ON 3-MONOTONE APPROXIMATION BY PIECEWISE POLYNOMIALS

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Abstract. We consider 3-monotone approximation by piecewise polynomials with prescribed knots. A general theorem is proved, which reduces the problem of 3-monotone uniform approximation of a 3-monotone function, to convex local L_1 approximation of the derivative of the function. As the corollary we obtain Jackson-type estimates on the degree of 3-monotone approximation by piecewise polynomials with prescribed knots. Such estimates are well known for monotone and convex approximation, and to the contrary, they in general are not valid for higher orders of monotonicity. Also we show that any such convex piecewise polynomial can be modified to be, in addition, interpolatory, while still preserving the degree of the uniform approximation. Alternatively, we show that we may smooth the approximating piecewise polynomials to be twice continuously differentiable, while still being 3-monotone and still keeping the same degree of approximation.

1. Introduction

Let f be a real-valued function defined on the interval $I := [a, b]$, and ν a natural number. Denote by

$$f[x_0, \dots, x_\nu] := \sum_{i=0}^{\nu} \frac{f(x_i)}{\prod_{j=0, j\neq i}^{\nu}(x_i - x_j)}$$

the νth order divided difference of f at the distinct points $x_0, \dots, x_\nu$. The function f is called ν-monotone in $[a, b]$, if $f[x_0, \dots, x_\nu] \geq 0$ for all choices of $\nu + 1$ distinct points $x_0, \dots, x_\nu \in [a, b]$. We denote by $\Delta^\nu_{[a,b]}$ the set of all ν-monotone functions in $[a, b]$, so in particular, $\Delta^1_{[a,b]}$ and $\Delta^2_{[a,b]}$ are the sets of non-decreasing and convex functions in $[a, b]$, respectively. It is well known that $\Delta^3_{[a,b]}$ is the set of all bounded functions, having a convex derivative on (a, b). Note that if $f \in \Delta^\nu_{[a,b]}$, $\nu \geq 2$, then f is continuous on (a, b)

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and $f(a+)$, $f(b-)$ exist and are finite. Thus, in the sequel we assume that our functions are continuous on I.

The problems of monotone and convex approximation, on a finite interval, by piecewise polynomials with prescribed knots have been considered among others by DeVore [3], Beatson [1], Hu [5], Kopotun [8], and Shevchuk [11]. Higher-order shape-preserving approximation, i.e., ν-monotone approximation, $\nu \ge 3$, has been investigated in recent years, with somewhat surprising results. Namely, the pattern of positive and negative results, that experts had believed prevail, which goes back to Shvedov [13] and shown to be valid for $\nu = 1, 2$, breaks down completely for $\nu \ge 4$ (see [7]). In fact, recent results by Konovalov and Leviatan [7] about shape-preserving widths demonstrate that, for $\nu \ge 4$, the statement *"If $f \in \Delta^{\nu}_{[-1,1]} \cap C^{(\nu)}_{[-1,1]}$, and $0 \le f^{(\nu)}(x) \le 1$, $x \in [-1,1]$, then there is a piecewise polynomial $s \in \Delta^{\nu}_{[-1,1]}$ of degree $\le \nu - 1$ with n equidistant knots such that $|f(x) - s(x)| \le c(\nu)n^{-\nu}$, $x \in [-1,1]$"*, is invalid. Moreover, for $\nu \ge 4$ the best order of approximation one can achieve for the statement is n^{-3}, and we have a loss of order of $n^{\nu-3}$. It is easy to construct splines providing this estimate for $\nu = 1$ and $\nu = 2$. Indeed, one may take the interpolatory piecewise-constant function and the inscribed polygon, respectively. Therefore, the only outstanding question is the case $\nu = 3$. Does it follow the pattern known for $\nu = 1, 2$, or does it belong to the cases $\nu \ge 4$?

For $f \in C_{[a,b]}$, and an interval $I \subset [a,b]$, we denote by $\|f\|_I$ the usual sup-norm of f on I, and for $h > 0$ denote by $\omega_k(f, h; I)$, the kth modulus of smoothness of f on I, with the step h. For the interval $[a,b]$ itself we write $\|f\| := \|f\|_{[a,b]}$ and $\omega_k(f, h) := \omega_k(f, h; [a,b])$. Finally, we need the notation $\omega^{\varphi}_k(f, h) := \omega^{\varphi}_k(f, h; [a,b])$, for the Ditzian-Totik [4] kth modulus of smoothness of f associated with the interval $[a,b]$.

For a given function $F \in \Delta^3_{[a,b]} \cap C^{(2)}_{[a,b]}$, Konovalov and Leviatan [6] have constructed a 3-monotone quadratic spline S with n equidistant knots such that

$$\|F - S\| \le \frac{c}{n^2}\omega_1(F'', 1/n),$$

where $c = c(a,b)$ is an absolute constant independent of F and n. This estimate provides an exact order of 3-monotone approximation for certain Sobolev classes of functions, and it was applied by Konovalov and Leviatan [7] to prove upper bounds on shape-preserving widths.

Recently Prymak [10] has extended the result of [6], constructing a 3-monotone piecewise quadratic with arbitrary prescribed knots which give an estimate of the degree of approximation in terms of the third modulus of smoothness of the function. An immediate consequence for the equidistant knots is that for each $F \in \Delta^3_{[a,b]}$ there exists a piecewise quadratic $S \in \Delta^3_{[a,b]}$ with n equidistant knots, for which

$$\|F - S\| \le c\omega_3(F, 1/n), \tag{1}$$

for some absolute constant $c = c(a, b)$.

Can one achieve higher degree of approximation with 3-monotone piecewise polynomials of degree higher than 2? The main purpose of this paper is to give an affirmative answer to this question in most of the conjectured cases, and to explain when it is impossible. One case remains outstanding, we do not know whether an estimate involving the fourth modulus of smoothness of F is valid or not (see Remark 3 below).

In Section 2 we state the main results and in Section 3 we prove Theorem 1 after an auxiliary construction. In Section 4 we prove Theorem 2, followed by the proof of Theorem 5 in Section 5.

2. **The main results**

We begin with

Theorem 1. *Let $F \in \Delta^3_{[a,b]}$ and $f(x) := F'(x)$, $x \in (a, b)$. Given an integer $k \ge 2$, a partition $a =: x_0 < x_1 < \cdots < x_n := b$, and a piecewise polynomial $s \in \Delta^2_{[a,b]}$ of degree $\le k-1$, with knots x_i, $i = 1, \dots, n-1$, such that*

$$s(x_i) = f(x_i), \quad i = 1, \dots, n-1, \tag{2}$$

there exists a piecewise polynomial $S \in \Delta^3_{[a,b]}$ of degree $\le k$ with knots x_i, $i = 1, \dots, n-1$, for which

$$\|F - S\| \le c \max_{1 \le i \le n} \|f - s\|_{L_1[x_{i-1}, x_i]}, \tag{3}$$

where c is an absolute constant, and $\|\cdot\|_{L_1[x_{i-1},x_i]}$ denotes the L_1-norm on $[x_{i-1},x_i]$. In fact $c \le 25$.

Note that Theorem 1 reduces the problem of 3-monotone approximation of a 3-monotone function in the uniform norm to that of convex approximation of its derivative with the interpolation condition (2). Moreover the derivative is approximated locally in the L_1-norm. Since ordinary integration of s normally leads to a loss of an order of approximation in the estimate, due to this local estimates, Theorem 1 yields a "gain" of one order of approximation.

Furthermore, as we will show, we do not require (2), but then the constant c may depend on the partition. To this end, we prove that any convex piecewise polynomial, (approximating a convex function) can be modified in such a way that the modified piecewise polynomial interpolates the function at the knots, and the new approximation error differs from the old one by a constant factor which depends only on the knots. Specifically, we prove

Theorem 2. *Suppose $f \in \Delta^2_{[a,b]}$, $k \ge 2$, and $x_{-1} := a =: x_0 < x_1 < \cdots < x_n := b =: x_{n+1}$. Then for each piecewise polynomial $s \in \Delta^2_{[a,b]}$ of degree $\le k-1$ with knots x_i, $i = 1,\dots,n-1$, there is a piecewise polynomial $s_1 \in \Delta^2_{[a,b]}$ of degree $\le k-1$, with the same knots such that*

$$1) \quad f(x_i) = s_1(x_i), \quad i = 0,\dots,n,$$

$$2) \quad \|f - s_1\|_{[x_{i-1},x_i]} \le c(m)\,\|f - s\|_{[x_{i-2},x_{i+1}]}\,, \quad i = 1,\dots,n,$$

where $c(m)$ is a constant depending only on m, the scale of the partition $x_0,\dots,x_n$, i.e.,

$$m := \max_{1\le i\le n-1} \left\{ \frac{x_{i+1}-x_i}{x_i - x_{i-1}}; \frac{x_i - x_{i-1}}{x_{i+1}-x_i} \right\}. \tag{4}$$

Remark 1. *The proof implies that $c(m) \le 4(2m+1)$. In particular, since for equidistant knots $m = 1$, and for the Chebyshev knots $m \le 3$, in both cases $c(m)$ is an absolute constant.*

Remark 2. *One can show that, in general, it is impossible to replace $c(m)$ by an absolute constant. Indeed, for $n = 2$, $k = 3$, we have $c(m) \ge \frac{1}{9}m$.*

The following is an immediate consequence of Theorems 1 and 2.

