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Interpolatory estimates in monotone piecewise polynomial approximation[☆]

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Abstract

Given a monotone function $f \in C^r[-1,1]$, $r \geq 1$, we obtain pointwise estimates for its monotone approximation by piecewise polynomials involving the second order modulus of smoothness of $f^{(r)}$. These estimates are interpolatory estimates, namely, the piecewise polynomials interpolate the function at the endpoints of the interval. However, they are valid only for $n \geq N(f,r)$. We also show that such estimates are in general invalid with N independent of f.

MSC: 41A10; 41A25; 41A29; 41A30

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1. Introduction and the main result

For $r \in \mathbb{N}$, let $C^r[a,b]$, $-1 \leq a < b \leq 1$, denote the space of r times continuously differentiable functions on $[a,b]$, and let $C^0[a,b] = C[a,b]$ denote the space of continuous functions on $[a,b]$, equipped with the uniform norm $\|\cdot\|_{[a,b]}$. When dealing with $[-1,1]$, we

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omit the reference to the interval, that is, we denote $\|\cdot\| := \|\cdot\|_{[-1,1]}$. Let $\mathbb{P}_n$ be the space of algebraic polynomials of degree $\le n$.

For $f \in C[a,b]$ and any $k \in \mathbb{N}$, set

$$\Delta_u^k(f,x;[a,b]) := \begin{cases} \sum_{i=0}^k (-1)^i \binom{k}{i} f(x+(k/2-i)u), & x \pm (k/2)u \in [a,b] \\ 0, & \text{otherwise,} \end{cases}$$

and denote by

$$\omega_k(f,t;[a,b]) := \sup_{0<u\le t} \|\Delta_u^k(f,\cdot;[a,b])\|_{[a,b]},$$

its kth modulus of smoothness. For $[a,b]=[-1,1]$, write $\omega_k(f,t) := \omega_k(f,t;[-1,1])$.

Let $X_n := \{x_{j,n}\}_{j=0}^n$, $x_{j,n} = -\cos j\pi/n$, $0 \le j \le n$, be the Chebyshev partition of $[-1,1]$ (see, *e.g.*, [6]), and set $x_{n+1,n} := 1$, $x_{-1,n} := -1$.

Finally, let

$$\varphi(x) = \sqrt{1-x^2} \quad \text{and} \quad \rho_n(x) := \frac{\varphi(x)}{n} + \frac{1}{n^2}. \tag{1.1}$$

Pointwise estimates have mostly been investigated for polynomial approximation of continuous functions in $[-1,1]$ and involved usually the quantity $\rho_n(x)$. The first to deal with such estimates was Nikolskii, and he was followed by Timan, Dzjadyk, Freud and Brudnyi. Detailed discussion may be found in the survey paper [5], where an extensive list of references is given. Discussion and references to estimates on pointwise monotone polynomial approximation involving $\rho_n(x)$ also may be found there. Pointwise estimates of polynomial approximation involving $\varphi(x)$ are due originally to Teljakovskiĭ and Gopengauz, see [4] for extensions and many references. Finally, for some results on pointwise rational approximation, see [1].

The main result of this paper is the following.

Theorem 1.1. *Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ with the property that if a function $f \in C^r[-1,1]$, is monotone, then there is a number $N = N(f,r)$, depending on f and r, such that for $n \ge N$, there are monotone continuous piecewise polynomials s of degree $r+1$ with knots at the Chebyshev partition, satisfying*

$$|f(x)-s(x)| \le c(r)\left(\frac{\varphi(x)}{n}\right)^r \omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1,1], \tag{1.2}$$

and

$$|f(x)-s(x)| \le c(r)\varphi^{2r}(x)\omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1,x_{1,n}] \cup [x_{n-1,n},1]. \tag{1.3}$$

Remark 1.2. Theorem 1.1 is well known for $r=0$, in fact, with $N=1$. Indeed, the polygonal line, that is, the continuous piecewise linear s, interpolating f at the Chebyshev nodes, is nondecreasing and yields (1.2) with $r=0$ (see, *e.g.*, a similar construction in [2]). It may be worth mentioning that if f is convex, then the same polygonal line is convex, thus we have the estimate (1.2) with $r=0$ also for convex approximation. This was shown in [7].

In the sequel all constant c will depend on r, but may otherwise be different in each occurrence.

