Coconvex approximation in the uniform norm: the final frontier *

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1 Introduction

Our main interest in this paper is approximation of a continuous function, on a finite interval, which changes convexity finitely many times by algebraic polynomials which are coconvex with it. This topic has received much attention in recent years, and the purpose of this paper is to give final answers to open questions concerning the validity of Jackson type estimates involving the weighted Ditzian-Totik (D-T) moduli of smoothness.

Let $\mathbb{C}[a,b]$ denote the space of continuous functions f on $[a,b]$, equipped with the uniform norm $\|f\|_{[a,b]} := \max_{x\in[a,b]} |f(x)|$. When dealing with the generic interval $[-1,1]$, we omit the special reference to the interval, namely, we write $\|f\| := \|f\|_{[-1,1]}$.

To make the notion of coconvexity more precise we first denote by $\mathbb{Y}_s$, $s \ge 1$, the set of all collections $Y_s := \{y_i\}_{i=1}^s$, such that $y_{s+1} := -1 < y_s < \ldots < y_1 < 1 =: y_0$, and $Y_0 := \{\emptyset\}$. Let $\Delta^2(Y_s)$ denote the collection of all functions $f \in \mathbb{C}[-1,1]$ that change convexity at the points of the set Y_s, and are convex in $[y_1, 1]$. In particular, $\Delta^2 := \Delta^2(Y_0)$ is the set of all convex functions $f \in \mathbb{C}[-1,1]$. Also with $\Pi(x) := \prod_{i=1}^s (x - y_i)$, if $f \in \mathbb{C}^2(-1,1) \cap \mathbb{C}[-1,1]$, then $f \in \Delta^2(Y_s)$ if and only if

$$f''(x)\Pi(x) \ge 0, \quad x \in (-1,1). \tag{1.1}$$

In fact, in this paper we will be able to use (1.1), as the results for functions that are not in $\mathbb{C}^2(-1,1)$, are already known.

We say that functions f and g are coconvex if both of them belong to the same class $\Delta^2(Y_s)$ (note that it is possible for a function to belong to more than one class $\Delta^2(Y_s)$, for example, $f \equiv 0$ is in $\Delta^2(Y_s)$ for *all* sets Y_s).

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Let $\mathbb{P}_n$ be the space of all algebraic polynomials of degree $\le n-1$, and denote by

$$E_n(f) := \inf_{p_n \in \mathbb{P}_n} \|f - p_n\| \quad \text{and} \quad E_n^{(2)}(f, Y_s) := \inf_{p_n \in \mathbb{P}_n \cap \Delta^2(Y_s)} \|f - p_n\|$$

the degrees of best uniform polynomial approximation and best uniform coconvex polynomial approximation of f, respectively. In particular,

$$E_n^{(2)}(f) := E_n^{(2)}(f, Y_0) = \inf_{p_n \in \mathbb{P}_n \cap \Delta^2} \|f - p_n\|$$

is the degree of best uniform convex polynomial approximation of $f \in \Delta^2$.

It is now known that the following equivalence relation is valid.

Theorem 1.1 *For $f \in \Delta^2$ and any $\alpha > 0$, we have*

$$E_n(f) = O\left(n^{-\alpha}\right), \quad n \to \infty \quad \Longleftrightarrow \quad E_n^{(2)}(f) = O\left(n^{-\alpha}\right), \quad n \to \infty.$$

Despite the simplicity of its statement Theorem 1.1 remained unresolved for quite some time, and while its particular cases have been known from as early as 1986, in its final form it appeared only very recently in [6] where the case for $\alpha = 4$ (which surprisingly turned out to be the most evasive case of all) has been proved (see [6] for more details).

One of the consequences of the results of this paper is an analog of Theorem 1.1 for coconvex polynomial approximation.

Theorem 1.2 *For any $s \ge 0$, $Y_s \in \mathbb{Y}_s$, $f \in \Delta^2(Y_s)$, and any $\alpha > 0$, we have*

$$E_n(f) = O\left(n^{-\alpha}\right), \quad n \to \infty \quad \Longleftrightarrow \quad E_n^{(2)}(f, Y_s) = O\left(n^{-\alpha}\right), \quad n \to \infty.$$

Theorem 1.2 follows from the Jackson type estimates involving the weighted D-T moduli of smoothness (see, e.g., [12]), which we now introduce together with some related function spaces.

Throughout this paper we will have parameters k, l, m, r, s all of which will denote nonnegative integers, with $k + r > 0$.

With $\varphi(x) := \sqrt{1 - x^2}$, we denote by $\mathbb{B}^r$, $r \ge 1$, the space of all functions $f \in \mathbb{C}[-1, 1]$ with locally absolutely continuous $(r-1)$st derivative in $(-1, 1)$ such that $\left\|\varphi^r f^{(r)}\right\| < \infty$, where for $g \in \mathbb{L}_\infty[-1, 1]$, we write

$$\|g\| = \operatorname{ess\,sup}_{x \in [-1,1]} |g(x)|.$$

This obviously conforms with our previous notation of the norm for $g \in \mathbb{C}[-1, 1]$.

Let

$$\varphi_\delta(x) := \sqrt{(1 - x - \delta\varphi(x)/2)\,(1 + x - \delta\varphi(x)/2)} = \sqrt{(1 - \delta\varphi(x)/2)^2 - x^2}\,.$$

The weighted D-T modulus of smoothness of a function $f \in \mathbb{C}(-1, 1)$, is defined by

$$\omega_{k,r}^{\varphi}(f, t) := \sup_{0 < h \le t} \left\|\varphi_{kh}^r(\cdot) \Delta_{h\varphi(\cdot)}^k(f, \cdot)\right\|,$$

where

$$\Delta_h^k(f,x) := \begin{cases} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih), & \text{if } |x \pm kh/2| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

is the kth symmetric difference.

If $r = 0$ and $f \in \mathbb{C}[-1,1]$, then

$$\omega_k^\varphi(f,t) := \omega_{k,0}^\varphi(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi(\cdot)}^k(f,\cdot)\|,$$

is the (usual) D-T modulus. Also, if $\varphi(\cdot)$ in the above definition is replaced by 1, then we get the ordinary kth modulus of smoothness:

$$\omega_k(f,t) := \sup_{0<h\le t} \|\Delta_h^k(f,\cdot)\|.$$

Since $\varphi_\delta(x) \le \varphi(x) \le 1$, it is clear from the above definitions that, if $f \in \mathbb{C}[-1,1]$, then

$$\omega_{k,r}^\varphi(f,t) \le \omega_k^\varphi(f,t) \le \omega_k(f,t). \tag{1.2}$$

Also, for $f \in \mathbb{C}(-1,1)$ and $k \ge 1$ we have

$$\omega_{k+1,r}^\varphi(f,t) \le c\omega_{k,r}^\varphi(f,t), \tag{1.3}$$

and

$$\omega_{k,r}^\varphi(f,t) \le c\|\varphi^r f\|. \tag{1.4}$$

Here and in the sequel, we write c for positive constants which may depend only on k, r, and s, while the constants C may depend on other parameters.

Finally, we need $\omega_k(f,t,[a,b])$, the ordinary kth modulus of smoothness on $[a,b] \subseteq [-1,1]$, *i.e.*,

$$\omega_k(f,t,[a,b]) := \sup_{0<h\le t} \|\Delta_h^k(f,\cdot)\|_{[a+kh/2,b-kh/2]}.$$

The modulus $\omega_{k,r}^\varphi$ has many of the properties of the usual and D-T moduli of smoothness. In particular, for any $k \ge 1$, $r \ge 0$, and $f \in \mathbb{C}(-1,1)$,

$$\omega_{k,r}^\varphi(f,\lambda t) \le c(\lambda+1)^k \omega_{k,r}^\varphi(f,t), \quad \lambda > 0.$$

This, in turn, implies that if a function f is not a polynomial of degree $\le k-1$, then, for some $C = C(f) > 0$,

$$\omega_{k,r}^\varphi(f,t) \ge Ct^k, \quad \text{for all} \quad 0 < t \le 1. \tag{1.5}$$

For arbitrary $f \in \mathbb{C}(-1,1)$, the function $\omega_{k,r}^\varphi(f,t)$ may be unbounded. However, it was shown in [8,12] that a necessary and sufficient condition for $\omega_{k,r}^\varphi(f,t)$ to be bounded for all $t > 0$ is that $\varphi^r f \in \mathbb{L}_\infty[-1,1]$. Moreover, if $r \ge 1$, then $\omega_{k,r}^\varphi(f,t) \to 0$, as $t \to 0$, if and only if $\lim_{x\to\pm1} \varphi^r(x) f(x) = 0$. Therefore, we denote $\mathbb{C}_\varphi^0 := \mathbb{C}[-1,1]$ and, for $r \ge 1$,

$$\mathbb{C}_\varphi^r := \{f \in \mathbb{C}^r(-1,1) \cap \mathbb{C}[-1,1] \mid \lim_{x\to\pm1} \varphi^r(x) f^{(r)}(x) = 0\}.$$

Clearly

$$\mathbb{C}_\varphi^r \subset \mathbb{B}^r, \tag{1.6}$$

while if $f \in \mathbb{B}^r$, then $f \in \mathbb{C}_\varphi^l$ for all $0 \le l < r$, and

$$\omega_{r-l,l}^\varphi(f^{(l)}, t) \le ct^{r-l} \left\| \varphi^r f^{(r)} \right\|, \qquad t > 0. \tag{1.7}$$

Note that for $f \in \mathbb{C}_\varphi^r$, and any $0 \le l \le r$ and $k \ge 1$, the following inequalities hold (see [12]).

$$\omega_{k+r-l,l}^\varphi(f^{(l)}, t) \le c\, t^{r-l} \omega_{k,r}^\varphi(f^{(r)}, t), \qquad t > 0, \tag{1.8}$$

in particular, if $l = 0$, then

$$\omega_{k+r}^\varphi(f, t) \le ct^r \omega_{k,r}^\varphi(f^{(r)}, t), \qquad t > 0. \tag{1.9}$$

Finally for $0 \le l < r/2$,

$$\mathbb{B}^r \subset \mathbb{C}^l[-1,1]. \tag{1.10}$$

In this paper, we are interested in determining for which values of the parameters k, r, and s, the statement

> *if* $f \in \mathbb{C}_\varphi^r \cap \Delta^2(Y_s)$, *then*
>
> $$E_n^{(2)}(f, Y_s) \le Cn^{-r} \omega_{k,r}^\varphi\big(f^{(r)}, 1/n\big), \quad n \ge N, \tag{1.11}$$
>
> *where* $C = const > 0$ *and* $N = const > 0$,

is valid, and for which it is invalid. Here and later in this paper, for clarity of exposition, we denote $\omega_{0,r}(f,t) := \|\varphi^r f\|$. Hence, in the case $k = 0$, (1.11) becomes:

$$E_n^{(2)}(f, Y_s) \le Cn^{-r} \|\varphi^r f^{(r)}\|, \quad n \ge N,$$

for $f \in \mathbb{B}^r \cap \Delta^2(Y_s)$.

The structure of our paper is as follows. In Section 2, our main results are stated. After collecting some auxiliary results in Section 3, we prove the positive results in Section 4 and the negative results in Section 5.

2 Main Results

In this section we state our main results devoted to investigating for which values of parameters k, r and s, the estimate (1.11) is valid, and for which it is invalid.

In particular, we wish to know the range of parameters k, r and s, for which (1.11) holds and, if it does hold, whether or not it is necessary for the constants C and N to essentially depend on Y_s (or even f), or whether it is true with C and N dependent only on the parameters k, r, and s.

For reader's convenience we describe our results using arrays in Figures 1 and 2 below. There, the symbols "$-$", "$\ominus$", "$\oplus$", and "$+$", have the following meaning.

- The symbol "$-$" in the position (k, r) means that, for each $Y_s \in \mathbb{Y}_s$ there is a function $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_s)$, such that

  $$\limsup_{n\to\infty} \frac{n^r E_n^{(2)}(f, Y_s)}{\omega^\varphi_{k,r}(f^{(r)}, 1/n)} = \infty.$$

  This means that the estimate (1.11) is invalid even if we allow both constants C and N to depend on the function f.

