65 YEARS SINCE THE PAPER "ON THE VALUE OF THE BEST APPROXIMATION OF FUNCTIONS HAVING A REAL SINGULAR POINT" BY I. I. IBRAGIMOV

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Abstract. In his famous paper [15] Ibragim Ibishievich Ibragimov has given asymptotic values of the best uniform approximation of functions of the form $(a-x)^s \ln^m(a-x), \quad (a \geq 1)$. These results have led to the development of a series of new directions in Approximation Theory, including the following ones, to which we devote this paper.

- Constructive characterization of approximation of functions on a closed interval.
- Babenko spaces.
- Ditzian-Totik moduli of smoothness.
- Constructive characterization of approximation of functions on the sets of complex plane.
- Shape preserving approximation.

In particular, we will show how we have applied the results by I. I. Ibragimov in our recent paper in the journal Constructive Approximation.

1. Introduction

Let

$$E_n[f] = \inf_{P_n} \|f - P_n\|_{C[-1,1]}$$

denote the error of best uniform approximation of a continuous function f on $[-1,1]$, by algebraic polynomials P_n of degree $< n$. In his famous paper [15] Ibragim Ibishievich Ibragimov has given asymptotic values of the best uniform approximation of functions of the form $(a-x)^s \ln^m(a-x), \quad (a \geq 1)$.

In particular, on p. 445 there Ibragimov states:

"**Theorem VII.** If p and m are integer positive numbers, then the best approximation of the function $(1-x)^p \ln^m(1-x)$ satisfies for sufficiently large n the inequality

$$C_p > \frac{n^{2p}}{(\ln n)^{m-1}} E_n[(1-x)^p \ln^m(1-x)] > \left(\frac{\pi}{4+\pi}\right)^{2p} \cdot \frac{C_p}{2\sqrt{2}}, \tag{45}$$

where

$$C_p = m2^{m-p+1} \int_0^\infty \frac{u^{2p-1}du}{e^u + e^{-u}}.$$

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Corollary. In the case $p = m = 1$ the inequality (45) takes the form

$$C_1 > n^2 E_n[(1-x)\ln(1-x)] > \left(\frac{\pi}{4+\pi}\right)^2 \cdot \frac{C_1}{2\sqrt{2}},$$

where

$$C_1 = 2\int_0^\infty \frac{udu}{e^u + e^{-u}}.$$

Inequality (45) confirms the unproven statement of S. N. Bernstein that the order of decrease of $E_n[(1-x)\ln(1-x)]$ when $n \to \infty$ is equal to $\frac{1}{n^2}$ [3, p. 91]."

For the functions

$$f_p(x) = (1-x)^p \ln(1-x), \quad m \in \mathbf{N}$$

$(f_p(1) := 0)$, inequalities (45) take a form

$$\frac{C'_p}{n^{2p}} < E_n[f_p] < \frac{C_p}{n^{2p}}. \tag{1}$$

These results have led to the development of a series of new directions in Approximation Theory, including the following ones, to which we devote this paper.

- Constructive characterization of approximation of functions on a closed interval.
- Babenko spaces.
- Ditzian-Totik moduli of smoothness.
- Constructive characterization of approximation of functions on the sets of complex plane.
- Shape preserving approximation.

In particular, we will show how we have applied the results by I. I. Ibragimov in the recent paper [19].

In the sequel $c(\cdot)$ denotes different positive constants, while all parameters on which c depends are indicated in the parentheses. If c is an absolute constant, then there will be no parentheses.

2. Constructive characterization of approximation of functions on a closed interval.

At the time of Ibragimov's paper, the Jackson estimates on the degree of approximation of continuous functions by algebraic polynomials had been known. (These estimates follow from Nikolskii[20] (1946) and Timan [22] (1951)). Namely, *if* $f \in C[-1,1]$, *then*

$$E_n[f] < c\omega(1/n, f),$$

where

$$\omega(t,f) := \inf_{h\in[0,t]} \max_{x\in[-1,1-h]} |f(x+h) - f(x)|$$

is the modulus of continuity of the function f.