Theorem 3. *Let $F \in \Delta^3_{[a,b]}$ and $f(x) := F'(x)$, $x \in (a,b)$. Given an integer $k \ge 2$, a partition $x_{-1} := a =: x_0 < x_1 < \cdots < x_n := b =: x_{n+1}$, and a piecewise polynomial $s \in \Delta^2_{[a,b]}$ of degree $\le k-1$, with knots x_i, $i = 1, \dots, n-1$, there exists a piecewise polynomial $S \in \Delta^3_{[a,b]}$ of degree $\le k$ with knots x_i, $i = 1, \dots, n-1$, for which*

$$\|F - S\| \le c(m) \max_{1\le i\le n} (x_i - x_{i-1}) \, \|f - s\|_{[x_{i-2}, x_{i+1}]}\,, \tag{5}$$

where m is the scale of the partition (4), and $c(m) \le cm$ for some absolute constant c.

Note that (5) is completely trivial if f is unbounded in (a,b). If f is bounded there, then $f(a+), f(b-) < \infty$, we put $f(a) := f(a+)$ and $f(b) := f(b-)$, and the conditions of Theorem 2 are satisfied.

In order to apply Theorem 3 to obtain Jackson-type inequalities for 3-monotone approximation by piecewise polynomials with equidistant knots, we summarize results by Hu [5], Kopotun [8], Leviatan and Shevchuk [9, Corollary 2.4], Shevchuk [12, p. 141], Shvedov [13] for convex approximation by piecewise polynomials. Namely,

Proposition. *Let $k \ge 1$ and $r \ge 0$, be integers such that either $r \ge 2$ or $2 \le k + r \le 3$. Then for each $f \in C^{(r)}_{[-1,1]} \cap \Delta^2_{[-1,1]}$ there exist piecewise polynomials $s_1, s_2 \in \Delta^2_{[-1,1]}$ of degree $\le k + r - 1$ such that s_1 has n equidistant knots, and satisfies*

$$\|f - s_1\|_{[-1,1]} \le \frac{c(k,r)}{n^r} \omega_k(f^{(r)}, 1/n; [-1,1]), \tag{6}$$

and s_2 has knots on the Chebyshev partition, and satisfies

$$\|f - s_2\|_{[-1,1]} \le \frac{c(k,r)}{n^r} \omega^{\varphi}_k(f^{(r)}, 1/n; [-1,1]). \tag{7}$$

Moreover, s_1 and s_2 interpolate f at the respective knots.
If, on the other hand, $0 \le r \le 1$ and $k + r \ge 4$, then, in general, (6) and (7) cannot be achieved.

This together with Theorem 3 immediately implies all except one of the affirmative statements of the following theorem.

Theorem 4. *Let $k \ge 1$ and $r \ge 0$, be integers such that either $r \ge 3$ or $3 \le k+r \le 4$, $(k,r) \ne (4,0)$. Then for each $F \in C^{(r)}_{[-1,1]} \cap \Delta^3_{[-1,1]}$ there exist piecewise polynomials $S_1, S_2 \in \Delta^3_{[-1,1]}$ of degree $\le k+r-1$, such that S_1 has n equidistant knots, and satisfies*

$$\|F - S_1\|_{[-1,1]} \le \frac{c(k,r)}{n^r}\omega_k(F^{(r)}, 1/n; [-1,1]), \tag{8}$$

and S_2 has knots on the Chebyshev partition, and satisfies

$$\|F - S_2\|_{[-1,1]} \le \frac{c(k,r)}{n^r}\omega_k^{\varphi}(F^{(r)}, 1/n; [-1,1]), \tag{9}$$

If $r \le 2$ and $k+r \ge 5$, then (8) and (9) in general cannot be achieved.

The only positive case claimed above which cannot be concluded from Theorem 3 is $(k,r) = (3,0)$, which is (1). The negative results follow from Shevchuk [12, Thm 16.1], who extended the original negative result of Shvedov [13].

Remark 3. *Note that we have left out one case. Namely, it is unknown to us whether it is possible to construct for an arbitrary 3-convex function F, a cubic piecewise polynomial $S \in \Delta^3_{[-1,1]}$ with n equidistant knots such that*

$$\|F - S\|_{[-1,1]} \le c\omega_4(F, 1/n; [-1,1]).$$

A 3-monotone function in $[a,b]$, necessarily possesses at least one continuous derivative in (a,b), and indeed all we can say about the piecewise polynomials we constructed in Theorems 1 and 3 is that they possess this minimal possible smoothness, namely, they are in $C^{(1)}_{[a,b]}$. However, this can be improved and it is possible to obtain smoother piecewise polynomials. We prove

Theorem 5. *Suppose $S \in \Delta^3_{[a,b]}$ is a piecewise polynomial of degree $\le k$, $k \ge 3$, with knots on the partition $x_{-1} := a =: x_0 < x_1 < \cdots < x_n := b =: x_{n+1}$. Then there is a piecewise polynomial S_1 of degree $\le k$ with the same knots, such that*

$$S_1 \in \Delta^3_{[a,b]} \cap C^{(2)}_{[a,b]},$$

and

$$\|S - S_1\| \le c(k,m,\mu) \max_{1 \le j \le n-1} \omega_{k+1}(S, (x_{j+1} - x_{j-1}); [x_{j-1}, x_{j+1}]), \tag{10}$$

where $c(k,m,\mu)$ depends only on k, m, μ, where m is given by (4), and

$$\mu = \max_{0\le i<j\le n} \frac{(j-i)(x_{i+1}-x_i)}{x_j-x_i}. \tag{11}$$

Remark 4. *For equidistant knots $m=1$ and $\mu=1$, and for the Chebyshev knots $m\le 3$ and $\mu\le\pi$. Thus, for these partitions $c(k,m,\mu)\le c^*(k)$, depending only on k.*

In view of this remark a standard proof combining Theorem 4 and Theorem 5 yields

Theorem 6. *Let $k\ge 1$ and $r\ge 0$, be integers such that either $r\ge 3$ or $k+r=4$, $(k,r)\ne(4,0)$. Then for each $F\in C^{(r)}_{[-1,1]}\cap\Delta^3_{[-1,1]}$ there exist piecewise polynomials $S_1,S_2\in\Delta^3_{[-1,1]}\cap C^{(2)}_{[a,b]}$ of degree $\le k+r-1$, such that S_1 has n equidistant knots, and satisfies*

$$\|F-S_1\|_{[-1,1]} \le \frac{c(k,r)}{n^r}\omega_k(F^{(r)},1/n;[-1,1]), \tag{12}$$

and S_2 has knots on the Chebyshev partition, and satisfies

$$\|F-S_2\|_{[-1,1]} \le \frac{c(k,r)}{n^r}\omega_k^{\varphi}(F^{(r)},1/n;[-1,1]), \tag{13}$$

If $r\le 2$ and $k+r\ge 5$, then (12) and (13) in general cannot be achieved.

3. **Auxiliary construction and the proof of Theorem 1.**

Given a real function f defined on $[a,b]$, let $L(\cdot;f;a,b)$ denote the linear Lagrange interpolation of f at the points a and b. Throughout this section we take $k\ge 2$.

We begin with

Lemma 1. *Let $f\in\Delta^2_{[a,b]}$, and suppose that $q\in\Delta^2_{[a,b]}$ is a polynomial of degree $\le k-1$, satisfying $f(a)=q(a)$ and $f(b)=q(b)$. Then there exists a polynomial $p\in\Delta^2_{[a,b]}$ of degree $\le k-1$, such that*

$$f(a)=p(a),\quad f(b)=p(b), \tag{14}$$

$$q'(a)\le p'(a),\quad p'(b)\le q'(b), \tag{15}$$

$$\left\|\int_a^{(\cdot)}(p(t)-f(t))\,dt\right\|_{[a,b]} \le 2\left\|\int_a^{(\cdot)}(q(t)-f(t))\,dt\right\|_{[a,b]}, \tag{16}$$

and

$$\int_a^b p(t)\,dt \geq \int_a^b f(t)\,dt. \tag{17}$$

Proof. If

$$\int_a^b q(t)\,dt \geq \int_a^b f(t)\,dt$$

then we take $p := q$ and (14) through (17) are self evident. Otherwise,

$$\int_a^b f(t)\,dt - \int_a^b q(t)\,dt =: A > 0.$$

Clearly,

$$A \leq \left\| \int_a^{(\cdot)} (q(t) - f(t))\,dt \right\|_{[a,b]}. \tag{18}$$

Let $l(\cdot) := L(\cdot; f; a, b)$. Then by the convexity of f, $l(x) \geq f(x)$, $x \in [a,b]$. Hence,

$$\int_a^x l(t)\,dt - \int_a^x f(t)\,dt \leq \int_a^b l(t)\,dt - \int_a^b f(t)\,dt =: B \geq 0, \quad x \in [a,b]. \tag{19}$$