2. Monotone approximation by piecewise polynomials on a general partition

Given $[a, b]$, let $X = \{x_j\}_{j=0}^n$ be a partition of the interval, such that

$$a =: x_0 < x_1 < \cdots < x_n := b.$$

Denote by $S(X, r+1)$ the set of continuous piecewise polynomials of order $r+1$ on the partition X, that is, $s \in S(X, r+1)$ if $s \in C[a, b]$ and s is a piecewise polynomial of degree $r \in \mathbb{N}$ with knots x_j, *i.e.*, on each interval $[x_{j-1}, x_j]$, $1 \le j \le n$, the function s is an algebraic polynomial of degree $\le r$.

We begin with a negative result, which is proved in a similar way to the proof of [4, Theorem 4].

Theorem 2.1. *For each $X = \{x_j\}_{j=0}^n$ and number $r \in \mathbb{N}$, there is a nondecreasing function $f \in C^\infty[a, b]$ such that, for any nondecreasing piecewise polynomial $s \in S(X, r+1)$, satisfying $s(a) = f(a)$ and $s(b) = f(b)$, we have*

$$s'(a) \ne f'(a).$$

Proof. Without loss of generality assume that $a = 0$ and $x_1 = 1 < b$. Let $g \in C^\infty[0, \infty)$ be a nondecreasing function such that

$$g(0) = 0, \quad g(x) = 1, \quad x \ge 1,$$

and

$$g'(0) =: d > 0.$$

We will show that a desired function f may be taken to be

$$f(x) = g\left(\frac{x}{\alpha}\right), \quad x \in [0, b],$$

with a suitable number $\alpha > 0$, sufficiently small, say, $\alpha < \frac{d}{2r^2}$.

Assume, to the contrary, that there is piecewise polynomial $s \in S(X, r+1)$, such that

$$s(0) = f(0), \quad s(b) = f(b) \quad \text{and} \quad s'(0) = f'(0).$$

The polynomial

$$p(x) := s(x), \quad x \in [0, 1],$$

is of degree $\le r$. Since $p'(0) = s'(0) = f'(0)$ and p is nondecreasing, we get by Markov's inequality

$$\frac{d}{\alpha} = p'(0) \le 2r^2 p(1),$$

so that

$$p(1) \ge \frac{d}{2r^2\alpha} > 1.$$

Hence,

$$s(b) \ge s(1) = p(1) > 1 = f(b),$$

that contradicts our assumption that $s(b) = f(b)$. □

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The following is an immediate consequence.

Corollary 2.2. *For each collection* $X = \{x_j\}_{j=0}^n$ *and number* $r \in \mathbb{N}$*, there is a nondecreasing function* $f \in C^\infty[a, b]$ *such that, for any nondecreasing piecewise polynomial* $s \in S(X, r+1)$*, satisfying*

$$|s(x) - f(x)| = o(x - a), \quad x \to a+,$$

we have

$$s(b) \neq f(b).$$

Since $\omega_k(g, t; [a, b]) = O(t^k)$, for every $k \geq 1$ and any $g \in C^\infty[a, b]$, we may conclude that (compare with Theorem 2.4)

Corollary 2.3. *For each collection* $X = \{x_j\}_{j=0}^n$ *and numbers* $r \in \mathbb{N}$ *and* $k \in \mathbb{N}$*, such that* $r + k > 2$*, there is a monotone function* $f \in C^\infty[a, b]$*, such that for any monotone piecewise polynomial* $s \in S(X, r+1)$*, satisfying*

$$|s(x) - f(x)| = O\left((x - a)^{r/2}\omega_k(f^{(r)}, \sqrt{x - a}; [a, b])\right), \quad x \to a+,$$

we have

$$s(b) \neq f(b).$$

One may salvage the interpolation of derivatives at the endpoints as we have also a positive result.

Theorem 2.4. *For each nondecreasing function* $f \in C^r[a, b]$*,* $r \in \mathbb{N}$*, there is a number* $H = H(f) > 0$*, such that for every collection* X*, satisfying*

$$x_1 - H < a \quad \text{and} \quad b < x_{n-1} + H,$$

there are a constant $c = c(r)$ *and a nondecreasing piecewise polynomial* $s \in S(X, r+2)$*, that yields,*

$$|s(x) - f(x)| \leq c(x - a)^r \omega_2(f^{(r)}, \sqrt{(x - a)(x_1 - a)}; [a, x_1]), \quad x \in [a, x_1], \tag{2.1}$$

$$|s(x) - f(x)| \leq c(b - x)^r \omega_2(f^{(r)}, \sqrt{(b - x)(b - x_{n-1})}; [x_{n-1}, b]), \quad x \in [x_{n-1}, b],$$

and

$$|s(x) - f(x)| \leq c(x_j - x_{j-1})^r \omega_2(f^{(r)}, x_j - x_{j-1}; [x_{j-1}, x_j]), \quad x \in [x_{j-1}, x_j], \quad 2 \leq j \leq n-1. \tag{2.2}$$