Finer estimates due to Nikolskii [20] and Timan [22], are the pointwise estimates. Namely, *if* $f \in C[-1,1]$*, then there exists a sequence of algebraic polynomials,* $\{P_n\}_{n=1}^\infty$, P_n *of degree* $< n$*, such that*

$$|f(x) - P_n(x)| < c\omega(\rho_n(x), f), \quad x \in [-1,1],$$

where

$$\rho_n(x) = \frac{1}{n^2} + \frac{1}{n}\sqrt{1-x^2}.$$

However, for the function $f_1(x) = (1-x)\ln(1-x)$ both estimates imply that

$$E_n(f_1) \le \frac{c\ln n}{n}, \qquad n > 1. \tag{2}$$

Far from Ibragimov's estimate. So, what can be done to improve (2), in particular, can one at least remove the extra logarithmic factor?

Almost simultaneously, Zygmund[23] (1945) showed that in the constructive theory of approximation of periodic functions, one can improve the estimates, applying, instead of the first modulus of continuity of f, its second modulus of smoothness, $\omega_2(\cdot, f)$.

It turns out that one may have similar estimates for the approximation of non-periodic functions. Indeed, Dzyadyk[8] (1958) and Freud[10] (1959) proved that *if* $f \in C[-1,1]$*, then there exists a sequence of algebraic polynomials,* $\{P_n\}_{n=2}^\infty$, P_n *of degree* $< n$*, such that*

$$|f(x) - P_n(x)| < c\omega_2(\rho_n(x), f), \qquad x \in [-1,1], \tag{3}$$

For the function $f_1 = (1-x)\ln(1-x)$, this yields

$$E_n(f_1) \le \frac{c}{n}, \qquad n \ge 1.$$

So, the application of the $\omega_2(\cdot, f)$ eliminates the logarithmic factor. Moreover (3) implies that near the endpoints of the interval $[-1,1]$ the order of the error is $\frac{1}{n^2}$, i.e., Ibragimov's estimate, but still we are far from it in the middle of the interval.

3. Babenko spaces

Babenko[2] (1985) introduced the spaces B^r. *A function* f *belongs to the space* B^r *if it has a locally absolutely continuous* $(r-1)$*-st derivative in* $(-1,1)$*, and*

$$\|f^{(r)}\varphi^r\|_{L_\infty[-1,1]} < +\infty,$$

where

$$\varphi(x) := \sqrt{1-x^2}.$$

(Babenko [2] introduced corresponding classes for the functions of several variables.)

It was proved that if $f \in B^r$, then

$$E_n(f) \le \frac{c(r)}{n^r}\|f^{(r)}\varphi^r\|_{L_\infty[-1,1]}, \quad n \ge r. \tag{4}$$

For $f_m = (1-x)^m \ln(1-x)$, this readily yields

$$E_n(f_m) \le \frac{c(m)}{n^{2m}}, \quad n \ge 1,$$

since $f_m \in B^{2m}$.

Thus, while for odd r, as was shown by Bernstein, the function $x^{r/2}$ has the exact order of approximation n^{-r}, making it a "proper representative" of the space B^r; for even r, a "proper representative" of the space B^r is, Ibragimov's $f_{r/2}$.

4. Ditzian-Totik moduli of smoothness

The next development was the Ditzian-Totik[7] moduli of smoothness (1987), $\omega_k^\varphi(\cdot, f)$. Denote

$$\Delta_h(f,x) := \begin{cases} f(x+h/2) - f(x-h/2), & x \pm h/2 \in [-1,1] \\ 0, & \text{otherwise,} \end{cases}$$

and for $k > 1$, let

$$\Delta_h^k := \Delta(\Delta_h^{k-1}).$$

Then, for $k \ge 1$, the kth Ditzian-Totik modulus of smoothness is defined by

$$\omega_k^\varphi(t,f) := \sup_{h\in[0,t]} \|\Delta_{h\varphi(\cdot)}^k(f,\cdot)\|_{C[0,1]}.$$

The Ditzian-Totik moduli of smoothness enabled the constructive characterization of function classes on the interval. In particular, for $f \in C[-1,1]$, they yielded the estimate,

$$E_n(f) \le c(k)\omega_k^\varphi\left(1/n, f\right), \quad n \ge k,$$

and since

$$\omega_{2m}^\varphi(t, f_m) \simeq c(m)t^{2m},$$

we conclude that

$$E_n(f_m) \le \frac{c(m)}{n^{2m}}, \quad n \ge 1.$$

Thus, in particular, the DT-moduli provide the correct estimates for f_m.