Let

$$p(x) := \frac{Al(x) + Bq(x)}{A + B}, \quad x \in [a,b].$$

Then p is a convex combination of l and q, and (14) and (15) are readily seen (note that for (15) we use the fact that q' is nondecreasing). For $x \in [a,b]$ we obtain by virtue of (19) and (18),

$$\begin{aligned}
\left| \int_a^x p(t)\,dt - \int_a^x f(t)\,dt \right| &= \left| \frac{A}{A+B} \int_a^x (l(t) - f(t))\,dt + \frac{B}{A+B} \int_a^x (q(t) - f(t))\,dt \right| \\
&\leq \frac{A}{A+B} \left| \int_a^x (l(t) - f(t))\,dt \right| + \frac{B}{A+B} \left| \int_a^x (q(t) - f(t))\,dt \right| \\
&\leq \frac{A}{A+B} B + \frac{B}{A+B} \left\| \int_a^{(\cdot)} (q(t) - f(t))\,dt \right\|_{[a,b]} \\
&\leq \frac{2B}{A+B} \left\| \int_a^{(\cdot)} (q(t) - f(t))\,dt \right\|_{[a,b]} \\
&\leq 2 \left\| \int_a^{(\cdot)} (q(t) - f(t))\,dt \right\|_{[a,b]},
\end{aligned}$$

that is, (16). Finally,

$$\begin{aligned}\int_a^b p(t)\,dt &= \frac{A}{A+B}\int_a^b l(t)\,dt + \frac{B}{A+B}\int_a^b q(t)\,dt \\ &= \frac{A}{A+B}\left(-B+\int_a^b l(t)\,dt\right) + \frac{B}{A+B}\left(\int_a^b q(t)\,dt + A\right) \\ &= \int_a^b f(t)\,dt,\end{aligned}$$

and (17) holds. This completes the proof. □

Next we show

Lemma 2. *Let $q \in \Delta^2_{[a,b]}$ be a polynomial of degree $\le k-1$, and let α and β be arbitrary nonnegative real numbers. Suppose that d_a, d_b are real numbers satisfying,*

$$d_a \le \frac{(q(b)-\beta)-(q(a)-\alpha)}{b-a} \le d_b, \tag{20}$$

and

$$d_a \le q'(a) \le q'(b) \le d_b.$$

Then there exists a polynomial $p \in \Delta^2_{[a,b]}$ of degree $\le k-1$, such that

$$p(a) = q(a)-\alpha, \quad p(b) = q(b)-\beta, \tag{21}$$

$$d_a \le p'(a) \le p'(b) \le d_b, \tag{22}$$

and

$$p(x) \le q(x), \quad x \in [a,b]. \tag{23}$$

Proof. If $\alpha = \beta$, then we take $p(x) := q(x)-\alpha$, $a \le x \le b$, and (21) through (23) are obvious. Otherwise, assume that $\alpha > \beta$ (the other case being similar). Let

$$\lambda := \frac{(b-a)d_b + q(a) - q(b)}{\alpha-\beta},$$

and note that the righthand side of (20) is equivalent to the inequality $\lambda \ge 1$. Put $l(x) := d_b(x-b) + q(b) - \lambda\beta$, $x \in [a,b]$. Then,

$$l(x) \le d_b(x-b) + q(b) \le q(x), \quad x \in [a,b]. \tag{24}$$

Now let

$$p(x) := \lambda^{-1}((\lambda-1)q(x)+l(x)), \quad x\in[a,b].$$

Then the polynomial p is convex being a linear combination of l and q, with nonnegative coefficients, and straightforward calculations yield (21) and (22) (again note that for (22) we use the fact that q' is nondecreasing). Finally by (24),

$$p(x) \le \lambda^{-1}((\lambda-1)q(x)+q(x)) = q(x), \quad x\in[a,b],$$

thus we have established (23). This completes the proof. □

Now we establish some relations between two convex functions in an interval. First

Lemma 3. *Let* $f\in\Delta^2_{[z_1,z_2]}$ *and* $g\in\Delta^2_{[z_1,z_2]}\cap C^{(1)}_{[z_1,z_2]}$, *be such that* $f(z_i)=g(z_i)$, $i=1,2$. *Let* $l_i(x):=(x-z_i)g'(z_i)+g(z_i)$, $i=1,2$, *and denote*

$$\Delta_i := \int_{z_1}^{z_2}(l_i(t)-f(t))_+dt, \quad i=1,2.$$

Then

$$\Delta_i \le \left\|\int_{z_i}^{(\cdot)}(f(t)-g(t))\,dt\right\|_{[z_1,z_2]}, \quad i=1,2. \tag{25}$$

Proof. We begin with $i=1$. Since g is convex, it follows that $l_1(x)\le g(x)$, $x\in[z_1,z_2]$. Since f is convex and l_1 is linear, there exists a $\theta\in[z_1,z_2]$, such that $f(x)\le l_1(x)$, $x\in[z_1,\theta]$, and $l_1(x)\le f(x)$, $x\in[\theta,z_2]$. Hence,

$$\begin{aligned}\Delta_1 &= \int_{z_1}^{\theta}(l_1(t)-f(t))\,dt \le \int_{z_1}^{\theta}(g(t)-f(t))\,dt\\ &\le \left\|\int_{z_1}^{(\cdot)}(f(t)-g(t))\,dt\right\|_{[z_1,z_2]},\end{aligned}$$

and (25) is proved for $i=1$. This in turn yields

$$\Delta_2 \le \left\|\int_{(\cdot)}^{z_2}(f(t)-g(t))\,dt\right\|_{[z_1,z_2]},$$

and the proof of (25) for $i=2$ is complete. □

We also have

Lemma 4. *Let* $f, g \in \Delta^2_{[a,b]}$*, be such that*

$$f(b) - f(a) = g(b) - g(a). \tag{26}$$

Then

$$f'(a+) \le g'(b-).$$

Proof. The functions f' and g' are non-decreasing on (a, b). Suppose to the contrary that $f'(a+) > g'(b-)$. Then

$$\begin{aligned} f(b) - f(a) &= \int_a^b f'(x)\,dx \ge f'(a+)(b-a) \\ &> g'(b-)(b-a) \ge \int_a^b g'(x)\,dx \\ &= g(b) - g(a), \end{aligned}$$

contradicting (26). □

An immediate consequence in the context of our paper is

Corollary 1. *Let* $f \in \Delta^2_{[a,b]}$ *and let* $s \in \Delta^2_{[a,b]}$ *be a piecewise polynomial of degree* $\le k-1$*, with knots on the partition* $a =: x_0 < x_1 < \ldots < x_n := b$ *satisfying* (2)*. Then for* $i = 2, \ldots, n-1$*,*

$$f'(x_{i-1}+) \le s'(x_i-), \quad i = 2, \ldots, n-1, \tag{27}$$

and

$$s'(x_{i-1}+) \le f'(x_i-), \quad i = 2, \ldots, n-1. \tag{28}$$

We are ready to begin our auxiliary construction. Given $f \in \Delta^2_{[a,b]}$ and $s \in \Delta^2_{[a,b]}$ as above, denote

$$M := \max_{1 \le i \le n} \|s - f\|_{L_1[x_{i-1}, x_i]}. \tag{29}$$

For a function g, we write $g \in A_{i,j}$, $1 \le i < j \le n-1$, if g is a convex piecewise polynomial of degree $\le k-1$, on $[x_i, x_j]$, with knots $x_{i+1}, \ldots, x_{j-1}$, and satisfies $s'(x_i+) \le g'(x_i+)$

and $g'(x_j-) \le s'(x_j-)$, and $g(x_i) = s(x_i)$ and $g(x_j) = s(x_j)$. For each $r = 1, \dots, n-1$ let

$$h_r(t) := \begin{cases} f'(x_i-), & \text{if } t \in (x_{i-1}, x_i], \quad i = 1, \dots, r-1, \\ s'(x_r-), & \text{if } t \in (x_{r-1}, x_r], \\ s'(x_r+), & \text{if } t \in (x_r, x_{r+1}), \\ f'(x_{i-1}+), & \text{if } t \in [x_{i-1}, x_i), \quad i = r+2, \dots, n, \end{cases}$$

and set

$$g_r(x) := f(x_r) + \int_{x_r}^{x} h_r(t)\, dt.$$

By virtue of Corollary 1, h_r is non-decreasing on (a, b), and g_r is convex there. It follows by (2) that $g_r(x_{r+1}) \le f(x_{r+1})$ and $g_r(x_{r-1}) \le f(x_{r-1})$. Hence,

$$g_r(x) \le f(x), \quad x \in [a, b] \setminus (x_{r-1}, x_{r+1}). \tag{30}$$

By Lemma 3,

$$\int_{x_r}^{x_{r+1}} (g_r(t) - f(t))_+\, dt \le M, \tag{31}$$

and

$$\int_{x_{r-1}}^{x_r} (g_r(t) - f(t))_+\, dt \le M. \tag{32}$$

For each pair $1 \le i < j \le n-1$, we will construct a function $g_{i,j} \in A_{i,j}$. To this end, if $j = i+1$, then we set $g_{i,i+1} := s\big|_{[x_i, x_{i+1}]}$, evidently belonging to $A_{i,i+1}$. Otherwise, we observe that by (30), we have $g_j(x_i) \le g_i(x_i)$ and $g_i(x_j) \le g_j(x_j)$. Also $g_j - g_i$ is continuous on $[x_i, x_j]$, therefore there exists a $\theta \in (x_i, x_j)$ such that $g_i(\theta) = g_j(\theta)$. In addition, by (27) and (28) $h_i(t) \le h_j(t)$, $t \in (x_i, x_j)$, whence $h_j - h_i$ is nonnegative on (x_i, x_j), and in turn $g_j - g_i$ is non-decreasing there. Hence