3. Auxiliary lemmas and proof of the theorems

We begin with some results for the interval [0, 1]. For $f \in C^r[0, 1]$ and $h \in (0, 1)$ denote by

$$L_{r,h}(x) := L_{r,h}(f, x) := f(0) + \frac{f'(0)}{1!}x + \cdots + \frac{f^{(r)}(0)}{r!}x^r + a_r(h; f)x^{r+1},$$

the Lagrange–Hermite polynomial of degree $r + 1$ such that

$$L_{r,h}^{(j)}(f, 0) = f^{(j)}(0), \quad j = 0, \ldots, r,$$

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and

$$L_{r,h}(f, h) = f(h).$$

The term $a_r(h; f)$ is equal to the generalized divided difference of f, *i.e.*,

$$\begin{aligned} a_r(h; f) &= [\underbrace{0, \dots, 0}_{r+1 \text{ times}}, h; f] \\ &= \frac{1}{h^{r+1}}\left(f(h) - f(0) - \frac{f'(0)}{1!}h - \cdots - \frac{f^{(r)}(0)}{r!}h^r\right) \\ &= \frac{1}{(r-1)!h^{r+1}}\int_0^h (h-t)^{r-1}(f^{(r)}(t) - f^{(r)}(0))dt. \end{aligned}$$

Hence,

$$|a_r(h; f)| \le \frac{1}{r!h}\omega_1(f^{(r)}, h; [0, h]). \tag{3.1}$$

Our first result is the following.

Lemma 3.1. *Let* $f \in C^r[0, 1]$, *be monotone nondecreasing on* $[0, 1]$. *Then there is a number* $H > 0$, *such that for all* $h \in (0, H)$ *the polynomials* $L_{r,h}(f, \cdot)$ *are monotone nondecreasing on* $[0, h]$.

Proof. Without loss of generality we may assume that $f(0) = 0$. If $f'(0) = \cdots = f^{(r)}(0) = 0$, then $a_r(h; f) = h^{-(r+1)}f(h)$, whence $L_{r,h}(x) = (x/h)^{r+1}f(h)$, that yields monotonicity of $L_{r,h}$ in $[0, 1]$.

Otherwise there is a number $0 < k \le r$, such that $f^{(j)}(0) = 0$ for all $j = 0, \dots, k-1$ and $f^{(k)}(0) > 0$. Put $\omega_1(t) := \omega_1(f^{(r)}, t; [0, 1])$, and take $H > 0$ so small that,

$$\sum_{i=k+1}^{r} \frac{|f^{(i)}(0)|}{(i-1)!}H^{i-k} + \frac{(r+1)\omega_1(H)}{r!}H^{r-k} < \frac{f^{(k)}(0)}{(k-1)!},$$

where we use the usual convention that an empty sum is zero.

Then, by (3.1), for $0 < x \le h \le H$,

$$\begin{aligned} L'_{r,h}(x) &= \sum_{i=k}^{r} \frac{f^{(i)}(0)}{(i-1)!}x^{i-1} + a_r(h; f)(r+1)x^r \\ &= x^{k-1}\left(\sum_{i=k}^{r} \frac{f^{(i)}(0)}{(i-1)!}x^{i-k} + a_r(h; f)(r+1)x^{r-k+1}\right) \\ &\ge x^{k-1}\left(\frac{f^{(k)}(0)}{(k-1)!} - \sum_{i=k+1}^{r} \frac{|f^{(i)}(0)|}{(i-1)!}H^{i-k} - \frac{(r+1)\omega_1(H)}{r!}H^{r-k}\right) \\ &> 0, \end{aligned}$$

and the proof is complete. □

Remark 3.2. Clearly, an analogous statement is valid if f is nonincreasing.

Next, we need an estimate on the distance between f and the polynomial $L_{r,1}(f, \cdot)$.