- The symbol "$\ominus$" in the position (k, r) means that (1.11) is valid with an absolute constant C, and N depending on the function f and, for any $Y_s \in \mathbb{Y}_s$, there are no constants C and N, both of which are independent of f, such that (1.11) holds for every function $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_s)$.

- The symbol "$\oplus$" in the position (k, r) means that (1.11) is valid with C depending only on k, r, and s, and N depending only k, r, and the set Y_s and, there are no constants C and N, both of which are independent of Y_s, such that (1.11) holds for all $Y_s \in \mathbb{Y}_s$ and $f \in \Delta^2(Y_s) \cap \mathbb{C}^r_\varphi$.

- The symbol "$+$" in the position (k, r) means that (1.11) is valid with C depending only on k, r, and s, and $N = k + r$.

Remark *Evidently, in the "cases $\ominus$", we also have (1.11) with $C(f)$ and $N = 1$. Taking into account the estimates by Pleshakov and Shatalina* [11] *for $n = k + r$, in the "cases $\oplus$", our results imply that (1.11) is valid also for $C = C(k, r, Y_s)$ and $N = k + r$.*

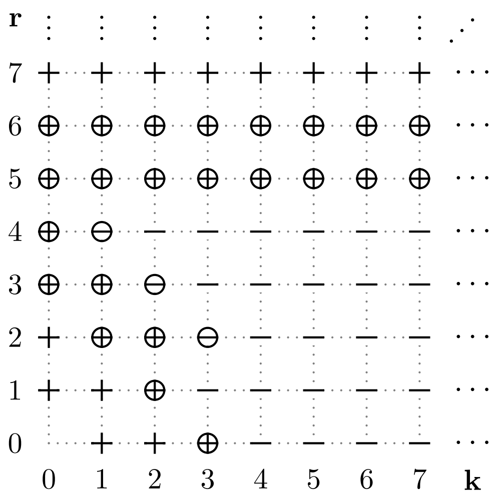

FIG. 1. $s = 1$

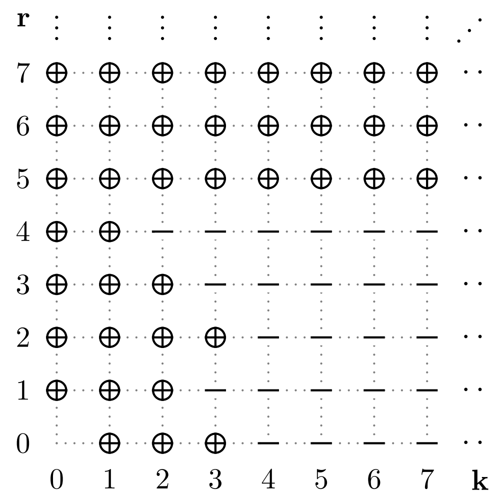

FIG. 2. $s \geq 2$

Remark *It follows from the inequalities* (1.3), (1.4), *and* (1.8) *that a positive result for a specific pair* (k_0, r_0) *implies positive results of the same type for all* (k, r) *with* $r_0 \leq r \leq k_0 + r_0 - k$. *Similarly, a negative result for* (k_0, r_0), $k_0 > 0$, *implies negative results of the same type for all* (k, r) *with* $k_0 + r_0 - k \leq r \leq r_0$, *and a negative result for* $(0, r_0)$, *implies negative results of the same type for all* (k, r) *with* $r_0 - k \leq r < r_0$.

This, in particular, implies the following:

(i) *If the symbol "$-$" appears in the position* (k_0, r_0), *then "$-$" should appear in all positions* (k, r) *with* $k_0 + r_0 - k \leq r \leq r_0$.

(ii) *If the symbol "$\ominus$" appears in the position* (k_0, r_0), *then, in all positions* (k, r) *with* $r_0 \leq r \leq k_0 + r_0 - k$, *we can have anything but "$-$", and, in all positions* (k, r) *with* $k_0 + r_0 - k \leq r \leq r_0$ *we can have only "$\ominus$" or "$-$".*

(iii) *If the symbol "$\oplus$" appears in the position* (k_0, r_0), *then in all positions* (k, r) *with* $r_0 \leq r \leq k_0 + r_0 - k$, *we can have only "$+$" or "$\oplus$" and, if* $k_0 = 0$, *then in all positions* (k, r) *with* $r_0 - k \leq r < r_0$, *we can have anything but "$+$". The latter leaves entries* $(k, 6)$, $k \geq 1$, *open. However, our counterexample can easily be modified to belong in the smaller space* $\mathbb{C}^6_\varphi \subsetneq \mathbb{B}^6$, *hence we have the negative result also for* $r = 6$ *and* $k \geq 1$.

(iv) *If the symbol "$+$" appears in the position* (k_0, r_0), *then "$+$" should appear in all positions* (k, r) *with* $r_0 \leq r \leq k_0 + r_0 - k$.

The results described by Figures 1 and 2 are obtained in or can be derived from our theorems and the papers listed in the table below.

Positive results: "+" in position (k,r)		
2002	$(2,0)$ for $s=1$; $\therefore \{(k,r) \mid k+r \le 2\}$ for $s=1$	Leviatan and Shevchuk [9]
—	$\{(k,7) \mid k \ge 0\}$ for $s=1$; $\therefore \{(k,r) \mid k \ge 0, r \ge 7\}$ for $s=1$	Theorem 2.3 ($k=0$) and Theorem 2.11 ($k \ge 1$)
Positive results: "⊕" in position (k,r)		
1999	$(3,0)$ for $s \ge 1$; $\therefore \{(k,r) \mid k+r \le 3\}$ for $s \ge 2$, and $\{(k,r) \mid k+r=3\}$ for $s=1$	Kopotun, Leviatan and Shevchuk [5]
—	$\{(k,5) \mid k \ge 0\}$ for $s \ge 1$; $\therefore \{(k,r) \mid k \ge 0, r \ge 5\}$ for $s \ge 2$, and $\{(k,r) \mid k \ge 0,\ 5 \le r \le 6\}$ for $s=1$	Theorem 2.1 ($k=0$) and Theorem 2.5 ($k \ge 1$)
—	$(3,2)$ for $s \ge 2$; $\therefore \{(k,r) \mid 2 \le r \le 5-k\}$ for $s \ge 2$	Theorem 2.7
—	$(2,2)$ for $s=1$; $\therefore \{(k,r) \mid 3 \le k+r \le 4, r \ge 2\}$ for $s=1$	Theorem 2.8
Positive results: "⊖" in position (k,r)		
—	$(3,2)$ for $s=1$; $\therefore \{(k,r) \mid k=5-r, 2 \le r \le 4\}$ for $s=1$	Theorem 2.8
Negative results: "−" in position (k,r)		
1993	$(3,1)$ for $s \ge 1$; $\therefore \{(k,r) \mid 4-k \le r \le 1\}$ for $s \ge 1$	Zhou[13]
2003	$(4,2)$ for $s \ge 1$; $\therefore \{(k,2) \mid 6-k \le r \le 2\}$ for $s \ge 1$	Gilewicz and Yushchenko[3]
—	$(2,4)$ for $s \ge 1$; $\therefore \{(k,r) \mid 6-k \le r \le 4\}$ for $s \ge 1$	Theorem 2.13
Negative results: "⊖" in position (k,r)		
—	$(1,4)$ for $s=1$; $\therefore (2,3)$ and $(3,2)$ for $s=1$	Theorem 2.15
Negative results: "+" CANNOT be in position (k,r)		
2000	$(2,1)$ for $s \ge 1$; $\therefore \{(k,r) \mid 3-k \le r \le 1\}$ for $s \ge 1$	Pleshakov and Shatalina [11]
2002	$\{(0,r) \mid 1 \le r \le 3\}$ for $s \ge 2$; $\therefore \{(k,r) \mid k \ge 0,\ 1-k \le r \le 2\}$ for $s \ge 2$	Leviatan and Shevchuk [9]
—	$\{(0,r) \mid r \ge 1\}$ for $s \ge 2$; $\therefore \{(k,r) \mid k+r \ge 1\}$ for $s \ge 2$	Theorem 2.2
—	$\{(0,r) \mid 3 \le r \le 6\}$ for $s=1$; $\therefore \{(k,r) \mid 3-k \le r \le 6\}$ for $s=1$	Theorem 2.4

We now give precise statements of the theorems yielding results summarized in the above arrays.

We begin with estimates for functions $f \in \mathbb{B}^r \cap \Delta^2(Y_s)$. Recall that we denote by c positive constants that may depend only on all or some of the parameters k, r, and s. We first have

Theorem 2.1 *Let $r \ge 1$, $s \ge 1$, and $Y_s \in \mathbb{Y}_s$, be given. If $f \in \mathbb{B}^r \cap \Delta^2(Y_s)$, then*

$$E_n^{(2)}(f, Y_s) \le c n^{-r} \|\varphi^r f^{(r)}\|, \quad n \ge N(r, Y_s), \tag{2.1}$$

where $N(r, Y_s)$ is a constant which may depend only on r and Y_s.

For $r \le 3$, Theorem 2.1 follows from [5], thus we only have to prove it for $r \ge 4$.

Remark *It is interesting to note that, in the case $s = 0$, the following result holds (see* [4, 7, 10]*):*

> *For any $f \in \Delta^2$ and $r \ne 4$,*
>
> $$E_n^{(2)}(f) \le c n^{-r} \|\varphi^r f^{(r)}\|, \quad n \ge r. \tag{2.2}$$
>
> *Moreover, the above statement is invalid for $r = 4$, however, it is valid, if the inequality $n \ge 4$ is replaced by $n \ge N(f)$.*

Unlike in the situation with (2.2)*, inequality* (2.1) *holds for all* $r \ge 1$*, that is, including the case* $r = 4$*.*

Next, we show that for $s \ge 2$, the constant $N(r, Y_s)$ in (2.1) cannot be replaced by a constant independent of Y_s. Namely,

Theorem 2.2 *Let* $s \ge 2$ *and* $r \ge 1$ *be given. Then for each* $n \ge 1$*, there are a collection* $Y_s \in \mathbb{Y}_s$ *and an* $f := f_n \in \mathbb{C}^r[-1, 1] \cap \Delta^2(Y_s)$*, such that*

$$E_n^{(2)}(f, Y_s) > cn \left(n^{-r} \|f^{(r)}\|\right). \tag{2.3}$$

For $s = 1$, we face a different situation. Depending on the value of r, it is sometimes possible to replace $N(r, Y_1)$ by $N(r)$, while for other r's it is impossible.

Theorem 2.3 *Suppose* $s = 1$*. If either* $r \le 2$ *or* $r \ge 7$*, then* (2.1) *is valid with* $N = r$*.*

For $r \le 2$, Theorem 2.3 follows from [9], thus we will prove it only for $r \ge 7$.

On the other hand, we show

Theorem 2.4 *Let* $s = 1$ *and* $3 \le r \le 6$*. Then for each* $n \ge 1$ *and every* $A > 0$*, there exist* $Y_1 := \{y_1\}$ *and a function* $f := f_{n,A} \in \mathbb{B}^r \cap \Delta^2(Y_1)$*, such that*

$$E_n^{(2)}(f, Y_1) > A\|\varphi^r f^{(r)}\|. \tag{2.4}$$

Moreover, for $r = 6$ *the function* $f_{n,A}$ *satisfying* (2.4)*, may be taken in* $\mathbb{C}^6_\varphi \cap \Delta^2(Y_1)$*.*

Note that the latter part of Theorem 2.4 provides the needed counterexample that implies that the symbol $\oplus$ in entries $(k, 6)$, $k \ge 1$, in Fig. 1, may not be replaced by $+$.