$$\max\{g_i(x), g_j(x)\} = \begin{cases} g_i(x), & \text{if } x \le \theta, \\ g_j(x), & \text{if } x > \theta, \end{cases}$$

and we set $\overline{g}_{i,j}(x) := \max\{g_i(x), g_j(x)\}$, $x \in [x_i, x_j]$. Note that $\overline{g}_{i,j}$ is convex in $[x_i, x_j]$ as the maximum of convex functions. For some integer m, $i+1 \le m \le j$, the point θ satisfies $\theta \in [x_{m-1}, x_m]$. Clearly, for each integer l, $l \ne m$, $i+1 \le l \le j$, the function $\overline{g}_{i,j}$

is linear on $[x_{l-1}, x_l]$, but it may not be so on the interval $[x_{m-1}, x_m]$. We wish to replace it on the latter with a suitable polynomial of degree $\leq k-1$. Since $\overline{g}_{i,j}$ is convex, we have

$$\overline{g}'_{i,j}(x_{m-1}+) \leq \frac{\overline{g}_{i,j}(x_m) - \overline{g}_{i,j}(x_{m-1})}{x_m - x_{m-1}} \leq \overline{g}'_{i,j}(x_m-).$$

Put

$$d_a := \begin{cases} s'(x_{m-1}+), & \text{if} \quad m-1=i, \\ f'(x_{m-2}+), & \text{otherwise,} \end{cases} \qquad \text{and} \quad d_b := \begin{cases} s'(x_m-), & \text{if} \quad m=j, \\ f'(x_{m+1}-), & \text{otherwise.} \end{cases}$$

Then it follows that

$$d_a \leq \overline{g}'_{i,j}(x_{m-1}+) \leq \overline{g}'_{i,j}(x_m-) \leq d_b.$$

Also, in view of (27) and (28),

$$d_a \leq s'(x_{m-1}+), \quad s'(x_m-) \leq d_b.$$

Applying Lemma 2 with $a := x_{m-1}$ and $b := x_m$, d_a and d_b as above, $q := s|_{[x_{m-1}, x_m]}$, and $\alpha := f(x_{m-1}) - \overline{g}_{i,j}(x_{m-1})$ and $\beta := f(x_m) - \overline{g}_{i,j}(x_m)$, we obtain a suitable polynomial p. Put

$$g_{i,j}(x) := \begin{cases} \overline{g}_{i,j}(x), & \text{if } x \notin [x_{m-1}, x_m], \\ p(x), & \text{if } x \in [x_{m-1}, x_m]. \end{cases}$$

Then (21) and (22) yield $g_{i,j} \in A_{i,j}$ and (23) gives

$$\int_{x_{m-1}}^{x_m} (g_{i,j}(t) - f(t))_+ \, dt \leq \int_{x_{m-1}}^{x_m} (s(t) - f(t))_+ \, dt \leq M. \tag{33}$$

By virtue (31) and (32) we have

$$\int_{x_i}^{x_{i+1}} (g_{i,j}(t) - f(t))_+ \, dt \leq M, \tag{34}$$

and

$$\int_{x_{j-1}}^{x_j} (g_{i,j}(t) - f(t))_+ \, dt \leq M. \tag{35}$$

Since (30) implies that $g_{i,j}(x) \leq f(x)$ for all $x \in [x_{l-1}, x_l]$, $i+1 < l < j$, $l \neq m$, we conclude from (33), (34) and (35) that

$$\int_{x_i}^{x_j} (g_{i,j}(t) - f(t))_+ \, dt \leq 3M. \tag{36}$$

If $\delta(\cdot)$ is a continuous function on $[x_i, x_j]$, then we have

$$\left\| \int_{x_i}^{(\cdot)} \delta(t)\,dt \right\|_{[x_i,x_j]} \le \left| \int_{x_i}^{x_j} \delta(t)\,dt \right| + \int_{x_i}^{x_j} \delta(t)_+\,dt. \tag{37}$$

Indeed, for $x_i < x < x_j$, if $\int_{x_i}^{x} \delta(t)\,dt \ge 0$, then

$$0 \le \int_{x_i}^{x} \delta(t)\,dt \le \int_{x_i}^{x} \delta_+(t)\,dt \le \int_{x_i}^{x_j} \delta_+(t)\,dt.$$

On the other hand, if $\int_{x_i}^{x} \delta(t)\,dt < 0$, then

$$\begin{aligned} \left| \int_{x_i}^{x} \delta(t)\,dt \right| &\le \int_{x_i}^{x} \delta_-(t)\,dt \le \int_{x_i}^{x_j} \delta_-(t)\,dt \\ &= -\int_{x_i}^{x_j} \delta(t)\,dt + \int_{x_i}^{x_j} \delta(t)_+\,dt. \end{aligned}$$

Thus, (37) is proved. Therefore, if we denote

$$\Delta_{i,j}(\cdot) := \int_{x_i}^{(\cdot)} (g_{i,j}(t) - f(t))\,dt,$$

then by (36),

$$\|\Delta_{i,j}\|_{[x_i,x_j]} \le |\Delta_{i,j}(x_j)| + 3M. \tag{38}$$

The next lemma establishes the existence of functions in $A_{i,j}$ with associated $\Delta_{i,j}$'s with desired properties.

Lemma 5. *Let $1 \le i \le n-2$ be a fixed integer. Then, there exist an integer $i+1 \le j \le n-1$, and a function $g^\star_{i,j} \in A_{i,j}$, such that for*

$$\Delta^\star_{i,j}(\cdot) := \int_{x_i}^{(\cdot)} (g^\star_{i,j}(t) - f(t))\,dt,$$

we have

$$\left\|\Delta^\star_{i,j}\right\|_{[x_i,x_j]} \le 12M. \tag{39}$$

If $j < n-1$, then, in addition,

$$\Delta^\star_{i,j}(x_j) < 0. \tag{40}$$

Proof. If $\Delta_{i,n-1}(x_{n-1}) \ge 0$, then by (36), $\Delta_{i,n-1}(x_{n-1}) \le 3M$, and setting $g^\star_{i,n-1} := g_{i,n-1}$, we see that (39) follows by (38). Otherwise, at least one of the above numbers $\Delta_{i,i+r}(x_{i+r})$, $1 \le r \le n-i-1$, is negative. If for some $1 \le r \le n-i-1$, $-6M \le \Delta_{i,i+r}(x_{i+r}) < 0$, then we take $j := i+r$ and $g^\star_{i,j} := g_{i,j}$. Then (40) is fulfilled, and again by (38), we obtain (39). Finally, if all negative numbers among the above are $< -6M$, then we let $1 \le r \le n-i-1$, be the smallest such that $\Delta_{i,i+r}(x_{i+r}) < -6M$. Evidently, $r \ge 2$, since $g_{i,i+1}(x) = s(x)$, $x \in [x_i, x_{i+1}]$, whence $|\Delta_{i,i+1}(x_{i+1})| \le M$. Set $j := i+r$, and let $p := s\big|_{[x_{j-1},x_j]}$. Then by (29),

$$\left\| \int_{x_{j-1}}^{(\cdot)} (p(t) - f(t))\, dt \right\|_{[x_{j-1},x_j]} \le M. \tag{41}$$

Denote

$$\tilde g_{i,j}(x) := \begin{cases} g_{i,j-1}(x), & \text{if } x \in [x_i, x_{j-1}), \\ p(x), & \text{if } x \in [x_{j-1}, x_j]. \end{cases}$$

Then $\tilde g_{i,j} \in A_{i,j-1}$ and $\tilde g_{i,j} \in A_{j-1,j}$, imply that $\tilde g_{i,j} \in A_{i,j}$. Let $g^\star_{i,j}(x) := \lambda g_{i,j}(x) + (1-\lambda)\tilde g_{i,j}(x)$, $x \in [x_i, x_j]$, where $\lambda := 6M|\Delta_{i,j}(x_j)|^{-1}$. Clearly, $\lambda \in (0,1)$, so that $g^\star_{i,j} \in A_{i,j}$. The choice of r implies that $0 \le \Delta_{i,j-1}(x_{j-1}) \le 3M$. Hence by (38)

$$\begin{aligned} \left\| \int_{x_i}^{(\cdot)} (\tilde g_{i,j}(t) - f(t))\, dt \right\|_{[x_i,x_{j-1}]} &= \|\Delta_{i,j-1}\|_{[x_i,x_{j-1}]} \\ &\le |\Delta_{i,j-1}(x_{j-1})| + 3M \\ &\le 6M. \end{aligned}$$

Also, by (41)

$$\begin{aligned} \left| \int_{x_i}^{x} (\tilde g_{i,j}(t) - f(t))\, dt \right| &= \left| \Delta_{i,j-1}(x_{j-1}) + \int_{x_{j-1}}^{x} (p(t) - f(t))\, dt \right| \\ &\le |\Delta_{i,j-1}(x_{j-1})| + M \\ &\le 4M, \quad x \in [x_{j-1}, x_j]. \end{aligned}$$