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Lemma 3.3. *Given $r \ge 1$ and $f \in C^r[0,1]$, we have*

$$|f(x) - L_{r,1}(f,x)| \le cx^{r+1} \int_{x/2}^{1} u^{-2}\omega_2(f^{(r)}, u; [0,1])du, \quad x \in [0,1], \tag{3.2}$$

where c is an absolute constant.

Proof. Set $\omega(t) := \omega_2(f^{(r)}, t; [0,1])$, $0 \le t \le 2$, and $m := r+2$. Recall (see [4, Section 2] or [3, Chapter 3, section 6]) that for $0 \le x_0 \le x_1 \le \cdots \le x_m$, such that $x_i < x_{i+r+1}$, $i = 0$ and 1, we may define the generalized divided difference $[x_0, x_1, \dots, x_m; f]$ of $f \in C^r[0,1]$. Also, by [4, Lemma 1, (2.1) and (2.2)] or [3, Chapter 3, (8.27) and (6.36)], we have

$$|[x_0, x_1, \dots, x_m; f]| \le c(r)\Lambda_r(x_0, x_1, \dots, x_m; \omega), \tag{3.3}$$

where (see [3, Chapter 3, (6.32) and (6.31)])

$$\Lambda_r(x_0, x_1, \dots, x_m; \omega) := \max_{0 \le p \le 1} \max_{p+r+1 \le q \le m} \Lambda_{p,q,r}(x_0, x_1, \dots, x_m; \omega)$$

and, with $d(p,q) := \min\{x_{q+1} - x_p, x_q - x_{p-1}\}$, $x_{-1} := -1$, $x_{m+1} := 2$,

$$\Lambda_{p,q,r}(x_0, x_1, \dots, x_m; \omega) := \frac{\int_{x_q - x_p}^{d(p,q)} u^{r+p-q-1}\omega(u)du}{\prod_{i=0}^{p-1}(x_q - x_i)\prod_{j=q+1}^{m}(x_j - x_p)}.$$

For $x \in (0,1)$, put $x_0 = x_1 = \cdots = x_r := 0$, $x_{r+1} := x$ and $x_{r+2} := 1$, and $p = 0$ or 1 and $p + r + 1 \le q \le m$. Then, for $p = 0$, $q = r+1$ or $r+2$, and for $p = 1$, $q = r+2$. Hence,

$$\prod_{i=0}^{p-1}(x_q - x_i) = 1 \quad \text{and} \quad \prod_{j=q+1}^{m}(x_j - x_p) = 1,$$

and

$$d(0, r+1) = 1, \quad d(0, r+2) = 2 \quad \text{and} \quad d(1, r+2) = 1.$$

Therefore

$$\begin{aligned} \Lambda_{0,r+1,r}(x_0, \dots, x_m; \omega) &= \int_x^1 u^{-2}\omega(u)du, \\ \Lambda_{1,r+2,r}(x_0, \dots, x_m; \omega) &= 0, \quad \text{and} \\ \Lambda_{0,r+2,r}(x_0, \dots, x_m; \omega) &= \int_1^2 u^{-3}\omega(u)du = \frac{3}{8}\omega(1) \\ &\le c\int_{1/2}^{1} u^{-2}\omega(u)du \\ &\le c\int_{x/2}^{1} u^{-2}\omega(u)du, \end{aligned} \tag{3.4}$$

where in the second inequality we used the fact that $\omega(t) = \omega(1)$, $1 \le t \le 2$, and $\omega(1) \le cu^{-2}\omega(u)$, $1/2 \le u \le 1$, and c, here and in the sequel, if it appears by itself, is an absolute constant.

Combining (3.3) and (3.4), we conclude that

$$|[x_0, \dots, x_m; f]| \le c\int_{x/2}^{1} u^{-2}\omega(u)du,$$

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which, in turn, implies

$$\begin{aligned}|f(x)-L_{r,1}(f,x)| &= f(x)-f(0)-\frac{f'(0)}{1!}x-\cdots-\frac{f^{(r)}(0)}{r!}x^r-a_r(1;f)x^{r+1}\\ &= ([x_0,\dots,x_r,x;f]-[x_0,\dots,x_r,1;f])\,x^{r+1}\\ &= x^{r+1}(1-x)|[x_0,\dots,x_m;f]| \le cx^{r+1}\int_{x/2}^1 u^{-2}\omega(u)du.\end{aligned}$$