We now consider analogous estimates for $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_s)$. First, we have

Theorem 2.5 *Let* $k \ge 1$*,* $r = 5$*,* $s \ge 1$*, and* $Y_s \in \mathbb{Y}_s$*, be given. If* $f \in \mathbb{C}^5_\varphi \cap \Delta^2(Y_s)$*, then*

$$E_n(f, Y_s) \le cn^{-5}\omega^\varphi_{k,5}(f^{(5)}, 1/n), \quad n \ge N(k, Y_s), \tag{2.5}$$

where $N(k, Y_s) = const$*, depends on* k *and* Y_s*.*

An immediate consequence of Theorem 2.5 and (1.8) is

Corollary 2.6 *Let* $k \ge 1$*,* $r \ge 5$*,* $s \ge 1$*, and* $Y_s \in \mathbb{Y}_s$*, be given. If* $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_s)$*, then*

$$E_n(f, Y_s) \le cn^{-r}\omega^\varphi_{k,r}(f^{(r)}, 1/n), \quad n \ge N(k, r, Y_s),$$

where $N(k, r, Y_s) = const$*, depends on* k*,* r *and* Y_s*.*

We also prove the following.

Theorem 2.7 **(**$s \ge 2$**)** *Let* $s \ge 2$*, and let* $Y_s \in \mathbb{Y}_s$ *be given. If* $f \in \mathbb{C}^2_\varphi \cap \Delta^2(Y_s)$*, then,*

$$E_n^{(2)}(f, Y_s) \le cn^{-2}\omega^\varphi_{3,2}(f'', 1/n), \quad n \ge N(Y_s), \tag{2.6}$$

where $N(Y_s) = const$*, depends on* Y_s*.*

Theorem 2.8 **($s=1$)** *Let $Y_1 \in \mathbb{Y}_1$ be given. If $f \in \mathbb{C}^2_\varphi \cap \Delta^2(Y_1)$, then,*

$$E_n^{(2)}(f,Y_1) \le cn^{-2}\omega^\varphi_{3,2}(f'',1/n) + cn^{-4}\omega^\varphi_{2,2}(f'',1/n), \quad n \ge N(Y_1), \tag{2.7}$$

where $N(Y_1) = const$, depends on Y_1. Hence

$$E_n^{(2)}(f,Y_1) \le cn^{-2}\omega^\varphi_{2,2}(f'',1/n), \quad n \ge N(Y_1). \tag{2.8}$$

Moreover,

$$E_n^{(2)}(f,Y_1) \le cn^{-2}\omega^\varphi_{3,2}(f'',1/n) + cn^{-6}\|f''\|_{[-1/2,1/2]}, \quad n \ge N(Y_1), \tag{2.9}$$

and therefore

$$E_n^{(2)}(f,Y_1) \le cn^{-2}\omega^\varphi_{3,2}(f'',1/n), \quad n \ge N(f). \tag{2.10}$$

By virtue of (1.8), immediate consequences of Theorems 2.7 and 2.8 are the following results.

Corollary 2.9 **($s \ge 2$)** *Let $s \ge 2$, $2 \le r \le 4$, $1 \le k \le 5-r$, and $Y_s \in \mathbb{Y}_s$, be given. If $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_s)$, then*

$$E_n^{(2)}(f,Y_s) \le cn^{-r}\omega^\varphi_{k,r}(f^{(r)},1/n), \quad n \ge N(Y_s). \tag{2.11}$$

Corollary 2.10 **($s=1$)** *Let $s=1$, $2 \le r \le 4$, and $Y_1 \in \mathbb{Y}_1$, be given. If $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_1)$, then*

$$E_n^{(2)}(f,Y_1) \le cn^{-r}\omega^\varphi_{5-r,r}(f^{(r)},1/n), \quad n \ge N(f).$$

and, for $1 \le k \le 4-r$,

$$E_n^{(2)}(f,Y_1) \le cn^{-r}\omega^\varphi_{k,r}(f^{(r)},1/n), \quad n \ge N(Y_1).$$

Remark *In view of* (1.7)*, it readily follows from* Theorem 2.2 *that, in the case $s \ge 2$, the condition that N in the above statements, depends on Y_s, is essential and cannot be removed. Thus, there cannot be the symbol "+" in any positions (k,r) in* Figure 2*. This is in contrast to the case $s=1$ where in* Figure 1 *we do have positions with "+" symbol (see* Theorem 2.11 *below).*

Theorem 2.11 *Let $k \ge 1$ and $Y_1 \in \mathbb{Y}_1$ be given. If $f \in \mathbb{C}^7_\varphi \cap \Delta^2(Y_1)$, then*

$$E_n^{(2)}(f,Y_1) \le cn^{-7}\omega^\varphi_{k,7}(f^{(7)},1/n), \quad n \ge k+7. \tag{2.12}$$

Again, by virtue of (1.8), an immediate consequence of Theorem 2.11 is

Corollary 2.12 *Let $k \ge 1$, $r \ge 7$, and $Y_1 \in \mathbb{Y}_1$, be given. If $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_1)$, then*

$$E_n^{(2)}(f,Y_1) \le cn^{-r}\omega^\varphi_{k,r}(f^{(r)},1/n), \quad n \ge k+r.$$

At the same time, we have the following negative result.

Theorem 2.13 *Let $s \geq 1$. For each $Y_s \in \mathbf{Y}_s$ there is a function $f \in \mathbb{C}^4_\varphi \cap \Delta^2(Y_s)$, such that*

$$\limsup_{n\to\infty} \frac{n^4 E_n^{(2)}(f, Y_s)}{\omega_{2,4}^\varphi(f^{(4)}, 1/n)} = \infty. \tag{2.13}$$

Therefore, (1.8) implies

Corollary 2.14 *For every $0 \leq r \leq 4$, $k \geq 6 - r$, and for each $Y_s \in \mathbb{Y}_s$, there is a function $f \in \mathbb{C}^r_\varphi \cap \Delta^2(Y_s)$, such that*

$$\limsup_{n\to\infty} \frac{n^r E_n^{(2)}(f, Y_s)}{\omega_{k,r}^\varphi(f^{(r)}, 1/n)} = \infty.$$

Furthermore, in the special case $s = 1$ and $r = 4$, we have

Theorem 2.15 *For every $Y_1 \in \mathbb{Y}_1$ and every $n \geq 1$, there is a function $f := f_n \in \mathbb{C}^4_\varphi \cap \Delta^2(Y_1)$, such that*

$$E_n^{(2)}(f, Y_1) > C \frac{\ln n}{n^4} \omega_{1,4}^\varphi(f^{(4)}, 1),$$

where $C = C(Y_1)$.

This shows that the symbols "$\ominus$" in Figure 1 cannot be replaced by "$\oplus$".

3 Auxiliary Results

The following results were proved in [10] (see Corollaries 2.4 and 2.6 there).

Lemma 3.1 *Let $k \geq 1$ and let $f \in \mathbb{C}^2[a, a+h]$, $h > 0$, be convex. Then there exists a convex polynomial P of degree $\leq k+1$ satisfying $P(a) = f(a)$, $P(a+h) = f(a+h)$, $P'(a) \geq f'(a)$, and $P'(a+h) \leq f'(a+h)$, and such that*

$$\|f - P\|_{[a,a+h]} \leq ch^2 \omega_k(f'', h, [a, a+h]).$$

Lemma 3.2 *Let $k > 1$ and let $a < \beta < a + h$ be fixed and assume that $f \in \mathbb{C}^2[a, a+h]$ is such that*

$$f''(x)(x - \beta) \geq 0, \quad a \leq x \leq a + h.$$

If a polynomial $p \in \mathbb{P}_{k-1}$ satisfies

$$p(x)(x - \beta) \geq 0, \quad a \leq x \leq a + h,$$

then there exists a polynomial $P \in \mathbb{P}_{k+1}$ such that $P'' = p$,

$$P(a) = f(a), \quad P'(a) \leq f'(a), \quad P'(a+h) \leq f'(a+h),$$

and

$$\|f - P\|_{[a,a+h]} \leq \frac{3}{2} h^2 \|f'' - p\|_{[a,a+h]}.$$

Let $x_j := \cos(j\pi/n)$, $0 \le j \le n$, be the Chebyshev knots, and denote $I_j := [x_j, x_{j-1}]$, and $|I_j| := x_{j-1} - x_j$, $1 \le j \le n$. Denote by $\Sigma_{k,n}$ the collection of all continuous piecewise polynomials of degree $k-1$, on the Chebyshev partition $\{x_j\}_{j=0}^n$.

Given $Y_s \in \mathbb{Y}_s$, let

$$O_i := O_{i,n}(Y_s) := (x_{j+1}, x_{j-2}), \quad \text{if} \quad y_i \in [x_j, x_{j-1}),$$

where $x_{n+1} := -1$, $x_{-1} := 1$, and denote

$$O = O(n, Y_s) := \bigcup_{i=1}^{s} O_i.$$

Finally, we write $j \in H = H(n, Y_s)$, if $I_j \cap O = \emptyset$, and denote by $\Sigma_{k,n}(Y_s)$ the subset of $\Sigma_{k,n}$ consisting of those continuous piecewise polynomials S for which

$$p_j \equiv p_{j+1} \quad \text{whenever} \quad j, j+1 \notin H\,,$$

where $p_j := S_{I_j}$. In other words, piecewise polynomials from $\Sigma_{k,n}(Y_s)$ do not have any knots "too close" to the points $y_i \in Y_s$ of convexity change.

Theorem 3.3 ([9, Theorem 3]) *For every $k \ge 1$ and $s \ge 1$ there are constants c and $c_* = c_*(k, s)$, such that if $n \ge 1$, $Y_s \in \mathbb{Y}_s$, and $S \in \Sigma_{k,n}(Y_s) \cap \Delta^2(Y_s)$, then there is a polynomial $P_n \in \Delta^2(Y_s)$ of degree $\le c_* n$, satisfying*

$$\|S - P_n\| \le c\omega_k^\varphi(S, 1/n). \tag{3.1}$$

Let $[z_0, \dots, z_m; g]$ stand for the m-th divided difference of a function g at the knots $z_0, \dots, z_m$.

Lemma 3.4 *Let $f \in \mathbb{C}(-1,1)$, let $k \ge 1$ and $r \ge 0$ be such that $k + r \ge 3$, and let $1 \le \mu \le n - k$ be fixed. Then, for all $1 \le j \le \mu$,*

$$\begin{aligned} &|[x_\mu, \dots, x_{\mu+k-1}; f] - [x_j, x_{j+1}, \dots, x_{j+k-1}; f]| \\ &\quad \le \; cn^{2k+r-2} \left(\frac{1}{\min\{j, n-\mu\}} \right)^{k+r-2} \omega_{k,r}^\varphi(f, 1/n). \end{aligned} \tag{3.2}$$

Moreover, if $k + r \ge 5$, then for all ν and j such that $1 \le j \le \nu \le \mu$, we also have

$$\begin{aligned} &\epsilon \left([x_\nu, \dots, x_{\nu+k-2}; f] - [x_j, x_{j+1}, \dots, x_{j+k-2}; f] \right) \\ &\quad \le \; cn^{2k+r-4} \left(1 + \frac{n^2}{(n-\mu)^{k+r-2}} \right) \omega_{k,r}^\varphi(f, 1/n), \end{aligned} \tag{3.3}$$

where $\epsilon := \operatorname{sgn}\left([x_\mu, \dots, x_{\mu+k-1}; f]\right)$.

Note that the righthand sides of the both inequalities (3.2) and (3.3) are finite if $\|\varphi^r f\| < \infty$. Otherwise both are infinite, while the lefthand sides are always finite, hence, the lemma is trivially valid in this case.

Proof. For convenience, everywhere in the proof below, we write $[x_j, \dots, x_{j+l}]$ instead of $[x_j, \dots, x_{j+l}; f]$, and we put $\mathbf{w} := \omega^{\varphi}_{k,r}(f, 1/n)$. Also, note that, for all $1 \le i \le n-1$, $\varphi(x_i) \sim \min\{i, n-i\}/n$, and $|I_i| \sim \min\{i, n-i\}/n^2$, where, as usual, $\alpha_i \sim \beta_i$ means that $\frac{\alpha_i}{\beta_i}$ is bounded away from 0 and ∞.