Therefore,

$$\left\| \int_{x_i}^{(\cdot)} (\tilde g_{i,j}(t) - f(t))\, dt \right\|_{[x_i,x_j]} \le 6M. \tag{42}$$

In particular,

$$\begin{aligned}\Delta^\star_{i,j}(x_j) &= \lambda\Delta_{i,j}(x_j) + (1-\lambda)\int_{x_i}^{x_j} (\tilde{g}_{i,j}(t) - f(t))\,dt \\ &\le -6M + (1-\lambda)6M < 0,\end{aligned}$$

so that (40) is verified. Finally, by virtue of (38) and (42),

$$\begin{aligned}\left\|\Delta^\star_{i,j}\right\|_{[x_i,x_j]} &\le \lambda\left\|\Delta_{i,j}\right\|_{[x_i,x_j]} + (1-\lambda)\left\|\int_{x_i}^{(\cdot)} (\tilde{g}_{i,j}(t) - f(t))\,dt\right\|_{[x_i,x_j]} \\ &\le \lambda(|\Delta_{i,j}(x_j)| + 3M) + 6(1-\lambda)M = 6M + 6M - 3\lambda M \\ &\le 12M.\end{aligned}$$

This proves (39) and completes the proof of Lemma 5. □

Proof of Theorem 1. We look for the required function S in the form

$$S(x) := F(x_1) + \int_{x_1}^{x} \overline{g}(t)\,dt, \quad x \in [a,b],$$

where

$$\overline{g}(t) = \begin{cases} s(t), & \text{if} \quad t \in [x_0, x_1) \cup (x_{n-1}, x_n], \\ g(t), & \text{if} \quad t \in [x_1, x_{n-1}], \end{cases}$$

is in $A_{1,n-1}$. The latter provides the 3-monotonicity of S. We are going to construct $g(t)$ by induction.

First we observe that when we apply Lemma 1 for $[x_{i-1}, x_i]$, $2 \le i \le n-1$, with $q := s\big|_{[x_{i-1},x_i]}$, then the resulting polynomial p is in $A_{i-1,i}$. Also, recall that if $g \in A_{i,j}$, $1 \le i < j < l \le n-1$, and $g \in A_{j,l}$, then $g \in A_{i,l}$. We construct g by induction. We apply Lemma 1 for $[x_1, x_2]$, with $q := s\big|_{[x_1,x_2]}$, obtain a polynomial $p \in A_{1,2}$, and put $g(x) := p(x)$, $x \in [x_1, x_2]$. Suppose that g is already defined on $[x_1, x_i]$ for some $2 \le i \le n-2$, it is in $A_{1,i}$, and satisfies for all $x \in [x_1, x_i]$,

$$\left|\int_{x_1}^{x} (g(t) - f(t))\,dt\right| \le 24M, \tag{43}$$

where M is given in (29), and

$$\left|\int_{x_1}^{x_i} (g(t) - f(t))\,dt\right| \le 12M. \tag{44}$$

Then we define g on some $[x_i, x_j]$, $i < j \le n-1$, so that $g \in A_{i,j}$, (43) remains valid, on the larger interval $[x_1, x_j]$, and if $j < n-1$, then also such that

$$\left| \int_{x_1}^{x_j} (g(t) - f(t))\, dt \right| \le 12M. \tag{45}$$

If

$$\int_{x_1}^{x_i} (g(t) - f(t))\, dt \le 0, \tag{46}$$

then we take $j = i+1$ and apply Lemma 1 for $[x_{j-1}, x_j]$, and $q := s\big|_{[x_{j-1},x_j]}$. We put $g(x) := p(x)$, $x \in [x_{j-1}, x_j]$, where p is the resulting polynomial. For $x \in [x_{j-1}, x_j]$, we have by (44) and (16),

$$\begin{aligned}\left| \int_{x_1}^{x} (g(t) - f(t))\, dt \right| &\le \left| \int_{x_1}^{x_i} (g(t) - f(t))\, dt \right| + \left| \int_{x_i}^{x} (p(t) - f(t))\, dt \right| \\ &\le 12M + 2M \le 14M.\end{aligned}$$

Hence, combining with (43) for $x \in [x_1, x_i]$, we see that (43) holds for $x \in [x_1, x_j]$. Moreover, (17) implies that

$$0 \le \int_{x_{j-1}}^{x_j} (g(t) - f(t))\, dt \le 2M.$$

which together with (44) and (46) yield

$$-12M \le \int_{x_1}^{x_j} (g(t) - f(t))\, dt \le 2M.$$

This proves (45). Note that here is the only place we make use of (17).

Otherwise,

$$\int_{x_1}^{x_i} (g(t) - f(t))\, dt > 0. \tag{47}$$

We apply Lemma 5, and get some integer j, $i+1 \le j \le n-1$, and $g^\star_{i,j} \in A_{i,j}$, satisfying (39), and (40) if $j < n-1$. We put $g(x) := g^\star_{i,j}(x)$, $x \in [x_i, x_j]$. If $j = n-1$, then (39) implies (43) for $x \in [x_1, x_{n-1}]$, and the construction is complete. Otherwise, for $x \in [x_i, x_j]$, by (39) and (44),

$$\begin{aligned}\left| \int_{x_1}^{x} (g(t) - f(t))\, dt \right| &\le \left| \int_{x_1}^{x_i} (g(t) - f(t))\, dt \right| + \left| \int_{x_i}^{x} (g^\star_{i,j}(t) - f(t))\, dt \right| \\ &\le 12M + 12M \le 24M.\end{aligned}$$

Hence, (43) holds for $x \in [x_1, x_j]$. Also, by (47) and (44),

$$0 < \int_{x_1}^{x_i} (g(t) - f(t))\, dt \le 12M,$$

which combined with (39) and (40) give

$$-12M < \int_{x_1}^{x_j} (g(t) - f(t))\, dt \le 12M.$$

This proves (45) and completes the induction step.

Finally, in view of the definition of S, we see by (43), that (3) holds with $c \le 25$. $\square$

4. **Proof of Theorem 2**

Recall that for f defined on $[a, b]$, we let $l(\cdot) := L(\cdot; f; a, b)$ denote the linear Lagrange interpolation of f at the points a and b. (Note that $l'(x) = f[a, b]$, $x \in [a, b]$.) We begin with some lemmas.

Lemma 6. *If $f \in \Delta^2_{[a,b]}$ and $s \in \Delta^2_{[a,b]}$, are such that either $s'(b-) \le f[a, b]$ or $s'(a+) \ge f[a, b]$, then*

$$\|f - l\| \le 2\|f - s\|.$$

Proof. Assume that $s'(b-) \le f[a, b]$, the case $s'(a+) \ge f[a, b]$ is symmetric. If $x_0 := \sup\{x \in (a, b) : f'(x) \le f[a, b]\}$, then $s'(x) \le s'(b-) \le f[a, b] \le f'(x)$, $x_0 \le x \le b$. Hence,

$$\begin{aligned}
\|f - l\| &= l(x_0) - f(x_0) \\
&= \int_{x_0}^{b} (f'(x) - l'(x))\, dx \\
&\le \int_{x_0}^{b} (f'(x) - s'(x))\, dx \\
&\le f(b) - s(b) - (f(x_0) - s(x_0)) \\
&\le 2\|f - s\|.
\end{aligned}$$

$\square$

The next lemma is essential to our proof.

Lemma 7. *Suppose that f is defined on $[a_1,b_1]$, and that s is a piecewise polynomial of degree $\le k-1$, with knots a and b, $a_1\le a<b\le b_1$, such that $s'(a+)\le f[a,b]\le s'(b-)$. If $f,s\in\Delta^2_{[a_1,b_1]}$, then there exists a piecewise polynomial $s_1\in\Delta^2_{[a_1,b_1]}$, of degree $\le k-1$, with knots a and b, satisfying*

$$1)\quad s'(a+)\le s_1'(a+),\quad s_1'(b-)\le s'(b-),$$

$$2)\quad s_1(a)=f(a),\quad s_1(b)=f(b),$$

$$3)\quad \|f-s_1\|_{[a,b]}\le 4\|f-s\|_{[a,b]},$$

$$4)\quad \|f-s_1\|_{[a_1,b_1]}\le 4\|f-s\|_{[a_1,b_1]}.$$

Note that if $[a,b]=[a_1,b_1]$, then both s and s_1 are polynomials of degree $\le k-1$ on $[a_1,b_1]$.

Proof. If $f(b)-f(a)=s(b)-s(a)$, we take $s_1(x):=s(x)+f(a)-s(a)$, $x\in[a_1,b_1]$. Then 1) and 2) are self evident, and

$$\|f-s_1\|_{[a,b]}\le\|f-s\|_{[a,b]}+|f(a)-s(a)|\le 2\|f-s\|_{[a,b]},$$

and

$$\|f-s_1\|_{[a_1,b_1]}\le\|f-s\|_{[a_1,b_1]}+|f(a)-s(a)|\le 2\|f-s\|_{[a_1,b_1]}.$$

Assume $f(b)-f(a)<s(b)-s(a)$, the case $f(b)-f(a)>s(b)-s(a)$ is symmetric. We first define s_1 in $[a,b]$, and then we extend it to $[a_1,b_1]$ if $[a,b]\ne[a_1,b_1]$.

