of solution arrived at, in the important field of capital budgeting. In order to protect the identity of the company involved, we have modified slightly some of the figures and the nature of the product.

Mrs. Efficiency was faced with the problem of baking an apple pie. Briefly, the background to her problem was this: Her husband was very fond of apple pie. She knew that unless she made a good pie, he might begin to buy his apple pie at the office cafeteria. At the same time, it was important that her total costs not exceed her monthly allowance. Because of the competition from other sources, such as the landlord, milkman, etc., she had only 50¢ to spend on apples.

So, Mrs. E. left the house to walk over to the grocery store to buy apples for her pie. On the way she pondered as to the brand of apples she would get. There were MacIntoshes, which were 10¢ each, and they were very good. Also, there were Jonathans, a little less expensive at 5¢ each—almost, but not quite as good as the MacIntoshes. Then finally the Winesaps, very reasonably priced at 2¢ each, and they could be made to do.

Mrs. E. hadn't quite made up her mind by the time she had reached the store. The more she thought about it, the more difficult it became to balance all the factors. The pressure had been on her recently, hearing about how Mrs. Spendthrift cooked steak two nights a week, and Mr. S. didn't make any more money than did Mr. E. She suddenly recalled having heard about this new thing called linear something-or-other . . . oh, yes, programming . . . about how it absolutely came up with the undisputably correct and best possible answer. No use in fooling around with guess-work and intuition, she thought, when the big brains have worked out a foolproof system. She remembered the blurb she had recently received in the mail about a new consultant who had just opened up an office. It was just down the street from the grocery. Maybe she could drop in and ask what he thought. According to the notice, he would be happy to discuss any problems she might have, free of charge. It wouldn't hurt to hear what he had to say.

She arrived in the office, and a nicely furnished one it was, too. The clatter of the calculating machines gave it an air of efficiency. She glanced at the impressive charts on the wall showing the successful results achieved by this new technique, linear programming. There didn't seem to be any problem it couldn't solve. She was glad she had come.

The secretary introduced her to Mr. Or, a pleasantly aggressive sort of man, very full of confidence. They sat down at a table and exchanged pleasantries about the business outlook, the weather, etc. She enjoyed the conversation and was forming a very good impression indeed of Mr. Or.

There was a slight lull in the conversation and it seemed like the right time for Mrs. E to pose her problem. She explained her situation, adding also that she needed an answer very quickly.

Mr. Or told her that because of the complexity of the problem, it would probably be necessary to make some simplifying assumptions. She appreciated this fact, and after some reluctant admissions about her husband's eating habits, agreed that maximizing the number of apples was a pretty fair statement of her objectives. Mr. Or set this up mathematically: Let

q_1 = quantity of MacIntoshes purchased,
q_2 = quantity of Jonathans purchased,
q_3 = quantity of Winesaps purchased:

$$\text{maximize } Q = q_1 + q_2 + q_3$$

"Yes," Mr. Or said, "so far it looks like a definite linear programming problem. What about the restrictions—let's keep our fingers crossed and see if they are linear too. You have a limited amount of capital to invest; is that correct, Mrs. E.?"

"Uh, huh," she answered. Not knowing who else might have been in to talk with Mr. Or, she was reluctant to state the exact amount. Mr. Or noticed this and tried to reassure her.

"Of course, you realize that all the information we obtain from customers, we hold in *strictest* confidence. It doesn't make any difference to the mathematics," he jokingly explained. "We just like to be sure we get a reasonable answer. Have to keep your feet on the ground, you know."

Mrs. E. thought quickly, about maybe using the figure \$5.00 or even \$50.00, but would she be on firm ground if she were to divide the final answer by 10 or 100? She decided not to chance it.

"Well, just for purposes of illustration, why don't you take a round number like 50¢." She was surprised when Mr. Or didn't even blink at this suggestion. He simply inserted the value in the waiting right hand side, to get

$$10q_1+5q_2+2q_3\leq 50.$$

"So far, so good," he said. "Any other restrictions, like size of an apple, or number of apples? For instance, what's the smallest number of apples that could be used in making a pie—one, two, ten?" This thought had never occurred to Mrs. E. She tried to picture the size of each of the apples, and of her smallest pie plate. She explained these factors to Mr. Or, mentioning also that it might be possible, although she had never heard of it, to bake half a pie. Or maybe she could use the tin that came when she bought frozen chicken pot pie. "I don't buy that very often," she confided, "because I don't think they put enough chicken in it for what they charge."

After some discussion, and under what she felt to be considerable pressure from Mr. Or, she finally agreed that it wouldn't be too bad to consider all the brands of apples to be about the same size. "Probably 3 or 4 would fill up that chicken pot pie tin." She wondered whether she still had it around the house. As she had said many times before, it just didn't pay to be throwing things out—a body never knows when there'll be use for something like that.