The following inequality is contained in the proof of Lemma 3.4 in [6]:

$$|[x_j, x_{j+1}, \dots, x_{j+k}]| \le cn^k \left(\frac{n}{\min\{j, n-j\}}\right)^{k+r} \mathbf{w}, \tag{3.4}$$

for all $1 \le j \le n-k-1$.

Now, for any $m \ge 0$ and $1 \le j \le \sigma < n-m$, we have

$$[x_\sigma, \dots, x_{\sigma+m}] - [x_j, x_{j+1}, \dots, x_{j+m}] = \sum_{i=j}^{\sigma-1} (x_{i+m+1} - x_i)[x_i, x_{i+1}, \dots, x_{i+m+1}]. \tag{3.5}$$

This, with $m = k-1$, $\sigma = \mu$, together with the inequality (3.4) for $1 \le j < \mu \le n-k$, implies

$$\begin{aligned}
&|[x_\mu, \dots, x_{\mu+k-1}] - [x_j, x_{j+1}, \dots, x_{j+k-1}]| \\
&\quad = \left| \sum_{i=j}^{\mu-1} (x_{i+k} - x_i)[x_i, x_{i+1}, \dots, x_{i+k}] \right| \\
&\quad \le c \sum_{i=j}^{\mu-1} |I_i| n^k \left(\frac{n}{\min\{i, n-i\}}\right)^{k+r} \mathbf{w} \\
&\quad \le cn^{2k+r-2} \mathbf{w} \sum_{i=j}^{\mu-1} \left(\frac{1}{\min\{i, n-i\}}\right)^{k+r-1} \\
&\quad \le cn^{2k+r-2} \mathbf{w} \sum_{i=\min\{j, n-\mu\}}^{\infty} \frac{1}{i^{k+r-1}} \\
&\quad \le cn^{2k+r-2} \left(\frac{1}{\min\{j, n-\mu\}}\right)^{k+r-2} \mathbf{w},
\end{aligned}$$

where for the last inequality we used $k + r \ge 3$. Thus, (3.2) is proved.

Now, suppose that $k + r \ge 5$. Applying (3.5) with $m = k-2$ and $\sigma = \nu$ and (3.2), for all $1 \le j \le \nu \le \mu$, yields

$$\begin{aligned}
&\epsilon \left([x_\nu, \dots, x_{\nu+k-2}] - [x_j, x_{j+1}, \dots, x_{j+k-2}]\right) \\
&\quad = \epsilon \sum_{i=j}^{\nu-1} (x_{i+k-1} - x_i)[x_i, x_{i+1}, \dots, x_{i+k-1}] \\
&\quad = \epsilon \sum_{i=j}^{\nu-1} (x_i - x_{i+k-1}) \left([x_\mu, \dots, x_{\mu+k-1}] - [x_i, x_{i+1}, \dots, x_{i+k-1}]\right)
\end{aligned}$$

$$
\begin{aligned}
& -\epsilon[x_\mu, \ldots, x_{\mu+k-1}] \sum_{i=j}^{\nu-1} (x_i - x_{i+k-1}) \\
\leq\ & \sum_{i=j}^{\nu-1} (x_i - x_{i+k-1}) \left| [x_\mu, \ldots, x_{\mu+k-1}] - [x_i, x_{i+1}, \ldots, x_{i+k-1}] \right| \\
\leq\ & c n^{2k+r-2} \mathbf{w} \sum_{i=j}^{\nu-1} |I_i| \left(\frac{1}{\min\{i, n-\mu\}} \right)^{k+r-2} \\
\leq\ & c n^{2k+r-4} \mathbf{w} \sum_{i=j}^{\nu-1} \frac{\min\{i, n-i\}}{(\min\{i, n-\mu\})^{k+r-2}} \\
\leq\ & c n^{2k+r-4} \mathbf{w} \sum_{i=1}^{\mu-1} \frac{\min\{i, n-i\}}{(\min\{i, n-\mu\})^{k+r-2}} =: \mathcal{S}.
\end{aligned}
$$

Now, since $k+r \geq 5$, if $\mu \leq \lfloor \frac{n}{2} \rfloor$, then

$$
\mathcal{S} \leq c n^{2k+r-4} \mathbf{w} \sum_{i=1}^{\infty} \frac{1}{i^{k+r-3}} \leq c n^{2k+r-4} \mathbf{w},
$$

and if $\mu > \lfloor \frac{n}{2} \rfloor$, then

$$
\begin{aligned}
\mathcal{S} \leq\ & c n^{2k+r-4} \mathbf{w} \left(\sum_{i=1}^{n-\mu} \frac{1}{i^{k+r-3}} + \sum_{i=n-\mu+1}^{\mu-1} \frac{\min\{i, n-i\}}{(n-\mu)^{k+r-2}} \right) \\
\leq\ & c n^{2k+r-4} \mathbf{w} \left(1 + \frac{1}{(n-\mu)^{k+r-2}} \sum_{i=1}^{n} i \right) \\
\leq\ & c n^{2k+r-4} \mathbf{w} \left(1 + \frac{n^2}{(n-\mu)^{k+r-2}} \right).
\end{aligned}
$$

This completes the proof of the lemma. □

Remark *Taking into account the inequality*

$$
|[x_\mu, \ldots, x_{\mu+k-1}; f]| \leq c n^{k-1} \left(\frac{n}{\min\{\mu, n-\mu\}} \right)^{k+r-1} \omega_{k-1,r}^{\varphi}(f, 1/n),
$$

(see (3.4)*), it follows from* (3.2) *that for any $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$ such that $k+r \geq 3$, all $f \in \mathbb{C}(-1,1)$, and every $1 \leq j \leq n-k$, the following estimate holds*

$$
\begin{aligned}
|[x_j, x_{j+1}, \ldots, x_{j+k-1}; f]| \leq\ & c n^{2k+r-2} \left(\frac{1}{\min\{j, n-\mu\}} \right)^{k+r-2} \omega_{k,r}^{\varphi}(f, 1/n) \\
& + c n^{k-1} \left(\frac{n}{\min\{\mu, n-\mu\}} \right)^{k+r-1} \omega_{k-1,r}^{\varphi}(f, 1/n).
\end{aligned}
$$

In particular, taking $j=1$ and $\mu=\lfloor \frac{n}{2}\rfloor$, we obtain

$$
(3.6)\qquad |[x_1,x_2,\dots,x_k;f]| \;\le\; cn^{2k+r-2}\omega^{\varphi}_{k,r}(f,1/n)+cn^{k-1}\omega^{\varphi}_{k-1,r}(f,1/n).
$$

Also, the same sequence of inequalities that was used to prove (3.3)*, in fact implies,*

$$
\begin{aligned}
&|[x_\nu,\dots,x_{\nu+k-2};f]-[x_j,x_{j+1},\dots,x_{j+k-2};f]|\\
&\le\; c|[x_\mu,\dots,x_{\mu+k-1};f]|+cn^{2k+r-4}\left(1+\frac{n^2}{(n-\mu)^{k+r-2}}\right)\omega^{\varphi}_{k,r}(f,1/n),
\end{aligned}
$$

if $k+r\ge 5$, and in particular,

$$
(3.7)\qquad
\begin{aligned}
&|[x_\nu,\dots,x_{\nu+k-2};f]-[x_1,x_2,\dots,x_{k-1};f]|\\
&\le\; cn^{2k+r-4}\omega^{\varphi}_{k,r}(f,1/n)+cn^{k-1}\omega^{\varphi}_{k-1,r}(f,1/n).
\end{aligned}
$$

Since $x_{n-j}=-x_j$ for all $0\le j\le n$, we may apply Lemma 3.4 to the function $f_1(x):=f(-x)$, observing that $[x_i,\dots,x_\sigma;f_1]=(-1)^{\sigma-i}[x_{n-i},\dots,x_{n-\sigma};f]$, and $\omega^{\varphi}_{k,r}(f,\delta)=\omega^{\varphi}_{k,r}(f_1,\delta)$. Hence we get the following corollary (note that while it is valid for general k, r and j we only give its statement for $k=3$, $r=2$, $j=1$ and $j=n-1$ which is what we need in this paper).

Corollary 3.5 *Let $f\in\mathbb{C}^2_\varphi$. Then*
(a) *For any index $1\le\mu\le n-3$, if $\operatorname{sgn}\{[x_\mu,x_{\mu+1},x_{\mu+2};f'']\}=\epsilon$, then*

$$
(3.8)\qquad -\epsilon[x_1,x_2,x_3;f'']\le cn^6\omega^{\varphi}_{3,2}(f'',1/n).
$$

Moreover, if an index $1\le\nu\le\mu$ is such that $\operatorname{sgn}\{[x_\nu,x_{\nu+1};f'']\}=\epsilon$, then we also have

$$
(3.9)\qquad -\epsilon[x_1,x_2;f'']\le cn^4\left(1+\frac{n^2}{(n-\mu)^3}\right)\omega^{\varphi}_{3,2}(f'',1/n).
$$

(b) *For any index $1\le\mu\le n-3$, if $\operatorname{sgn}\{[x_{n-\mu},x_{n-\mu-1},x_{n-\mu-2};f'']\}=\epsilon$, then*

$$
(3.10)\qquad -\epsilon[x_{n-1},x_{n-2},x_{n-3};f'']\le cn^6\omega^{\varphi}_{3,2}(f'',1/n).
$$

Moreover, if an index $1\le\nu\le\mu$ is such that $\operatorname{sgn}\{[x_{n-\nu},x_{n-\nu-1};f'']\}=-\epsilon$, then we also have

$$
(3.11)\qquad \epsilon[x_{n-1},x_{n-2};f'']\le cn^4\left(1+\frac{n^2}{(n-\mu)^3}\right)\omega^{\varphi}_{3,2}(f'',1/n)\,.
$$

We note that, for a set $Y_s\in\mathbb{Y}_s$, $s\ge 1$, if

$$
n\ge 4\left(\min_{1\le j\le s+1}\{y_{j-1}-y_j\}\right)^{-1}=:\mathcal{N}(Y_s),
$$

then there is at least one knot x_i between y_{j-1} and y_j, for all $1\le j\le s+1$.

The following are consequences of Corollary 3.5 for $f\in\Delta(Y_s)$, $s\ge 2$.

Corollary 3.6 **($s \geq 3$)** *Let $s \geq 3$, $f \in \mathbb{C}^2_\varphi \cap \Delta(Y_s)$, and*

$$n \geq \max\left\{\mathcal{N}(Y_s), (\min\{\varphi(y_i) \mid 1 \leq i \leq s\})^{-3}\right\}.$$

Then,

$$\max\{|[x_1,x_2,x_3;f'']|, |[x_{n-1},x_{n-2},x_{n-3};f'']|\} \leq cn^6 \omega^\varphi_{3,2}(f'',1/n), \tag{3.12}$$

and

$$\max\{|[x_1,x_2;f'']|, |[x_{n-1},x_{n-2};f'']|\} \leq cn^4 \omega^\varphi_{3,2}(f'',1/n). \tag{3.13}$$

Corollary 3.7 **($s = 2$)** *Let $f \in \mathbb{C}^2_\varphi \cap \Delta(Y_2)$, and*

$$n \geq \max\left\{\mathcal{N}(Y_2), (\min\{\varphi(y_1),\varphi(y_2)\})^{-3}\right\}.$$

Then,

$$\max\{-[x_1,x_2,x_3;f''], -[x_{n-1},x_{n-2},x_{n-3};f'']\} \leq cn^6 \omega^\varphi_{3,2}(f'',1/n), \tag{3.14}$$

and

$$\max\{-[x_1,x_2;f''], [x_{n-1},x_{n-2};f'']\} \leq cn^4 \omega^\varphi_{3,2}(f'',1/n). \tag{3.15}$$

Proof of Corollaries 3.6 and 3.7. For the sake of convenience denote $\mathcal{A} := \mathcal{A}(Y_s) := \min\{\varphi(y_i) \mid 1 \leq i \leq s\}$. Let $s \geq 2$ and $f \in \mathbb{C}^2_\varphi \cap \Delta(Y_s)$, be given. Observe that if an index i is such that $y_s \leq x_i \leq y_1$, then

$$\min\{i, n-i\} \geq n\sin(i\pi/n)/4 = n\varphi(x_i)/4 \geq n\min\{\varphi(y_s),\varphi(y_1)\}/4 \geq \mathcal{A}n/4.$$

Now, let indices μ_1, ν_1, ν_2, and μ_2 (if $s \geq 3$) be such that $f''(x_{\mu_1+1}) = \min\{f''(x_i) \mid y_2 \leq x_i \leq y_1\}$, $x_{\nu_1+1} \leq y_1 < x_{\nu_1}$, $x_{\nu_2+1} \leq y_2 < x_{\nu_2}$, and $f''(x_{\mu_2+1}) = \max\{f''(x_i) \mid y_3 \leq x_i \leq y_2\}$.