Let $\tilde s(x):=s(x)-s'(a+)(x-a)$, $x\in[a,b]$, and $\tilde f(x):=f(x)-s'(a+)(x-a)$, $x\in[a,b]$. Then $\|\tilde f-\tilde s\|_{[a,b]}=\|f-s\|_{[a,b]}$, $\tilde s'(a+)=0$, $\tilde s'(b-)=s'(b-)-s'(a+)$, and $\tilde f[a,b]=f[a,b]-s'(a+)\ge 0$. In particular $\tilde f(b)-\tilde f(a)\ge 0$, and since by our assumption $\tilde f(b)-\tilde f(a)<\tilde s(b)-\tilde s(a)$, it follows that $\tilde s(b)-\tilde s(a)>0$. Thus we may set $\tilde s_1(x):=\tilde f(a)+\lambda(\tilde s(x)-\tilde s(a))$, $x\in[a,b]$, where $\lambda:=(\tilde f(b)-\tilde f(a))(\tilde s(b)-\tilde s(a))^{-1}$. Then $0\le\lambda<1$ and $\tilde s_1$ is convex in $[a,b]$. Also $\tilde s_1(a)=\tilde f(a)$, $\tilde s_1(b)=\tilde f(b)$, $\tilde s_1'(a+)=0$, and $\tilde s_1'(b-)=\lambda\tilde s'(b-)<s'(b-)-s'(a+)$. We set $s_1(x):=\tilde s_1(x)+s'(a+)(x-a)$, $x\in[a,b]$, and it has the properties 1), 2). Finally, note that $\tilde s'\ge 0$ in $[a,b]$ so that $\tilde s$ is nondecreasing there, and $\|\tilde s(\cdot)-\tilde s(a)\|_{[a,b]}=\tilde s(b)-\tilde s(a)$.

Hence

$$\begin{aligned}\|f-s_1\|_{[a,b]} &= \left\|\tilde{f}-\tilde{s}_1\right\|_{[a,b]}\\ &= \left\|\tilde{f}(\cdot)-\tilde{s}(\cdot)+\tilde{s}(a)-\tilde{f}(a)+\tilde{s}(\cdot)-\tilde{s}(a)+\tilde{f}(a)-\tilde{s}_1(\cdot)\right\|_{[a,b]}\\ &\le 2\left\|\tilde{f}-\tilde{s}\right\|_{[a,b]}+\|\tilde{s}(\cdot)-\tilde{s}(a)-\lambda(\tilde{s}(\cdot)-\tilde{s}(a))\|_{[a,b]}\\ &\le 2\left\|\tilde{f}-\tilde{s}\right\|_{[a,b]}+(1-\lambda)|\tilde{s}(b)-\tilde{s}(a)|\\ &= 2\left\|\tilde{f}-\tilde{s}\right\|_{[a,b]}+|\tilde{s}(b)-\tilde{s}(a)-(\tilde{f}(b)-\tilde{f}(a))|\\ &\le 4\left\|\tilde{f}-\tilde{s}\right\|_{[a,b]}=4\,\|f-s\|_{[a,b]}\,,\end{aligned}$$

and 3) is done.

Further, if $[a,b]\ne[a_1,b_1]$, then we extend s_1 either to the left or to the right or both, as needed, by setting

$$s_1(x)=\begin{cases} s(x)+f(a)-s(a), & x\in[a_1,a)\\ s(x)+f(b)-s(b), & x\in(b,b_1].\end{cases}$$

Then it is easy to see that s_1 is a convex piecewise polynomial of degree $\le k-1$ on $[a_1,b_1]$, with knots a and b, which possesses the properties 1) and 2) and 3). We only have to estimate the distance between f and s_1 on the intervals $[a_1,a]$ and $[b,b_1]$. Now

$$\|f-s_1\|_{[b,b_1]}\le\|f-s\|_{[b,b_1]}+|f(b)-s(b)|\le 2\|f-s\|_{[b,b_1]},$$

and similarly

$$\|f-s_1\|_{[a_1,a]}\le 2\|f-s\|_{[a_1,a]}.$$

Combining these with 3), we establish 4), and the proof is complete. □

Next is a lemma which is needed in the proof of Lemma 9

Lemma 8. *Suppose $f\in\Delta^2_{[a,b_1]}$, and $s\in\Delta^2_{[a,b_1]}$, $a<b<b_1$, and $s'(b-)-f[b,b_1]>0$. Then*

$$(s'(b-)-f[b,b_1])(b_1-b)\le 2\,\|f-s\|_{[b,b_1]}\,.$$

Symmetrically, if $f\in\Delta^2_{[a_1,b]}$, and $s\in\Delta^2_{[a_1,b]}$, $a_1<a<b$, and $f[a_1,a]-s'(a+)>0$, then

$$(f[a_1,a]-s'(a+))(a-a_1)\le 2\,\|f-s\|_{[a_1,a]}\,.$$

Proof. We prove the first statement, the proof of the other is similar. Let $x_1 := \sup\{x \in (b, b_1) : f'(x) \le s'(b-)\}$. Then

$$\begin{aligned}(s'(b-) - f[b,b_1])(b_1 - b) &= \int_b^{b_1} (s'(b-) - f[b,b_1])\,dx = \int_b^{b_1} (s'(b-) - f'(x))\,dx \\ &\le \int_b^{x_1} (s'(b-) - f'(x))\,dx \\ &\le \int_b^{x_1} (s'(x) - f'(x))\,dx \\ &= s(x_1) - f(x_1) - (s(b) - f(b)) \\ &\le 2\,\|s - f\|_{[b,b_1]},\end{aligned}$$

where in the second inequality we used the fact that s' is nondecreasing so that $f'(x) \le s'(b-) \le s'(x)$, $x \in (b, x_1)$. $\square$

The following lemma plays a crucial role in the proof.

Lemma 9. *Let $a_1 < a < b < b_1$, $m := \max\left\{\frac{b-a}{b_1-b}; \frac{b-a}{a-a_1}\right\}$, and $f \in \Delta^2_{[a_1,b_1]}$, and suppose that $s \in \Delta^2_{[a_1,b_1]}$ is a piecewise polynomial of degree $\le k-1$ with knots a and b, satisfying $f(a) = s(a)$, $f(b) = s(b)$. Then, there is a polynomial $s_1 \in \Delta^2_{[a,b]}$ of degree $\le k-1$ such that*

1) $s'(a+) \le s_1'(a+), \quad s_1'(b-) \le s'(b-)$,

2) $f[a,a_1] =: k_a \le s_1'(a+), \quad s_1'(b-) \le k_b := f[b,b_1]$,

3) $s_1(a) = f(a), \quad s_1(b) = f(b)$,

4) $\|f - s_1\|_{[a,b]} \le c(m)\,\|f - s\|_{[a_1,b_1]}$,

where $c(m) \le 2m + 1$.

Proof. Subtracting a linear function if necessary, we may assume that $f(a) = f(b)$. If s is constant on $[a,b]$, take $s_1(x) := s(x)$, $x \in [a,b]$. Otherwise, since $s(b) = s(a)$ and s is convex, we have $s'(b-) > 0 > s'(a+)$. Denote

$$\lambda := \min\left\{\frac{k_b}{s'(b-)}, \frac{k_a}{s'(a+)}\right\} \ge 0.$$

If $\lambda \geq 1$, then take $s_1(x) := s(x)$, $x \in [a,b]$, and there is nothing to prove. Otherwise $\lambda < 1$, and without loss of generality we may assume that $\lambda = \frac{k_b}{s'(b-)} < 1$. Then let $s_1(x) := s(a) + \lambda(s(x) - s(a))$, $x \in [a,b]$, so that $s_1 \in \Delta^2_{[a,b]}$ and it is a polynomial of degree $\leq k-1$. It is readily seen that $s_1(a) = f(a) = f(b) = s_1(b)$. Also, since $s'(a+) < 0$ and $\frac{k_a}{s'(a+)} \geq \frac{k_b}{s'(b-)} > 0$, we have

$$s_1'(a+) = \lambda s'(a+) = \frac{k_b}{s'(b-)} s'(a+) \geq \frac{k_a}{s'(a+)} s'(a+) = k_a,$$

and

$$s_1'(b-) = \lambda s'(b-) = \frac{k_b}{s'(b-)} s'(b-) = k_b.$$

Let $x_0 := \sup\{x \in (a,b) : s'(x) \leq 0\}$. Since $0 = s(b) - s(a) = \int_a^b s'(t)\,dt$, we have

$$\|s - s(a)\|_{[a,b]} = \int_{x_0}^{a} s'(t)\,dt = \int_{x_0}^{b} s'(t)\,dt \leq (b - x_0)s'(b-) \leq (b-a)s'(b-).$$

This in turn implies by virtue of Lemma 8,

$$\begin{aligned}
\|s - s_1\|_{[a,b]} &= \max_{x\in[a,b]} |s(x) - s(a) - \lambda(s(x) - s(a))| \\
&= (1-\lambda)\,\|s - s(a)\|_{[a,b]} \leq (1-\lambda)(b-a)s'(b-) \\
&= \frac{s'(b-) - k_b}{s'(b-)}(b-a)s'(b-) \leq (s'(b-) - k_b)(b-a) \\
&\leq m(s'(b-) - k_b)(b_1 - b) \leq 2m\,\|f - s\|_{[b,b_1]}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|f - s_1\|_{[a,b]} &\leq \|f - s\|_{[a,b]} + \|s - s_1\|_{[a,b]} \\
&\leq (2m+1)\,\|f - s\|_{[a_1,b_1]}\,,
\end{aligned}$$

and Lemma 9 is proved with $c(m) = 2m+1$. □

Finally, we need a one-sided (weaker) version of Lemma 9. This version is required when f may not be extended to the left of a as a convex function, i.e., when $f'(a+) = -\infty$.