Mr. Or automatically suggested that for a minimum-size pie they use $3\frac{1}{2}$. But when she objected that it would be difficult to get the grocer to sell half an apple, he consented to use 4.

Mr. Or put down the restriction

$$q_1+q_2+q_3\geq 4.$$

"Well, I think we've got the problem pretty well stated now. Are there any other factors that you think we should consider before putting it in the mill and coming up with the apple pie?" He laughed, quite pleased with his pun.

Mrs. E. did not search her mind this time. It was getting late and she was anxious to see how this whole business would work out. "It looks pretty good to me," she answered.

"Well, suppose we see whether we can't solve it then. Of course you realize that we have oversimplified the problem considerably. But we can use this as a test run so that next time we will be able to set up a detailed and realistic model with all the variables and restrictions. Even so, as it stands I think we will come up with some extremely interesting things. If you'll pardon me a minute, I'll take this into the other room and give it to my assistant. He's an expert at this stuff and should have it finished before long. We can discuss future plans while he's working out the answer. Ordinarily I would do the first run myself to get a feel for how the variables interact, but I know it would bore you to death."

The thought of actually watching Mr. Or work out the solution to her problem had not occurred to Mrs. E., but the idea now sounded very appealing. In fact she found herself getting quite excited at the prospect. As Mr. Or started to get up from the table, she stopped him.

"It would really be no bother at all. In fact I would quite like to see how you work it out, although I'm sure I shan't understand a thing. That is, if you have no objection."

Mr. Or was quite displeased, although he didn't show it. He had found out long ago that it wasn't smart to let customers in on details of his work. Trying to explain them either confused the customer or raised many doubts about the innumerable assumptions, most of which didn't affect the solution one way or the other. They just left him wide open unnecessarily, and invariably the customer left less happy than he otherwise would have been. Then too, often the customer would decide that the computations were so simple he couldn't see why he himself shouldn't be able to do it if he just had a good reference book. And when the bill for the work arrived, he always felt overcharged.

Mr. Or quickly appraised his situation and saw that he could not easily escape from working out the solution. And with Mrs. E. looking over his shoulder at that. He resigned himself, and therewith put the conflict completely from his mind.

"OK, that's fine. Let's see now what our equations are." He took out a clean sheet of paper. Verbalizing as he wrote, he started in. "We are maximizing the total number of apples Q subject to two restrictions: the total cost of the apples must not exceed 50¢, and the total number of apples must be at least 4.

"First thing we do is alter each of the restrictions to make them into equations rather than inequalities. We make the first into

$$10q_1+5q_2+2q_3+X_1=50$$

so that X_1 represents the amount of money that is not spent. That's why we call it a slack variable, since it takes up the slack. If in our final solution X_1 comes out to zero, that means you spend all of the 50¢.

"Now we do the same thing to the second restriction and write

$$q_1+q_2+q_3-X_2=4.$$

X_2 represents the number of apples beyond the minimum of 4 that we decide to buy. Of course we have to use the minus sign since we work only with positive variables in linear programming. X_2 is also a slack variable, and if in the final solution it is zero, it means we only buy the minimum number of apples."

Mrs. E. wasn't sure that she was understanding everything Mr. Or was saying, but it all sounded reasonable enough. His last remark though, she understood well. She hoped that it wouldn't come out with X_2 equal zero. She'd probably have to buy a chicken pot pie, use that for dinner, and make the apple pie tomorrow. She wondered whether the apple pie would have a chicken flavor.

Mr. Or was continuing his explanation. "Now that minus sign in front of the X_2 is a nuisance because it prevents us from getting an initial solution the way we would like to. (You'll see what I mean in a moment.) So we have to add another variable Y to the equation, giving

$$q_1+q_2+q_3-X_2+Y=4.$$

Now Y is going to be zero in the final answer. The way we make sure of this is to attach to it a very, very large cost, which we can represent by M. Y is called an artificial vector and you can see why."

Mrs. E. heartily agreed that Y certainly did seem artificial.

"Now we are ready to set up the simplex tableau which will enable us to compute the solution automatically." He drew out another sheet of paper and arranged it like this.

			Y	X_1	X_2	q_1	q_2	q_3
			$-M$	0	0	1	1	1
$-M$	Y	4	1		-1	1	1	1
0	X_1	50		1		10	5	2

Mr. Or explained that the entries in the table were simply a shorthand way of writing the equations. The numbers at the top and the left margin were the 'costs' attached to each variable as given from the original equation for Q. The M of course was part of the device of using Y.

He continued the explanation. "We also get from this automatically an initial solution, implied by the Y and X_1 here (and he pointed to the left margin) which is interpreted as $Y=4$, $X_1=50$ with all the other variables zero. This means that we can satisfy the requirements of the problem by spending all of the 50¢ on slack X_1, which means that we don't spend anything at all. We satisfy our minimum requirement of 4 apples by using artificial ones, or maybe stealing them, but by doing so, our total quantity of apples Q, which is also in this case the negative of cost, is minus $4M$."