Then, using $f''(x)(x-y_1)(x-y_2) \geq 0$ for all $x \geq y_3$ (or $x > -1$ if $s = 2$), we conclude that the following inequalities hold

$$\begin{aligned}
&1 \leq \nu_1 \leq \mu_1 < \nu_2 \leq n-2, \quad \nu_2 \leq \mu_2 \leq n-3 \text{ (if } s \geq 3),\\
&[x_{\mu_1}, x_{\mu_1+1}, x_{\mu_1+2}; f''] \geq 0, \quad [x_{\nu_1}, x_{\nu_1+1}; f''] \geq 0,\\
&[x_{\mu_2}, x_{\mu_2+1}, x_{\mu_2+2}; f''] \leq 0, \quad [x_{\nu_2}, x_{\nu_2+1}; f''] \leq 0.
\end{aligned}$$

Now by Corollary 3.5(a) with $\mu = \mu_1$ and $\nu = \nu_1$, taking into account that $n - \mu_1 + 1 \geq \mathcal{A}n/4$, it follows that

$$-[x_1,x_2,x_3;f''] \leq cn^6 \omega^\varphi_{3,2}(f'',1/n), \tag{3.16}$$

and

$$-[x_1,x_2;f''] \leq cn^4\left(1+\frac{n^2}{(n-\mu_1)^3}\right)\omega^\varphi_{3,2}(f'',1/n) \leq cn^4 \omega^\varphi_{3,2}(f'',1/n). \tag{3.17}$$

Further, if $s \geq 3$, then Corollary 3.5(a) with $\mu = \mu_2$ and $\nu = \nu_2$ and the observation that $n - \mu_2 \geq \mathcal{A}n/4$ imply

$$[x_1,x_2,x_3;f''] \leq cn^6 \omega^\varphi_{3,2}(f'',1/n), \tag{3.18}$$

and

$$(3.19)\qquad [x_1,x_2;f''] \le cn^4\left(1+\frac{n^2}{(n-\mu_2)^3}\right)\omega^{\varphi}_{3,2}(f'',1/n) \le cn^4\omega^{\varphi}_{3,2}(f'',1/n).$$

This in turn implies that

$$|[x_1,x_2,x_3;f'']| \le cn^6\omega^{\varphi}_{3,2}(f'',1/n) \quad \text{and} \quad |[x_1,x_2;f'']| \le cn^4\omega^{\varphi}_{3,2}(f'',1/n),$$

and the analogous inequalities for $|[x_{n-1},x_{n-2},x_{n-3};f'']|$ and $|[x_{n-1},x_{n-2};f'']|$, follow by symmetry. This completes the proof of Corollary 3.6.

In order to complete the proof of Corollary 3.7 it suffices to use Corollary 3.5(b) with $\mu = n-\mu_1-2$ and $\nu = n-\nu_2-1$, and the estimate $\mu_1+1 \ge \mathcal{A}n/4$, and to combine the resulting inequalities with (3.16) and (3.17). □

In the case $s=1$, let $f \in \mathbb{C}^2_\varphi \cap \Delta(Y_1)$. Then, just as in the proof above, for the index ν_1 such that $x_{\nu_1+1} \le y_1 < x_{\nu_1}$, we have $[x_{\nu_1},x_{\nu_1+1};f''] \ge 0$. Hence, by virtue of (3.6) and (3.7) with $k=3$, $r=2$, and $\nu=\nu_1$, we obtain the following result (the estimates for $[x_{n-1},x_{n-2},x_{n-3};f'']$ and $[x_{n-1},x_{n-2};f'']$ follow by symmetry), that will be used in the proof of (2.7) and (2.8).

Corollary 3.8 ($s=1$) *Let $f \in \mathbb{C}^2_\varphi \cap \Delta(Y_1)$, and $n \ge 7\,(\varphi(y_1)\})^{-3}$. Then,*

$$(3.20)\qquad \begin{aligned} &\max\{|[x_1,x_2,x_3;f'']|,|[x_{n-1},x_{n-2},x_{n-3};f'']|\} \\ &\quad\le\ cn^6\omega^{\varphi}_{3,2}(f'',1/n)+cn^2\omega^{\varphi}_{2,2}(f'',1/n) \end{aligned}$$

and

$$(3.21)\qquad \begin{aligned} &\max\{-[x_1,x_2;f''],-[x_{n-1},x_{n-2};f'']\} \\ &\quad\le\ cn^4\omega^{\varphi}_{3,2}(f'',1/n)+cn^2\omega^{\varphi}_{2,2}(f'',1/n)\,. \end{aligned}$$

The following lemma is an immediate consequence of [6, Corollary 3.5] and will be used in the proof of estimates (2.9) and (2.10).

Lemma 3.9 *Let $n \ge 9$, $m=1$ or $m=2$, and $f \in \mathbb{C}^2_\varphi$. Then,*

$$(3.22)\qquad \begin{aligned} &\max\{|[x_1,x_2,\dots,x_{m+1};f'']|,|[x_{n-1},x_{n-2},\dots,x_{n-m-1};f'']|\} \\ &\quad\le\ cn^{2m+2}\,\omega^{\varphi}_{3,2}(f'',1/n)+c\|f''\|_{[-1/2,1/2]}\,. \end{aligned}$$

Let

$$\mathfrak{l}_1(x) := f''(x_1)+(x-x_1)[x_1,x_2;f'']+(x-x_1)(x-x_2)[x_1,x_2,x_3;f''],$$

be the quadratic polynomial function which interpolates f'' at x_1, x_2 and x_3; and symmetrically, let

$$\mathfrak{l}_n(x) := f''(x_{n-1})+(x-x_{n-1})[x_{n-1},x_{n-2};f'']+(x-x_{n-1})(x-x_{n-2})[x_{n-1},x_{n-2},x_{n-3};f'']$$

be the quadratic polynomial which interpolates f'' at x_{n-1}, x_{n-2} and x_{n-3}.

The following lemma is a consequence of [6, Lemma 3.1].

Lemma 3.10 *Let $f \in \mathbb{C}^2_\varphi$, $n \ge 4$, and let a polynomials $\mathfrak{p}_1$ and $\mathfrak{p}_n$ of degree ≤ 4 be such that $\mathfrak{p}_1^{(i)}(x_1) = f^{(i)}(x_1)$ and $\mathfrak{p}_n^{(i)}(x_{n-1}) = f^{(i)}(x_{n-1})$, for $i = 0, 1$, and $\mathfrak{p}_1''(x) = \mathfrak{l}_1(x)$, and $\mathfrak{p}_n''(x) = \mathfrak{l}_n(x)$. Then,*

$$\|f - \mathfrak{p}_1\|_{I_1} \le cn^{-2}\omega^\varphi_{3,2}(f'', 1/n), \tag{3.23}$$

and

$$\|f - \mathfrak{p}_n\|_{I_n} \le cn^{-2}\omega^\varphi_{3,2}(f'', 1/n). \tag{3.24}$$

We end this section by recalling that for $f \in \mathbb{C}^r_\varphi$, it was shown in [6] (see inequalities (3.4) and (3.5) there) that

$$|I_j|^l \omega_{k+r-l}(f^{(l)}, |I_j|, I_j) \le cn^{-r}\omega^\varphi_{k,r}\left(f^{(r)}, n^{-1}\right), \tag{3.25}$$

where either $1 < j < n$ and $0 \le l \le r$, or $1 \le j \le n$ and $0 \le l < r/2$.

4 Proofs of the positive results

Proof of Theorem 2.5. In view of Theorem 3.3 and the estimate

$$\omega^\varphi_{k+5}(s_n, 1/n) \le c\|f - s_n\| + c\omega^\varphi_{k+5}(f, 1/n) \le c\|f - s_n\| + cn^{-5}\omega^\varphi_{k,5}(f^{(5)}, 1/n)$$

(see (1.9)), we only need to construct a spline $s_n \in \Sigma_{k+5,n}(Y_s) \cap \Delta^2(Y_s)$, such that

$$\|f - s_n\| \le cn^{-5}\omega^\varphi_{k,5}(f^{(5)}, 1/n). \tag{4.1}$$

Inequality (3.25) with $l = 3$ and $r = 5$ implies

$$|I_j|^3\omega_{k+2}(f^{(3)}, |I_j|, I_j) \le cn^{-5}\omega^\varphi_{k,5}\left(f^{(5)}, 1/n\right) \tag{4.2}$$

for $1 < j < n$, while, with $l = 2$ and $r = 5$, it implies

$$|I_j|^2\omega_{k+3}(f'', |I_j|, I_j) \le cn^{-5}\omega^\varphi_{k,5}\left(f^{(5)}, 1/n\right) \tag{4.3}$$

for all $1 \le j \le n$.

Taking these estimates into account, the same construction as in [10, Proof of Theorems 4.1 and 4.2] yields a spline $s_n \in \Sigma_{k+5,n}(Y_s)$ which is coconvex with f on $[-1, 1]$ and such that (4.1) holds. For the sake of completeness, we briefly describe this construction.

We take $N(Y_s)$ to be so large that, for $n \ge N$, the sets O_i, $1 \le i \le s$, are all disjoint and do not contain the endpoints of the interval $[-1, 1]$. Now, if $I_j \not\in O$, then f does not change its convexity on I_j, and Lemma 3.1 implies that there exists a polynomial $p_j \in \mathbb{P}_{k+5}$ which is coconvex with f, interpolates it at the endpoints of I_j, and such that $p_j'(x_j) \ge f'(x_j)$ and $p_j'(x_{j-1}) \le f'(x_{j-1})$ (if f is convex on I_j), or $p_j'(x_j) \le f'(x_j)$ and $p_j'(x_{j-1}) \ge f'(x_{j-1})$ (if f is concave on I_j), and satisfies

$$\|f - p_j\|_{I_j} \le c|I_j|^2\omega_{k+3}(f'', |I_j|, I_j) \le cn^{-5}\omega^\varphi_{k,5}\left(f^{(5)}, 1/n\right).$$

Now, it is convenient to denote the endpoints of O_i by a_i and b_i, *i.e.*, $O_i = (a_i, b_i)$, $1 \le i \le s$. For each $1 \le i \le s$, there exists a polynomial $\tilde{p}_i \in \mathbb{P}_{k+3}$ which is copositive with f'' on O_i (*i.e.*, $\tilde{p}_i(x) f''(x) \ge 0$ for all $x \in O_i$) and such that (see [2, Corollary 3.1])

$$\|f'' - \tilde{p}_i\|_{O_i} \le c|O_i|\omega_{k+2}(f^{(3)}, |O_i|, O_i) \,.$$

Lemma 3.2 implies that there exists a polynomial $\overline{p}_i \in \mathbb{P}_{k+5}$ such that $\overline{p}_i'(a_i) \le f'(a_i)$ and $\overline{p}_i'(b_i) \le f'(b_i)$ (if f is such that $f''(x)(x-y_i) \ge 0$ for $x \in O_i$), or $\overline{p}_i'(a_i) \ge f'(a_i)$ and $\overline{p}_i'(b_i) \ge f'(b_i)$ (if f is such that $f''(x)(x-y_i) \le 0$ for $x \in O_i$), and satisfying

$$\|f - \overline{p}_i\|_{O_i} \le c|O_i|^2 \|f'' - \tilde{p}_i\|_{O_i} \le c|O_i|^3 \omega_{k+2}(f^{(3)}, |O_i|, O_i) \le c n^{-5} \omega_{k,5}^{\varphi}\left(f^{(5)}, 1/n\right) \,,$$

where the last inequality follows from (4.2), the observation that $|O_i| \sim |I_j|$ where j is such that $y_i \in I_j$, and the fact that O_i is "far" from ± 1.