Lemma 10. *Let $a < b < b_1$, $\tilde{m} := \frac{b-a}{b_1-b}$, and $f \in \Delta^2_{[a,b_1]}$, and suppose that $s \in \Delta^2_{[a,b_1]}$ is a piecewise polynomial of degree $\leq k-1$ with knot b, satisfying $f(a) = s(a)$ and $f(b) = s(b)$.*

Then, there is a polynomial $s_1 \in \Delta^2_{[a,b]}$ *of degree* $\le k-1$ *such that*

$$1)\quad s_1'(b-) \le s'(b-),$$

$$2)\quad s_1'(b-) \le k_b := f[b, b_1],$$

$$3)\quad s_1(a) = f(a), s_1(b) = f(b),$$

$$4)\quad \|f - s_1\|_{[a,b]} \le c(\tilde m)\, \|f - s\|_{[a,b_1]},$$

where $c(\tilde m) \le 2\tilde m + 1$.

Symmetrically, let $a_1 < a < b$, $\tilde m := \frac{b-a}{a-a_1}$, *and* $f \in \Delta^2_{[a_1,b]}$, *and suppose that* $s \in \Delta^2_{[a_1,b]}$ *is a piecewise polynomial of degree* $\le k-1$ *with knot* a, *satisfying* $f(a) = s(a)$ *and* $f(b) = s(b)$. *Then, there is a polynomial* $s_1 \in \Delta^2_{[a,b]}$ *of degree* $\le k-1$ *such that*

$$1)\quad s'(a+) \le s_1'(a+),$$

$$2)\quad f[a, a_1] =: k_a \le s_1'(a+),$$

$$3)\quad s_1(a) = f(a), \quad s_1(b) = f(b),$$

$$4)\quad \|f - s_1\|_{[a,b]} \le c(\tilde m)\, \|f - s\|_{[a_1,b]},$$

where $c(\tilde m) \le 2\tilde m + 1$.

Proof. We indicate the proof for the first case, the second is completely analogous. We repeat the proof of Lemma 9, except that this time we simply take $\lambda := \frac{k_b}{s'(b-)}$. Properties 3) and 4) are the same and for 1) and 2), we deal only with the point b. $\square$

We are ready with the

Proof of Theorem 2. Denote

$$l_r(\cdot) := L(\cdot; f; x_{r-1}, x_r), \quad r = 0, \dots, n+1.$$

Let $A \subset \{1, \dots, n\}$ be the set of all integers j, satisfying $s'(x_{j-1}+) \le l_j' \le s'(x_j-)$. For all $j \notin A$ we set $s_1(x) := l_j(x)$, $x \in [x_{j-1}, x_j]$. By Lemma 6

$$\|f - s_1\|_{[x_{j-1},x_j]} \le 2\, \|f - s\|_{[x_{j-1},x_j]}. \tag{48}$$

In order to define s_1 on $[x_{j-1}, x_j]$, $j \in A$, we first assume $1 < j < n$ and apply to the interval $[x_{j-2}, x_{j+1}]$, first Lemma 7 and then Lemma 9, with $a = x_{j-1}$ and $b = x_j$. We conclude the existence of $s_1 \in \Delta^2_{[x_{j-1},x_j]}$, such that

$$\|f - s_1\|_{[x_{j-1},x_j]} \le 4(2m+1)\,\|f - s\|_{[x_{j-2},x_{j+1}]}\,, \tag{49}$$

and $f(x_{j-1}) = s_1(x_{j-1})$, $f(x_j) = s_1(x_j)$.

Finally we have to deal with the possibility that either $j = 1$ or $j = n$ is in A. To this end, assume $1 \in A$, the case $n \in A$ being symmetric, so that $s'(a+) \le f[a, x_1] \le s'(x_1-)$. Then by Lemma 7 we have a convex piecewise polynomial $\tilde{s}_1$ in $[a, x_2]$ which interpolates f at a and x_1, satisfies $\tilde{s}_1'(x_1-) \le s'(x_1-)$, and is such that

$$\|f - \tilde{s}_1\|_{[a,x_2]} \le 4\|f - s\|_{[a,x_2]}.$$

We now apply Lemma 10 and obtain a polynomial s_1 on $[a, x_1]$, of degree $\le k-1$, which interpolates f at a and x_1, satisfies $s_1'(x_1-) \le \tilde{s}_1'(x_1-)$, and is such that

$$\|f - s_1\|_{[a,x_1]} \le 4(2m+1)\,\|f - s\|_{[a,x_2]}\,. \tag{50}$$

We are left with having to show that combining the various pieces we have an $s_1 \in \Delta^2_{[a,b]}$. To this end, all we should show is that

$$s_1'(x_j-) \le s_1'(x_j+), \quad j = 1, \dots, n-1. \tag{51}$$

Indeed, if $j, j+1 \notin A$, then $s_1'(x_j-) = l_j'$ and $s_1'(x_j+) = l_{j+1}'$, and the inequality $l_j' \le l_{j+1}'$ is evident in view of the convexity of f.

If $j, j+1 \in A$, then by virtue of Lemma 7 and Lemmas 9 or 10, and the convexity of s we conclude that

$$s_1'(x_j-) \le s'(x_j-) \le s'(x_j+) \le s_1'(x_j+).$$

If $j \in A$, $j+1 \notin A$, then by Lemmas 9 or 10,

$$s_1'(x_j-) \le l_j' = s_1'(x_j+),$$

and the case $j \notin A$, $j+1 \in A$, is symmetric. Thus (51) is proved.

In conclusion, s_1 is a convex piecewise-polynomial function of degree $\le k-1$, satisfying $s_1(x_j) = f(x_j)$, $j = 0, \dots, n$, and (48) through (50) imply

$$\|f - s_1\|_{[x_{i-1},x_i]} \le 4(2m+1)\,\|f - s\|_{[x_{i-2},x_{i+1}]}\,, \quad 1 \le i \le n.$$

$\square$

5. **Proof of Theorem 5**

The following lemma is a modification of a lemma by Bondarenko [2, Lemma 3] for arbitrary partitions, it can be proved in the same way, so we omit the proof.

Lemma 11. *Let $B \ge 1$ and μ be given by* (11)*. Then for every step function*

$$g(x) = \sum_{i=1}^{n-1} \alpha_i (x - x_i)_+^0, \quad x \in [a,b],$$

with $\alpha_i \ge 0$, there exists a polygonal line

$$p(x) = \sum_{i=1}^{n-1} \frac{\beta_i}{(x_{i+1} - x_i)} (x - x_i)_+,$$

satisfying

$$|\beta_i| < \frac{\alpha_i}{B}, \quad i = 1, \dots, n-1, \tag{52}$$

and such that

$$|g(x) - p(x)| < 8\mu B A, \quad x \in [a,b], \tag{53}$$

where

$$A := \max_{i=1,\dots,n-1} \alpha_i.$$

Lemma 12. *Let $x_0 < x_1 < \dots < x_n$ be a given partition, $\delta_1, \dots, \delta_{n-1}$ a sequence of non-negative numbers, satisfying*

$$\delta_i \le (x_{i+1} - x_{i-1})^{-2}\Omega, \quad 1 \le i \le n-1,$$

where Ω is some positive constant. Then there exists a cubic piecewise polynomial q with the knots $x_1, \dots, x_{n-1}$, such that $q \in C^{(1)}_{[a,b]}$,

$$q''(x_i+) - q''(x_i-) = -\delta_i, \quad i = 1, \dots, n-1, \tag{54}$$

$$q \in \Delta^3_{(x_{i-1}, x_i)}, \quad i = 1, \dots, n, \tag{55}$$

$$\|q\|_{[a,b]} \le c(m, \mu)\Omega, \tag{56}$$

where $c(m,\mu)$ is a constant depending on m, the scale of the partition, given in (4) *and μ, defined by* (11).