This completes the proof. □

Corollary 3.4. *Given $r\ge 1$ and $f\in C^r[0,1]$, we have*

$$|f(x)-L_{r,1}(f,x)| \le cx^r\omega_2(f^{(r)},\sqrt{x},[0,1]),\quad x\in[0,1]. \tag{3.5}$$

Proof. Recall that $\omega(t)=\omega_2(f^{(r)},t,[0,1])$. We need the following estimate.

$$\begin{aligned}x\int_{x/2}^1 u^{-2}\omega(u)du &= x\int_{x/2}^{\sqrt{x}} u^{-2}\omega(u)du + x\int_{\sqrt{x}}^1 u^{-2}\omega(u)du\\ &\le x\omega(\sqrt{x})\int_{x/2}^{\sqrt{x}} u^{-2}du + 4\omega(\sqrt{x})\int_{\sqrt{x}}^1 du\\ &\le x\omega(\sqrt{x})\int_{x/2}^{\infty} u^{-2}du + 4\omega(\sqrt{x})\\ &= 6\omega(\sqrt{x}).\end{aligned}$$

In view of (3.2), this proves (3.5). □

Applying the linear substitution $x=yh$, we obtain

Corollary 3.5. *Let $r\ge 1$ and $h>0$. If $f\in C^r[0,h]$, then*

$$|f(x)-L_{r,h}(f,x)| \le cx^r\omega_2(f^{(r)},\sqrt{xh};[0,h]),\quad x\in[0,h].$$

It readily follows from Corollary 3.5 that

Corollary 3.6. *Let $r\ge 1$ and $h>0$. If $f\in C^r[1-h,1]$ and $g(x):=f(1-x)$, then*

$$|f(x)-L_{r,h}(g,1-x)| \le c(1-x)^r\omega_2(f^{(r)},\sqrt{(1-x)h};[1-h,1]),\quad x\in[1-h,1].$$

We also need a result that follows immediately from [8, Lemma 2, p. 58].

Lemma 3.7. *Let $r\ge 1$, $x^*<y^*$. If $f\in C^r[x^*,y^*]$, is nondecreasing in $[x^*,y^*]$, then there exists a polynomial $p\in\mathbb{P}_{r+1}$, nondecreasing in $[x^*,y^*]$, interpolating f at both x^* and y^*, and such that*

$$\begin{aligned}\|f-p\|_{[x^*,y^*]} &\le c(r)(y^*-x^*)\omega_{r+1}(f',y^*-x^*;[x^*,y^*])\\ &\le c(r)(y^*-x^*)^r\omega_2(f^{(r)},y^*-x^*;[x^*,y^*]).\end{aligned} \tag{3.6}$$

Proof of Theorem 2.4. Since without loss of generality we may assume that $[a,b]=[0,1]$, and we let $h<H$, with H from Lemma 3.1, then the polynomials obtained in Corollaries 3.5 and 3.6 are nondecreasing. Thus, combining Corollaries 3.5 and 3.6 with (3.6) completes the proof of Theorem 2.4. □

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Proof of Theorem 1.1. Recall the Chebyshev partition $X_n := \{x_{j,n}\}_{j=0}^n$, $-1 = x_{0,n} < x_{1,n} < \cdots < x_{n-1,n} < x_{n,n} = 1$, and denote $I_{j,n} := [x_{j,n}, x_{j+1,n}]$, $j = -1, \ldots, n$, and set $h_{j;n} := x_{j+1,n} - x_{j,n}$, the length of $I_{j,n}$. Then we have (see, *e.g.*, [8, (1.2) and (1.3)])

$$\frac{\varphi(x)}{n} < \rho_n(x) < h_{j,n} < 5\rho_n(x), \quad x \in I_{j,n}, \quad 0 \le j \le n-1,$$

$$h_{j,n} < 8\frac{\varphi(x)}{n}, \quad x \in I_{j,n}, \quad 1 \le j \le n-2, \quad \text{and}$$

$$h_{j\pm1,n} < 3h_{j,n}, \quad 0 \le j \le n-1.$$

Given a nondecreasing $f \in C^r[-1,1]$, let H be obtained by Lemma 3.1. Then for $n \ge N(f,r)$, we have $\max_{j=0,n-1} h_{j,n} < H$. Therefore, by Theorem 2.4, there exists a nondecreasing piecewise polynomial $S \in S(X_n, r+2)$ yielding (1.2) and (1.3). This completes the proof of Theorem 1.1. □

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