Now, the piecewise polynomial continuous approximant $s_n \in \Sigma_{k+5,n}(Y_s) \cap \Delta^2(Y_s)$ is constructed from the polynomial pieces p_j and $\overline{p}_i$ in such a way that, if s_n is constructed for all $x \le x_\nu$, then, on $[x_\nu, x_{\nu-1}]$ (or $[x_\nu, x_{\nu-3}] = O_\mu$ if x_ν happens to be the left endpoint of some interval O_μ) it is defined to be p_ν (or $\overline{p}_\mu + \alpha$, where the constant α is chosen in such a way as to make s_n continuous). It is not difficult to see now that s_n is coconvex with f and (4.1) holds. □

Proof of Theorems 2.7 and 2.8. Suppose that n is such that

$$n \ge \max\left\{ 4\left(\min_{1\le j\le s+1}\{y_{j-1} - y_j\}\right)^{-1}, \left(\min_{1\le j\le s}\{\varphi(y_j)\}\right)^{-3} \right\}.$$

Then, in particular, f is of fixed convexity in $[x_2, 1]$ and in $[-1, x_{n-2}]$.

Again, we use the same construction as in [10, Proof of Theorem 4.1] which we described in the Proof of Theorem 2.5 above. The only difference now is that, on each interval O_i, $1 \le i \le s$, the polynomial $\tilde{p}_i$ is defined to be the quadratic polynomial interpolating f'' at a_i, y_i and b_i, whence, by Whitney's inequality,

$$\|f'' - \tilde{p}_i\|_{O_i} \le c\omega_3(f'', |O_i|, O_i).$$

Hence, using the inequality

$$|I_j|^2 \omega_3(f'', |I_j|, I_j) \le c n^{-2} \omega_{3,2}^{\varphi}(f'', 1/n), \quad 1 < j < n,$$

which follows from (3.25), we conclude that there exists a spline $s_n \in \Sigma_{5,n}(Y_s)$ which is coconvex with f on $[x_1, x_{n-1}]$, satisfies the inequality

$$\|f - s_n\|_{[x_{n-1}, x_1]} \le c n^{-2}\, \omega_{3,2}^{\varphi}(f'', 1/n), \tag{4.4}$$

and is such that $s_n(x_{n-1}+) = f(x_{n-1})$, $(-1)^{s+1} s_n'(x_{n-1}+) \le (-1)^{s+1} f'(x_{n-1})$, and $s_n'(x_1-) \le f'(x_1)$.

We now extend the construction of s_n to the intervals I_1 and I_n preserving its co-convexity with the original function f, as well as keeping it close to f.

To this end, on I_1 and I_n, s_n is defined as follows

$$s_n(x_1+) = s_n(x_1-), \quad s_n'(x_1+) = f'(x_1), \quad \text{and} \quad s_n^{(i)}(x_{n-1}-) = f^{(i)}(x_{n-1}), \quad i = 0, 1,$$

$$\begin{aligned} s_n''(x) \;:=\; & f''(x_1) + (x - x_1)\max\{0, [x_1, x_2, f'']\} \\ & +(x - x_1)(x - x_2)\max\{0, [x_1, x_2, x_3, f'']\}, \quad x \in I_1, \end{aligned}$$

and

$$\begin{aligned} s_n''(x) \;:=\; & f''(x_{n-1}) + (x - x_{n-1})(-1)^{s+1}\max\{0, (-1)^{s+1}[x_{n-1}, x_{n-2}; f'']\} \\ & +(x - x_{n-1})(x - x_{n-2})(-1)^s\max\{0, (-1)^s[x_{n-1}, x_{n-2}, x_{n-3}; f'']\}, \quad x \in I_n. \end{aligned}$$

(We wish to emphasize that in the case $s \geq 3$, we could alternatively define $s_n''(x) := f''(x_1)$, $x \in I_1$, and $s_n''(x) := f''(x_{n-1})$, $x \in I_n$, which is somewhat simpler than the current construction, but would force us to consider the case $s \leq 2$ separately.)

Evidently, s_n is continuous on $[-1, 1]$ and is in $\Delta^2(Y_s)$ (since s_n' and $(-1)^s s_n'$ are nondecreasing on I_1 and I_n, respectively, we have that $(-1)^s s_n'(x_{n-1}-) \leq (-1)^s s_n'(x_{n-1}+)$, and $s_n'(x_1-) \leq s_n'(x_1+)$).

Hence, it remains to estimate $\|f - s_n\|_{I_1}$ and $\|f - s_n\|_{I_n}$. First, we note that (4.4) implies that $\alpha := f(x_1) - s_n(x_1-)$ satisfies $|\alpha| \leq cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n)$. Therefore, by Lemma 3.10 we have for every $x \in I_1$

$$\begin{aligned} |f(x) - s_n(x)| &\leq \|f - \mathfrak{p}_1\|_{I_1} + |\mathfrak{p}_1(x) - s_n(x)| \\ &\leq cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n) + \left| f(x_1) - s_n(x_1+) + \int_{x_1}^{x} (x - u)(\mathfrak{l}_1(u) - s_n''(u))\, du \right| \\ &\leq cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n) + |\alpha| + \left| \int_{x_1}^{x} (x - u)(\mathfrak{l}_1(u) - s_n''(u))\, du \right| \\ &\leq cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n) + cn^{-4}\|\mathfrak{l}_1 - s_n''\|_{I_1}. \end{aligned}$$

Similarly (except that $s_n(x_{n-1}-) = f(x_{n-1}) = \mathfrak{p}_n(x_{n-1})$), for every $x \in I_n$, we have

$$|f(x) - s_n(x)| \leq cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n) + cn^{-4}\|\mathfrak{l}_n - s_n''\|_{I_n}.$$

Now, for $x \in I_1$,

$$\begin{aligned} (4.5) \quad 0 &\leq s_n''(x) - \mathfrak{l}_1(x) \\ &= (x - x_1)\left(\max\{0, [x_1, x_2, f'']\} - [x_1, x_2, f'']\right) \\ &\quad +(x - x_1)(x - x_2)\left(\max\{0, [x_1, x_2, x_3, f'']\} - [x_1, x_2, x_3, f'']\right) \\ &= (x - x_1)\max\{0, -[x_1, x_2, f'']\} + (x - x_1)(x - x_2)\max\{0, -[x_1, x_2, x_3, f'']\}. \end{aligned}$$

Hence, for $s \geq 2$, we conclude by Corollaries 3.6 and 3.7, that

$$\begin{aligned} 0 \leq s_n''(x) - \mathfrak{l}_1(x) &\leq (x - x_1)cn^4\omega_{3,2}^{\varphi}(f'', 1/n) + (x - x_1)(x - x_2)cn^6\omega_{3,2}^{\varphi}(f'', 1/n) \\ &\leq cn^2\omega_{3,2}^{\varphi}(f'', 1/n), \quad x \in I_1. \end{aligned}$$

For $s = 1$, we apply Corollary 3.8, and similarly conclude that

$$0 \le s_n''(x) - \mathfrak{l}_1(x) \;\; \le \;\; cn^2\omega_{3,2}^{\varphi}(f'', 1/n) + c\omega_{2,2}^{\varphi}(f'', 1/n), \quad x \in I_1.$$

Analogously, for $x \in I_n$,

$$\begin{aligned} 0 &\le (-1)^s \left(s_n''(x) - \mathfrak{l}_n(x)\right) \\ &= (x_{n-1} - x) \max\{0, (-1)^s[x_{n-1}, x_{n-2}; f'']\} \\ &\quad + (x - x_{n-1})(x - x_{n-2}) \max\{0, (-1)^{s+1}[x_{n-1}, x_{n-2}, x_{n-3}; f'']\}. \end{aligned}$$

Hence, for $s \ge 2$, by Corollaries 3.6 and 3.7, we obtain

$$\begin{aligned} 0 &\le (-1)^s \left(s_n''(x) - \mathfrak{l}_n(x)\right) \\ &\le (x_{n-1} - x)cn^4\omega_{3,2}^{\varphi}(f'', 1/n) + (x - x_{n-1})(x - x_{n-2})cn^6\omega_{3,2}^{\varphi}(f'', 1/n) \\ &\le cn^2\omega_{3,2}^{\varphi}(f'', 1/n), \quad x \in I_n, \end{aligned}$$

and for $s = 1$, by Corollary 3.8 we get

$$0 \le -\left(s_n''(x) - \mathfrak{l}_n(x)\right) \;\; \le \;\; cn^2\omega_{3,2}^{\varphi}(f'', 1/n) + c\omega_{2,2}^{\varphi}(f'', 1/n), \quad x \in I_n.$$

Also, in the case $s = 1$, applying Lemma 3.9 instead of Corollary 3.8 we have for $x \in I_1$

$$\begin{aligned} |s_n''(x) - \mathfrak{l}_1(x)| &\le (x - x_1)\,|[x_1, x_2, f'']| + (x - x_1)(x - x_2)\,|[x_1, x_2, x_3, f'']| \\ &\le n^{-2}|[x_1, x_2, f'']| + n^{-4}|[x_1, x_2, x_3, f'']| \\ &\le cn^2\omega_{3,2}^{\varphi}(f'', 1/n) + cn^{-2}\|f''\|_{[-1/2,1/2]}, \end{aligned}$$

and the estimate for $\|s_n'' - \mathfrak{l}_n\|_{I_n}$ is derived analogously.

To summarize, in the case $s \ge 2$ we have

$$\|f - s_n\| \le cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n), \tag{4.6}$$

and in the case $s = 1$ we have

$$\|f - s_n\| \le cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n) + cn^{-6}\|f''\|_{[-1/2,1/2]}, \tag{4.7}$$

and

$$\|f - s_n\| \le cn^{-2}\,\omega_{3,2}^{\varphi}(f'', 1/n) + cn^{-4}\,\omega_{2,2}^{\varphi}(f'', 1/n). \tag{4.8}$$

By virtue of Lemma 3.3 and the estimate

$$\omega_5^{\varphi}(s_n, 1/n) \le c\|f - s_n\| + c\omega_5^{\varphi}(f, 1/n) \le c\|f - s_n\| + cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n)$$

(see (1.9)), we conclude that there exists a polynomial $P_n \in \Delta^2(Y_s)$ of degree $\le cn$ such that

$$\begin{aligned} \|f - P_n\| &\le \|f - s_n\| + \|s_n - P_n\| \le \|f - s_n\| + c\omega_5^{\varphi}(s_n, 1/n) \\ &\le c\|f - s_n\| + cn^{-2}\omega_{3,2}^{\varphi}(f'', 1/n). \end{aligned} \tag{4.9}$$

Combining this with the inequalities (4.6)–(4.8) we get (2.6), (2.7) and (2.9).