Proof. For $1 \le i \le n-1$, we construct an auxiliary function $q_i(\delta, x)$, $\delta, x \in \mathbb{R}$, as follows. Put

$$\delta_i^+ := \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} \delta_i, \quad \delta_i^- := \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \delta_i,$$

and $\delta_i^* := \min\{\delta_i^+, \delta_i^-\}$. Clearly $\delta_i^+(x_i - x_{i-1}) = \delta_i^-(x_{i+1} - x_i)$, and $\delta_i^+ + \delta_i^- = \delta_i$. Define

$$g_i(\delta, x) := \begin{cases} 0, & x \notin (x_{i-1}, x_{i+1}), \\ \frac{\delta_i^+ + \delta}{x_i - x_{i-1}}(x - x_{i-1}), & x \in (x_{i-1}, x_i], \\ \frac{\delta_i^- - \delta}{x_{i+1} - x_i}(x - x_{i+1}), & x \in (x_i, x_{i+1}), \end{cases}$$

and let

$$q_i(\delta, x) := \int_{x_0}^{x} \int_{x_0}^{t} g_i(\delta, \tau)\, d\tau\, dt.$$

It follows by straightforward calculations that

$$q_i(\delta, x) = \begin{cases} 0, & x \in [a, x_{i-1}], \\ \frac{\delta_i^+ + \delta}{6(x_i - x_{i-1})}(x - x_{i-1})^3, & x \in (x_{i-1}, x_i], \\ \frac{\delta_i^- - \delta}{6(x_{i+1} - x_i)}(x - x_{i+1})^3 + w_i(\delta)(x - x_i) + h_i, & x \in (x_i, x_{i+1}), \\ w_i(\delta)(x - x_i) + h_i, & x \in [x_{i+1}, b], \end{cases}$$

where

$$w_i(\delta) = \frac{\delta}{2}(x_{i+1} - x_{i-1}),$$

and

$$h_i = \frac{\delta_i^+ + \delta}{6}(x_i - x_{i-1})^2 + \frac{\delta_i^- - \delta}{6}(x_i - x_{i+1})^2.$$

Clearly, $q_i(\delta,(\cdot)) \in C^{(1)}_{[a,b]}$, and

$$q_i''(\delta, x_i+) - q_i''(\delta, x_i-) = -\delta_i, \tag{57}$$

moreover, x_i is the only point of discontinuity of the second derivative of $q_i(\delta,(\cdot))$. Finally, if $|\delta| \le \delta_i^*$, then

$$q_i^{(3)}(\delta, x) \ge 0, \quad x \in [a,b], \tag{58}$$

thus we take $|\delta| \le \delta_i^*$.

Let

$$(x)_+ = \begin{cases} x, & x > 0, \\ 0, & x \le 0, \end{cases} \quad \text{and} \quad (x)_+^0 := \begin{cases} 1, & x > 0, \\ 0, & x \le 0, \end{cases}$$

and denote

$$q_i(\delta, x) =: r_i(\delta, x) + w_i(\delta)(x - x_i)_+ + h_i(x - x_i)_+^0, \quad x \in [a,b].$$

Then

$$r_i(\delta, x) = 0, \quad x \notin (x_{i-1}, x_{i+1}), \tag{59}$$

and for $x \in (x_{i-1}, x_i)$

$$\begin{aligned} |r_i(\delta, x)| = \left| \frac{\delta_i^+ + \delta}{6(x_i - x_{i-1})}(x - x_{i-1})^3 \right| &\le \frac{\delta_i}{6}(x_i - x_{i-1})^2 \\ &\le \frac{1}{6}\left(\frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}\right)^2 \Omega \\ &\le \frac{\Omega}{6}. \end{aligned}$$

The same inequality holds for $x \in [x_i, x_{i+1})$. Hence,

$$|r_i(\delta, x)| \le \frac{\Omega}{6}, \quad x \in (x_{i-1}, x_{i+1}). \tag{60}$$

Also

$$\begin{aligned} 0 \le h_i &\le \frac{\delta_i}{6}(x_i - x_{i-1})^2 + \frac{\delta_i}{6}(x_i - x_{i+1})^2 \\ &\le \frac{1}{6}\delta_i (x_{i+1} - x_{i-1})^2. \end{aligned} \tag{61}$$

Put $B := \frac{4m^2}{3}$, where m is the scale of the partition $x_0, \dots, x_n$, see (4). We will show that for β_i, satisfying

$$|\beta_i| < \frac{\frac{1}{6}\delta_i(x_{i+1} - x_{i-1})^2}{B}, \tag{62}$$

we may choose δ in a way that guarantees

$$-w_i(\delta) = \frac{\beta_i}{x_{i+1} - x_i},$$

i.e.,

$$\delta = \frac{-2\beta_i}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})}, \tag{63}$$

and such that

$$|\delta| \le \delta_i^*. \tag{64}$$

Indeed,

$$\frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \ge \frac{1}{2m}, \quad \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}} \ge \frac{1}{2m},$$

so that

$$\delta_i^* \ge \frac{\delta_i}{2m}.$$

Hence, (62) and (63) yield,

$$\begin{aligned} |\delta| &\le \frac{2|\beta_i|}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})} \le \frac{4m|\beta_i|}{(x_{i+1} - x_{i-1})^2} \\ &\le \frac{4m\delta_i(x_{i+1} - x_{i-1})^2}{6B(x_{i+1} - x_{i-1})^2} = \frac{\delta_i}{2m} \le \delta_i^*. \end{aligned}$$

For our purposes we apply Lemma 11 with $\alpha_i := h_i$, $i = 1, \dots, n-1$, and $B := \frac{4m^2}{3}$. Then by (52) and (61) we clearly have (62). Thus we take $\tilde{\delta}_i$ to satisfy

$$-w_i(\tilde{\delta}_i) = \frac{\beta_i}{x_{i+1} - x_i},$$

so that in view of (64), we have

$$|\tilde{\delta}_i| \le \delta_i^*.$$

Also, by (61) we see that $A \le \frac{\Omega}{6}$.

Define

$$q(x) := \sum_{i=1}^{n-1} q_i(\tilde{\delta}_i, x), \quad x \in [a, b].$$

Then clearly $q \in C^{(1)}_{[a,b]}$, (55) follows from (58), and (57) together with the observation that x_i is the only discontinuity of q_i'', yields (54). Finally, by virtue of (59), (60) and (53),

$$\begin{aligned}
\|q\|_{[x_0,x_n]} &= \left\|\sum_{i=1}^{n-1} q_i(\tilde\delta_i,\cdot)\right\|_{[a,b]} \\
&\le \left\|\sum_{i=1}^{n-1} r_i(\tilde\delta_i,\cdot)\right\|_{[a,b]} + \left\|\sum_{i=1}^{n-1} h_i(\cdot - x_i)_+^0 - \frac{\beta_i}{x_{i+1}-x_i}(\cdot - x_i)_+\right\|_{[a,b]} \\
&\le \frac{\Omega}{3} + c_1(m,\mu)(\frac{4m^2}{3}+1)\frac{\Omega}{6} \\
&\le c(m,\mu)\Omega,
\end{aligned}$$

we establish (56). This completes the proof of Lemma 12. □

Proof of Theorem 5. Let

$$\delta_i := S''(x_i+) - S''(x_i-), \quad i = 1,\dots,n-1.$$

Since $S \in \Delta^3_{[x_0,x_n]}$, $\delta_i \ge 0$, $1 \le i \le n-1$. By Whitney's inequality there is a polynomial p_k of degree $\le k$, satisfying

$$\|S - p_k\|_{[x_{i-1},x_{i+1}]} \le c(k)\omega_{k+1}(S,(x_{i+1}-x_{i-1});[x_{i-1},x_{i+1}]).$$

star This in turn implies by Markov's inequality on $[x_i, x_{i+1}]$,

$$|p_k''(x_i) - S''(x_i+)| \le \frac{c(k)}{(x_{i+1}-x_i)^2} \max_{1\le j\le n-1} \omega_{k+1}(S,(x_{j+1}-x_{j-1});[x_{j-1},x_{j+1}]).$$

By the same argument

$$|p_k''(x_i) - S''(x_i-)| \le \frac{c(k)}{(x_i-x_{i-1})^2} \max_{1\le j\le n-1} \omega_{k+1}(S,(x_{j+1}-x_{j-1});[x_{j-1},x_{j+1}]).$$

Thus,

$$\delta_i \le c(m,k)(x_{i+1}-x_{i-1})^{-2} \max_{1\le j\le n-1} \omega_{k+1}(S,(x_{j+1}-x_{j-1});[x_{j-1},x_{j+1}]).$$

Denote

$$\Omega := c(m,k) \max_{1\le j\le n-1} \omega_{k+1}(S,(x_{j+1}-x_{j-1});[x_{j-1},x_{j+1}])$$

and apply Lemma 12 to obtain the piecewise polynomial q. Now set

$$S_1(x) := S(x) + q(x), \quad x \in [x_0,x_n].$$

Evidently, S_1 is a piecewise polynomial of degree $\leq k$ with the knots $x_0, \dots, x_n$, satisfying

$$S_1''(x_i-) = S_1''(x_i+), \quad i = 1, \dots, n-1, \tag{65}$$

so that $S_1 \in C^{(2)}_{[a,b]}$. Also, since $S \in \Delta^3_{[x_0,x_n]}$, we conclude by (55) that S_1'' is non-decreasing on each interval (x_{i-1}, x_i), $1 \leq i \leq n$. Thus combining with (65), we have that S_1'' is non-decreasing on the whole $[a,b]$, so that $S_1 \in \Delta^3_{[a,b]}$. Finally, (10) follows from (56). This completes the proof. □

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