Moreover, Leviatan [18] (1986), has proved that there exists a sequences of convex polynomials $\{P_n\}$, such that

$$\|f_1 - P_n\| \le \frac{c}{n^2}, \quad n \ge 1.$$

In fact, in order to enable us to characterize functions in Babenko classes via their approximation properties, we have extended the DT-moduli.

Given $r \ge 1$, $f \in B^r$ and $k \ge 1$, let

$$\omega_{k,r}^\varphi(f^{(r)}, t) := \sup_{h\in[0,t]} \|W_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)},\cdot)\|_{C[-1,1]},$$

where

$$W_\delta(x) := \begin{cases} (1-x-\delta\varphi(x)/2)^{1/2}(1+x-\delta\varphi(x)/2)^{1/2}, & x \pm \delta\varphi(x)/2 \in [-1,1] \\ 0, & \text{otherwise.} \end{cases}$$

We write $f \in C^r_\varphi$, if $f \in C^{(r)}(-1,1)$ and

$$\lim_{x\to\pm1} f^{(r)}(x)\varphi^r(x) = 0.$$

Then, it follows that for $f \in C(-1,1)$,

$$\lim_{t\to0} \omega^\varphi_{k,r}(f^{(r)},t) = 0 \quad \Longleftrightarrow \quad f \in C^r_\varphi.$$

For functions $f \in C^r_\varphi$ the generalized moduli of smoothness $\omega^\varphi_{k,r}$ have similar properties as the ordinary modulus of smoothness ω_k, e.g.,

$$\omega^\varphi_{k,r}(f^{(r)},nt) \le c(k,r)n^k\omega^\varphi_{k,r}(f^{(r)},t), \quad t \ge 0,$$

and if $f \in C^{r+1}_\varphi$ and $k > 1$, then

$$\omega^\varphi_{k,r}(f^{(r)},t) \le c(k,r)t\omega^\varphi_{k-1,r+1}(f^{(r+1)},t), \quad t \ge 0.$$

Moreover, in a forthcoming paper, with Kopotun, we extend the definition to the L_p metric $1 \le p < \infty$ by setting

$$\omega^\varphi_{k,r}(f^{(r)},t) := \sup_{h\in[0,t]} \|W^r_{kh}(\cdot)\Delta^k_{h\varphi(\cdot)}(f^{(r)},\cdot)\|_{L_p[-1,1]}.$$

5. Approximation of functions on sets in the complex plane

In this section we discuss the possibility of having an analog of (4) for complex approximation. It turns out that it is indeed possible.

Let $G \subset \mathbb{C}$ be a domain with a Jordan boundary ∂G, consisting of l smooth curves Γ_j, such that $z_j := \Gamma_{j-1} \cap \Gamma_j \ne \emptyset$, $j = 1,\dots,l$, where $\Gamma_0 := \Gamma_l$. Denote by $\alpha_j\pi$, $0 < \alpha_j \le 2$, the angle between the curves Γ_{j-1} and Γ_j, at z_j, exterior with respect to the domain G. Set $\overline{G} := G \cup \partial G$, the closure of G.

For a function $g : G \mapsto \mathbb{C}$, denote as usual

$$\|g\|_G := \sup_{z\in G} |g(z)|,$$

and let

$$E_n(g,G) := \inf_{P_n\in\mathbb{P}_n} \|g - P_n\|_G,$$

be the error of the best (complex) polynomial approximation of g. Finally, let Φ be the conformal mapping of the exterior $\mathbb{C} \setminus \overline{G}$ of $\overline{G}$ onto the exterior of the closed unit disk, normalized by $\Phi'(\infty) > 0$. Assume, that there is a neighborhood U of $\overline{G}$, such that

$$c \le \varphi(z)|\Phi'(z)| \le C, \quad z \in U \setminus \overline{G}, \tag{5}$$