Finally, in order to prove (2.10), note that (1.5) implies that

$$n^3\omega^{\varphi}_{3,2}(f'',1/n) \ge C(f), \quad \text{for all} \quad n\in\mathbb{N}.$$

Hence, for $n \ge \|f''\|_{[-1/2,1/2]}/C(f) =: N(f)$,

$$\frac{1}{n}\|f''\|_{[-1/2,1/2]} \le C(f) \le n^3\omega^{\varphi}_{3,2}(f'',1/n).$$

Therefore, it follows from (4.7) and (4.9) that

$$\|f-P_n\| \le cn^{-2}\omega^{\varphi}_{3,2}(f'',1/n), \quad n\ge N(f),$$

and (2.10) is proved. □

Proof of Theorem 2.1. As was mentioned above, Theorem 2.1 for $r\le 3$ is due to [5]. For $r=4$, it follows from (1.7) and Theorems 2.7 and 2.8, and for $r\ge 6$, Theorem 2.1 follows from (1.7) and Theorem 2.5. Finally, if $r=5$, then, for $s\ge 2$, it follows from (1.7) and Theorem 2.7, and, for $s=1$, we repeat the arguments of the proof of Theorem 2.5, replacing $\omega^{\varphi}_{k,5}(f^{(5)},1/n)$ by $\|\varphi^5 f^{(5)}\|$. □

Proof of Theorem 2.11. We follow the proof of Theorem 2.5, where we observe that since $s=1$, there is no need to separate the points of inflection. This time we construct an $S\in\Sigma_{k+7,n}(Y_s)\cap\Delta^2(Y_s)$. Also, it follows by virtue of (1.10) that $f\in\mathbb{C}^3[-1,1]$, and by (3.25) with $l=3$ and $r=7$, we have

$$|I_j|^3\omega_{k+4}(f^{(3)},|I_j|,I_j) \le cn^{-7}\omega_{k,7}(f^{(7)},1/n), \quad 1\le j\le n,$$

which we use instead of (4.2). Hence, even if $I_1\in O_1$ or $I_n\in O_1$, we are on safe grounds and we don't need to make sure that O_1 is "far" from ±1. We omit the details. □

Proof of Theorem 2.3. As mentioned above, Theorem 2.3 for $r\le 2$ was proved in [9]. For $r>7$, Theorem 2.3 readily follows from Corollary 2.12 and (1.7). The case $r=7$, is proved by applying the same arguments as in the proof of Theorem 2.11, replacing $\omega_{k,7}(f^{(7)},1/n)$ by $\|\varphi^7 f^{(7)}\|$. □

In order to prove Theorem 1.2, we need the following corollary which readily follows from the positive results described in Section 2 (see Figs 1 and 2).

Corollary 4.1 *Let $r\ge 0$ and let $Y_s\in\mathbb{Y}_s$. If $f\in\mathbb{C}^r_\varphi\cap\Delta^2(Y_s)$, then*

$$E^{(2)}_n(f,Y_s) = O(n^{-r}\omega^{\varphi}_{1,r}(f^{(r)},1/n)), \quad n\to\infty, \tag{4.10}$$

and if, in addition, $r\ne 4$, then

$$E^{(2)}_n(f,Y_s) = O(n^{-r}\omega^{\varphi}_{2,r}(f^{(r)},1/n)), \quad n\to\infty. \tag{4.11}$$

Proof of Theorem 1.2. Let $\alpha > 0$, and $Y_s \in \mathbb{Y}_s$, and let $f \in \Delta^2(Y_s)$, be such that

$$E_n(f) = O(n^{-\alpha}), \quad n \to \infty.$$

Then the well known inverse theorem [1] (see also [12]) implies that for each pair (k, r) such that $r < \alpha < k + r$, we have that $f \in \mathbb{C}^r_\varphi$, and

$$\omega^\varphi_{k,r}(f^{(r)}, t) = O(t^{\alpha - r}), \quad t \to 0. \tag{4.12}$$

Hence, if $\alpha \notin \mathbb{N}$, then we put $r := [\alpha]$, and (4.10) yields,

$$E^{(2)}_n(f, Y_s) = O(n^{-\alpha}), \quad n \to \infty. \tag{4.13}$$

If $\alpha \in \mathbb{N}$, then we put $r := \alpha - 1$. Then for $\alpha \neq 5$, (4.13) follows from (4.11) and (4.12). The proof for $\alpha = 5$, needs some modification of the proof of Theorem 2.5, we will not elaborate here. □

5 Proofs of negative results

We begin with two lemmas which we need for the proof of Theorem 2.2. It is possible that the following lemma is known but we have failed to find any similar result in the literature.

Lemma 5.1 *Given a monotone odd function $g \in \mathbb{L}_1[-1, 1]$. Then, for every polynomial $P_{n-1} \in \mathbb{P}_{n-1}$, the following inequality holds*

$$\|g(\cdot/n)\|_{\mathbb{L}_1[-1,1]} \|P_{n-1}\|_{\mathbb{L}_1[-1,1]} \le 2 \|gP_{n-1}\|_{\mathbb{L}_1[-1,1]}, \tag{5.1}$$

where, as usual, $\|f\|_{\mathbb{L}_1[-1,1]} := \int_{-1}^1 |f(x)|\, dx$.

Note that inequality (5.1) is sharp in that the constant 2 is exact since, for the function $g(x) = \operatorname{sgn}(x)$, (5.1) becomes an equality.

Proof. Without loss of generality assume that $\|P_{n-1}\|_{\mathbb{L}_1[-1,1]} = 1$ and $g(x) = 1$ for $1/n \le x \le 1$. We may further assume that g is absolutely continuous on $[-1, 1]$. Integration by parts, together with the observation that g' is an even function on $[-1, 1]$, yields

$$\left\| g\left(\frac{\cdot}{n}\right) \right\|_{\mathbb{L}_1[-1,1]} = \int_{-1}^1 \left| g\left(\frac{x}{n}\right) \right| \, dx = n \int_{-1/n}^{1/n} |g(x)| \, dx = 2 - 2n \int_0^{1/n} x g'(x)\, dx,$$

and

$$\begin{aligned} \|gP_{n-1}\|_{\mathbb{L}_1[-1,1]} &= \int_{-1}^1 |P_{n-1}(x)|\, dx - \int_{-1}^1 (1 - |g(x)|)\, |P_{n-1}(x)|\, dx \\ &= 1 - \int_{-1/n}^{1/n} (1 - |g(x)|)\, |P_{n-1}(x)|\, dx \\ &= 1 - \int_0^{1/n} g'(x) \int_{-x}^{x} |P_{n-1}(u)| \, du\, dx. \end{aligned}$$

Therefore, (5.1) is equivalent to

$$\int_0^{1/n} g'(x) \int_{-x}^{x} |P_{n-1}(u)|\, du\, dx \le n \int_0^{1/n} x g'(x)\, dx. \tag{5.2}$$

Since g' is nonnegative, the proof will be complete if we show that, for any $0 \le x \le 1/n$,

$$\int_{-x}^{x} |P_{n-1}(u)|\, du \le nx = \frac{n}{2}\int_{-x}^{x} du,$$

which, in turn, will be proved if we verify that

$$|P_{n-1}(x)| \le \frac{n}{2} \quad \text{for all} \quad -1/n \le x \le 1/n. \tag{5.3}$$

Now, let $-1 < \alpha < 1$ be such that $\int_{-1}^{\alpha} |P_{n-1}(x)|\, dx = \int_{\alpha}^{1} |P_{n-1}(x)|\, dx = 1/2$, and define $Q_n(x) := \int_{\alpha}^{x} P_{n-1}(u)\, du$. Then, $Q_n \in \mathbb{P}_n$ and $\|Q_n\| \le 1/2$. Therefore, by the Bernstein inequality, for all $-1/n \le x \le 1/n$,

$$|P_{n-1}(x)| = |Q_n'(x)| \le \frac{n-1}{\sqrt{1-n^{-2}}}\|Q_n\| \le \frac{n-1}{2\sqrt{1-n^{-2}}} \le \frac{n}{2},$$

and the proof of the lemma is complete. □

Taking $g(x) = x|x|$ in the statement of Lemma 5.1 we get the following corollary.

Corollary 5.2 *For every polynomial $P_{n-1} \in \mathbb{P}_{n-1}$, we have*

$$\|P_{n-1}\|_{\mathbb{L}_1[-1,1]} \le 3n^2 \left\|x^2 P_{n-1}\right\|_{\mathbb{L}_1[-1,1]}. \tag{5.4}$$

Lemma 5.3 *Let $h \le \frac{1}{3n}$, and let $P \in \mathbb{P}_{n+1}$ be such that*

$$(x^2 - h^2)P''(x) \ge 0, \quad x \in [-1,1]. \tag{5.5}$$

Then

$$P(-1) - 2P(0) + P(1) \ge 0. \tag{5.6}$$

Proof. First of all, note that (5.5) implies that $P''(\pm h) = 0$ and, therefore, $P''(x) = (x^2 - h^2)Q(x)$, where $Q \in \mathbb{P}_{n-3}$ is nonnegative on I. Now, taking into account that

$$(1 - |x|)(x^2 - h^2) \ge \frac{1}{2}(1 - x^2)(x^2 - 2h^2), \quad x \in [-1,1],$$

we have

$$\begin{aligned} P(-1) - 2P(0) + P(1) &= \int_{-1}^{1} (1 - |x|)P''(x)\, dx \\ &= \int_{-1}^{1} (1 - |x|)(x^2 - h^2)Q(x)\, dx \\ &\ge \frac{1}{2}\int_{-1}^{1} (1 - x^2)(x^2 - 2h^2)Q(x)\, dx \ge 0, \end{aligned}$$

where the last inequality follows from Corollary 5.2 taking into account that the polynomial $R(x) := (1-x^2)Q(x)$ of degree $\le n-2$ is nonnegative on $[-1,1]$ and $2h^2 \le \frac{1}{3n^2}$. $\square$

Using linear transformation of the interval $[-1,1]$ to $[-1/2,1/2]$, and change of variables we immediately get the following consequence.

Corollary 5.4 *Let $h \le \frac{1}{6n}$, and let $Q \in \mathbb{P}_n$ be such that*

$$(x^2-h^2)Q''(x) \ge 0\,, \quad x \in [-1/2,1/2].$$

Then

$$Q(-1/2) - 2Q(0) + Q(1/2) \ge 0.$$

We are ready with

Proof of Theorem 2.2. Suppose that $s \ge 2$ and $r \ge 1$ are given. Let $Y_s = \{y_i\}_{i=1}^s$ be such that $-1 < y_s < \ldots < y_{s-2} \le -1/2$, $y_2 = -h$ and $y_1 = h$, where $h = \frac{1}{6n}$. Now, let f be such that

$$f(x) = \int_0^x (x-t) f''(t)\,dt\,,$$

where

$$f''(t) := \begin{cases} -(h^2-t^2)^r, & |t| \le h, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $f \in \mathbb{C}^r[-1,1] \cap \Delta^2(Y_s)$, and

$$\|f^{(r)}\| \le c h^{r+2}. \tag{5.7}$$

Also, $f(0)=0$, and

$$\begin{aligned} -f(1/2) - f(1/2) &= -\int_0^{1/2} (1/2-t) f''(t)\,dt - \int_0^{-1/2} (-1/2-t) f''(t)\,dt \\ &= \int_0^h (1-2t)(h^2-t^2)^r\,dt \\ &\ge \frac{1}{3h}\int_0^h (h^2-t^2)^r 2t\,dt = \frac{h^{2r+1}}{3(r+1)}. \end{aligned}$$

If $Q_n \in \mathbb{P}_n$ is in $\Delta^2(Y_s)$ (whence, in particular, $(x^2-h^2)Q''(x) \ge 0$ on $[-1/2,1/2]$), then applying Corollary 5.4, we conclude that

$$\begin{aligned} \frac{h^{2r+1}}{3(r+1)} &\le -f(1/2) - f(1/2) \\ &\le Q_n(-1/2) - f(-1/2) - 2\left(Q_n(0) - f(0)\right) + Q_n(1/2) - f(1/2) \\ &\le |Q_n(-1/2) - f(-1/2)| + 2\,|Q_n(0) - f(0)| + |Q_n(1/2) - f(1/2)| \\ &\le 4\|Q_n - f\|, \end{aligned}$$

implying that

$$E_n^{(2)}(f, Y_s) \geq \frac{h^{2r+1}}{12(r+1)}. \tag{5.8}$$

Now, by (5.7) and (5.8) and recalling that $h = 1/(6n)$, we have

$$\frac{n^r E_n^{(2)}(f, Y_s)}{\|f^{(r)}\|} \geq \frac{n^r h^{2r+1}}{12(r+1)ch^{r+2}} = cn.$$

This completes our proof. □

We now construct counterexamples which prove our claims in Theorem 2.4.