Mrs. E. was staring open-mouthed. She couldn't quite believe that she had heard what she thought she had. Mr. Or must be playing a joke, but a glance at his engrossed and utterly serious expression convinced her that he had meant every word of it.

"Anyway," he went on, "it isn't important to give a non-mathematical interpretation of the initial solution, in fact, it needn't even have to make sense. (Mrs. E. breathed a deep sigh of relief.) We just use it as a first stepping stone, an opening wedge if you please, to go on to better and better solutions. The way we decide how to improve the solution is to see what happens to the cost when one unit of each of the other variables is included in the previous solution.

"Now there's a very simple way to do this from the tableau. One merely . . ." He stopped, trying to remember exactly what it was that he merely did. This always happened to him whenever he had been away from an actual computation for more than a week.

"Let me just check with our little bible to make sure I do this right," he said, trying to give his failing a light joking touch. He reached over to the row of books sitting at the end of the table, picked out a tall, thin, paper-bound one titled *Introduction to Linear Programming* by Charnes, Cooper and Henderson. The book opened almost by itself to the first tableau of the nut-mix problem.

He studied the entries for not more than half a minute. "Oh yes, I have it now. We multiply the entries in each column by each of the costs in the left margin and then subtract the cost at the top. We record the net result at the bottom."

. . .

Mr. Or continued on and on, tossing around more and more q's and m's and minuses, as he warmed up to his subject. Actually he went through only four more tableaus, although to Mrs. E it had seemed more like a dozen. During the course of the explanation, she had experienced a variety of reactions. After the initial feeling of incredulity, she rapidly passed through phases of enlightenment, bewilderment, and finally resignation. Mr. Or seemed to be nearing the end. He was saying: "This is a surprise—looks like we have our answer. None of the numbers in the last row are negative, which means that there is no way to improve the solution. So $q_3=25$ and $X_2=21$. Buy 25 Winesaps, 21 above the minimum of 4. $X_1=0$, so spend all of the 50¢. q_1 and q_2 are zero, so don't buy any MacIntoshes or Jonathans. And of course Y is zero. Pretty simple, eh, Mrs. E.? Do you have any questions about the procedure?"

The implications of the final solution had not yet registered with Mrs. E. She was about to say no in answer to Mr. Or, but suddenly changed her mind. "Well, this is probably a very foolish question, but when you were talking at the beginning about an initial solution, which you then go on to improve, I was wondering why you couldn't have started with a guess at the final answer, rather than go through that artificial business—I mean with the Y?"

Mr. Or was strangely silent. This thought had never occurred to him, yet it certainly sounded reasonable enough. Of course he wasn't going to admit this. "Well, you see, there would be no simple way to adjust the equations so that you could go on to use this automatic simplex method. At least I've never seen anybody show how it could be done."

His answer didn't particularly satisfy Mrs. E. She was glad though that she had asked the question.

Mr. Or decided it was time to get on to the main point. "You see now, Mrs. E., how linear programming works. The procedure we went through will always guarantee that you are doing the very best possible. I think though, that the next step would be to improve on some of the assumptions. For instance, that all the apples are the same size, or that the minimum is four. We might want to bring in the quality of the apples and consider other criteria besides maximum number of apples. Then also, it would be interesting to see what would happen if we used less than 50¢ or more than 50¢.

"You might even want to go beyond the apple pie. For instance, should you be making apple pie at all? Maybe you'd be better off making strawberry pie instead. Eventually you might even want to get into where and how would be the most efficient way to spend your entire monthly allowance. We might even be able to show that your husband doesn't give you a large enough allowance."

Mr. Or proceeded to relate the details of his "convenient charging system." Mrs. E. came very close to signing up right then and there, but for some unexplained reason (perhaps she recalled her husband's anger two months ago when she brought home a foot warmer at a bargain price of $15) she caught herself. "I'll tell you what. Why don't I think about this for a day or two and talk it over with my husband? I'll call and let you know."

Mr. Or was disappointed. "Certainly Mrs. E. Take your time, and I'll be expecting a call from you. Here's a card with my number. Remember now, that in this day and age we can't afford to make decisions in the slip-shod way that our parents and grandparents did. A person has to do things scientifically nowadays if he's going to keep up with his competitors."

Mrs. E. left and walked back to the grocery store. It was already quite late in the afternoon, but if she hurried she could get her apple pie finished in time.

She walked confidently through the door, went straight to the bin where the Winesaps were sold. No guesswork this time for me, she told herself. Suddenly she froze, stunned at what her eye saw. Winesaps had gone up to 3¢ each.