Proof of Theorem 2.4. Given $A > 0$, let

$$g_r(x) := \begin{cases} (-1)^{(r-1)/2} c_r (1+x)^{r/2}, & r = 3, 5, \\ c_4 (1+x)^2 \ln(1+x), & r = 4, \\ c_6 (1+x)^3 (3 - \ln(1+x)), & r = 6, \end{cases}$$

where the normalizing constants c_r are so chosen that

$$\|g_r^{(r)} \varphi^r\| = 1, \quad 3 \leq r \leq 6. \tag{5.9}$$

Thus, in particular, $g_r \in \mathbb{B}^r$.

First observe that

$$g_r^{(3)}(x) > 0, \quad 3 \leq r \leq 6, \quad \text{and} \quad g_r^{(5)}(x) > 0, \quad r = 5, 6, \quad x \in (-1, 1]. \tag{5.10}$$

Denote $M_r := \|g_r\|$, $3 \leq r \leq 6$, let $m := \max\{4, n-1\}$, and take $b \in (-1, 0)$ to be such that

$$|g_r''(b)| > m^4 (A + M_r), \; r = 3, 4, \quad \text{and} \quad g_r^{(3)}(b) > m^6 (A + M_r), \; r = 5, 6. \tag{5.11}$$

Finally, let

$$f_b(x) := \begin{cases} \frac{1}{2!} \int_b^x g_r^{(3)}(t)(x-t)^2 \, dt, & r = 3, 4, \\ \frac{1}{4!} \int_b^x g_r^{(5)}(t)(x-t)^4 \, dt, & r = 5, 6, \end{cases}$$

that is, $f_b(x) = g_r(x) - T_r(x)$ where T_r is the Taylor polynomial about $x = b$, of degree 2, for $r = 3, 4$, and of degree 4, for $r = 5, 6$, respectively. Then in view of (5.10), it readily follows that f_b changes its convexity once in $(-1, 1)$, at $y_1 := b$. Now assume that some $p_n \in \mathbb{P}_n$ satisfying

$$p_n''(x)(x - b) \geq 0, \quad -1 \leq x \leq 1, \tag{5.12}$$

is such that

$$\|f_b - p_n\| \leq A \|g_r^{(r)} \varphi^r\| = A. \tag{5.13}$$

Then

$$\|T_r + p_n\| \le A + M_r,$$

which by Markov's inequality implies,

$$\|T_r'' + p_n''\| \le m^4(A + M_r), \tag{5.14}$$

and

$$\|T_r^{(3)} + p_n^{(3)}\| \le m^6(A + M_r). \tag{5.15}$$

On the other hand, if $r = 3$ or $r = 4$, then by (5.11),

$$\|T_r'' + p_n''\| \ge |T_r''(b) + p_n''(b)| = |T_r''(b)| = |g_r''(b)| > m^4(A + M_r),$$

a contradiction to (5.14). If $r = 5$ or $r = 6$, then by (5.11),

$$\|T_r^{(3)} + p_n^{(3)}\| \ge T_r^{(3)}(b) + p_n^{(3)}(b) \ge T_r^{(3)}(b) = g_r^{(3)}(b) > m^6(A + M_r),$$

contradicting (5.15). Note that in the second inequality we used the fact that p_n'' passes from negative to positive at b, and therefore $p_n^{(3)}(b) \ge 0$.

We conclude that no polynomial satisfying (5.12), also verifies (5.13). This completes the first part of the proof.

What is left is to modify g_6 so that it will be in $\mathbb{C}_\varphi^6$, and still preserve (2.4). To this end, for $0 < \epsilon < 1/2$, set

$$g_\epsilon := g_6(x + \epsilon).$$

Then $g_\epsilon \in \mathbb{C}_\varphi^6$, $\|g_\epsilon^{(6)}\varphi^6\| < 1$, $g_\epsilon^{(3)}(x) > 0$, and $g_\epsilon^{(5)}(x) > 0$, $x \in [-1, 1]$, and finally $M_\epsilon := \|g_\epsilon\| \le 2M_6$. Now we take ϵ so small that

$$g_\epsilon^{(3)}(-1) > m^6(A + 2M_6),$$

where we recall that $m := \max\{4, n - 1\}$, and we proceed with the above arguments to obtain a contradiction. □

In order to prove Theorem 2.13, we let $b \in (0, 1)$, and set

$$g_b(x) := \Pi(x) \ln \frac{b}{1 + x + b}, \quad x \in [-1, 1],$$

where we recall that $\Pi(x) := \prod_{i=1}^s (x - y_i)$. Finally, we denote

$$G_b(x) := \int_{-1}^x (x - u) g_b(u)\, du\,, \quad x \in [-1, 1],$$

so that clearly, $G_b \in \mathbb{C}^\infty[-1, 1]$.

First, we prove

Lemma 5.5 *The following estimate holds:*

$$\omega_{2,4}^{\varphi}(G_b^{(4)},t) \le c\left(1+t^2\ln\frac{1}{b}\right), \tag{5.16}$$

and

$$|g_b(x)|b\ln\frac{1}{b} \le |\Pi(x)|(1+x)\ln\frac{3e^2}{1+x}, \qquad x\in(-1,1]. \tag{5.17}$$

Proof. First, since $G_b''(x) = g_1(x)+g_2(x)$, where $g_1(x) := \Pi(x)\ln b$ and $g_2(x) := -\Pi(x)\ln(1+x+b)$, we have

$$\omega_{2,4}^{\varphi}(G_b^{(4)},t) \le \omega_{2,4}^{\varphi}(g_1'',t)+\omega_{2,4}^{\varphi}(g_2'',t) \le \omega_2(g_1'',t)+c\|\varphi^4 g_2''\|,$$

where we used the inequalities (1.3) and (1.4).

Now,

$$\omega_2(g_1'',t) \le t^2\ln\frac{1}{b}\|\Pi''\| = ct^2\ln\frac{1}{b},$$

and since $|(1+x)\ln(1+x+b)|\le 3$, and $(1+x)/(1+x+b)\le 1$, we conclude that

$$\|\varphi^4 g_2''\| \le c(\|\Pi\|+\|\Pi'\|+\|\Pi''\|) \le c.$$

This completes the proof of (5.16). Inequality (5.17) is proved in Lemma 5.1 in [6]. □

Denote by $\mathbb{P}_n^*$ the subset of polynomials $p_n\in\mathbb{P}_n$, such that

$$\Pi(-1)p_n''(-1)\ge 0.$$

Clearly, every polynomial p_n from $\mathbb{P}_n\cap\Delta^2(Y_s)$, is also in $\mathbb{P}_n^*$.

Lemma 5.6 *For each $b\in(0,n^{-2})$, and every polynomial $p_n\in\mathbb{P}_n^*$, we have*

$$\|G_b-p_n\| \ge \frac{C}{n^4}\ln\frac{1}{n^2b}-\frac{1}{n^4},$$

where $C=C(Y_s)$.

Proof. Put

$$g_b^*(x) := -\Pi(x)\ln\left(n^2(1+x+b)\right), \quad l(x) := g_b(x)-g_b^*(x) = \Pi(x)\ln n^2 b,$$

so that l is a polynomial of degree s. Let

$$G_b^*(x) := \int_{-1}^{x}(x-u)g_b^*(u)\,du \quad\text{and}\quad L(x) := \int_{-1}^{x}(x-u)l(u)\,du.$$

Then we have

$$G_b^*(x)+L(x) = G_b(x).$$

Also, for every $p_n \in P_n^*$,

$$\Pi(-1)p_n''(-1) - \Pi(-1)L''(-1) \geq -\Pi(-1)l(-1) = \Pi^2(-1)\ln 1/n^2 b. \tag{5.18}$$

Straightforward computations yield

$$\int_{-1}^{x} |g_b^*(u)|\, du \leq c/n^2 + cn^2(1+x)^2, \quad -1 \leq x \leq 1,$$

whence

$$|G_b^*(x)| \leq \frac{c}{n^4}(1+n^2(1+x))^3.$$

Hence,

$$|p_n(x) - L(x)| \leq \|p_n - G_b\| + \|G_b^*\| \leq cn^6(\|p_n - G_b\| + \frac{1}{n^4})(1/n^2 + (1+x))^3.$$

We may apply now the Dzjadyk-type inequality, we used in [6], to obtain

$$|p_n''(-1) - L''(-1)| \leq cn^4(\|p_n - G_b\| + \frac{1}{n^4}).$$

This combined with (5.18), in turn completes the proof of the lemma. □

We are now ready to prove Theorem 2.13 by constructing a counterexample.

Proof of Theorem 2.13. The proof follows along the lines of the proof of Theorem 2.3 in [6], and we will only sketch it.

We begin with $b_n \in (0, 1/e)$, $n \geq 2$, such that

$$b_n \ln \frac{1}{b_n} = \frac{1}{n^2},$$

and set

$$f_n(x) := c\frac{1}{n^2}G_{b_n}(x),$$

where $c < 1$ (which is independent of n) is taken so small that (5.21) and (5.22) below are fulfilled. We summarize the properties of f_n as follows from Lemma 5.5. Namely, for every $n \geq 2$,

$$f_n \in \mathbb{C}^\infty[-1,1].$$

$$|f_n''(x)| \leq |\Pi(x)|(1+x)\ln\frac{3e^2}{1+x}\,. \tag{5.19}$$

$$f_n(-1) = f_n'(-1) = f_n''(-1) = 0, \tag{5.20}$$

$$\|f_n^{(j)}\| < 1, \quad j = 0,1,2, \quad \text{and} \quad \|\varphi^{2j-4} f_n^{(j)}\| < 1, \quad j = 3,4, \tag{5.21}$$

and

$$\omega_{2,4}^{\varphi}(f_n^{(4)}, 1/n) \leq n^{-2}. \tag{5.22}$$

The proof proceeds with no change constructing a subsequence f_{n_j} and an infinite sum which we continue to denote $\mathfrak{f}_1(x)$, and which differs from the one in [6] in that we multiply the second derivative of the latter by $\Pi(x)$. Therefore we have

$$|\mathfrak{f}_1''(x)| \le 2|\Pi(x)|(1+x)\ln\frac{3e^2}{1+x},$$

so that if we put

$$\mathfrak{f}_2''(x) := 2\Pi(x)(1+x)\ln\frac{3e^2}{1+x},$$

and

$$\mathfrak{f}_2(x) := \int_{-1}^{x}(x-u)\mathfrak{f}_2''(u)du,$$

and if we denote

$$\mathfrak{f}(x) := \mathfrak{f}_1(x) + \mathfrak{f}_2(x)\,,$$

then $\mathfrak{f} \in \Delta^2(Y_s)$. The rest of the proof follows exactly as the proof of Theorem 2.3 in [6]. □

Finally, we have

Proof of Theorem 2.15. For $s = 1$, $Y_1 := \{y_1\}$, and $\Pi(x) = (x - y_1)$ is a polynomial of degree 1. We observe that Lemma 5.5 may be strengthened to yield

$$\omega_{1,4}(G_b^{(4)}, t) \le c.$$

Let

$$F_b(x) := \frac{1}{b}\int_{-1}^{x}(x-u)\Pi(u)(u+1)\,du,$$

and set $f_b := G_b + F_b$. Since $F_b^{(4)}(x) = const$, its modulus of continuity vanishes, so that we have

$$\omega_{1,4}^{\varphi}(f_b^{(4)}, t) = \omega_{1,4}^{\varphi}(G_b^{(4)}, t) \le c.$$

At the same time

$$\Pi(x)f_b''(x) = \Pi^2(x)\left(\frac{x+1}{b} + \ln\frac{b}{x+1+b}\right) \ge 0, \quad x \in [-1,1],$$

so that $f_b \in \Delta^2(Y_1)$.

Since $F_b \in \mathbb{P}_n^*$ for $n \ge 5$, we may apply Lemma 5.6 and conclude that for every $p_n \in \mathbb{P}_n^*$,

$$\|f_b - p_n\| \ge \frac{C}{n^4}\ln\frac{1}{n^2 b} - \frac{1}{n^4}.$$

Hence, with $b = n^{-5/2}$ we obtain,

$$E_n^{(2)}(f_b, Y_1) \ge C\frac{\ln n}{n^4}\omega_{1,4}(f_b^{(4)}, 1).$$

□

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