where, $c = c(G)$ and $C = C(G)$ are positive constants, that depend only on G, and

$$\varphi(z) := \prod_{j=1}^{l} |z - z_j|^{1-\frac{1}{\alpha_j}}, \quad z \in \mathbb{C},$$

is defined for $z \ne z_j$, if $\alpha_j < 1$. To satisfy (5) one should require that all l smooth curves Γ_j, constituting the boundary ∂G, be "a little more, than smooth", e.g., they may be required to be Ljapunov curves, or somewhat less smooth than Ljapunov curves, the so called Dini-type curves.

Abdullayev and Shevchuk[1] (2005) have proved that *if* $r \in \mathbb{N}$ *and* f *is an analytic function in* G, *then*

$$E_n(f,G) \le \frac{c(r,G)}{n^r}\left\|f^{(r)}\varphi^r\right\|_G, \quad n \ge r. \tag{6}$$

They also proved some inverse theorems. Although it is impossible to have a strong inverse, we have a weak one, with additional $\varepsilon > 0$. Specifically, *let* $r \in \mathbb{N}$, $\varepsilon > 0$ *and* $\alpha_j \ge 1$ *for all* $j = 1, \dots, l$. *If* $f : G \mapsto \mathbb{C}$, *then*

$$\left\|\varphi^r f^{(r)}\right\|_G \le \frac{c(r,G,\varepsilon)}{\varepsilon} \sup_{n \ge r} n^{r+\varepsilon} E_n(f,G). \tag{7}$$

Evidently, if the right hand side of (7) is finite, then f is analytic in G. However, if at least one $\alpha_j < 1$, then even this weak inverse fails to hold as counterexamples are provided by the simplest functions, $f(z) = z^r$ and $f(z) = e^z$.

Furthermore, there are polynomials yielding, in addition to (6), also

$$\left\|P_n^{(r)}\varphi^r\right\|_G \le c\left\|f^{(r)}\varphi^r\right\|_G. \tag{8}$$

For these polynomials the following inverse inequality is valid

$$\left\|f^{(r)}\varphi^r\right\|_G \le \liminf_{n\to\infty}\left(r!\, n^r \|f - P_n\|_G + \left\|P_n^{(r)}\varphi^r\right\|_G\right). \tag{9}$$

Remark. In contrast to the notion "constructive characterization", there is a concept called "approximative characterization" (in Russian "approximativnaya characteristika", see, e.g., [9, p.267]). It is a pair of direct and inverse theorems, involving additional conditions, in our case (6) and (8) correspond to the direct conditions and (9) corresponds to the inverse one. Thus, direct and inverse theorems provide the approximative characterization of the class of analytic functions f, in G, satisfying $\left\|f^{(r)}\varphi^r\right\|_G < +\infty$. Inequalities (6), (8) and (9) imply that *for* $r \in \mathbb{N}$, *if* f *is an analytic in* G *function, then there is a sequence* $\{P_n\}_{n=1}^{\infty}$ *of polynomials* $P_n \in \mathbb{P}_n$, *such that*

$$\exists \lim_{n\to\infty}\left(r!\; n^r \|f - P_n\|_G + \left\|P_n^{(r)}\varphi^r\right\|_G\right) = c(r,G)\left\|f^{(r)}\varphi^r\right\|_G.$$

Actually, more general results are proved in [1]. They show that one "may divide inside the norm sign" both sides of (6) and of (8) by φ^ν, for some ν.

Theorem 1. *Let* $r \in \mathbb{N}$ *and* $0 \le \beta \le r$, *and denote* $\alpha := \min\{1, \alpha_1, \dots, \alpha_l\}$. *If* f *is an analytic function in* G, *then for each* $n \ge lr/\alpha$, *there is a polynomial* $P_n \in \mathbb{P}_n$, *such that*

$$n^r\left\|\frac{(f-P_n)\varphi^\beta}{\varphi^r}\right\|_G + \left\|P_n^{(r)}\varphi^\beta\right\|_G \le c\left\|f^{(r)}\varphi^\beta\right\|_G.$$

Remark. For $\beta = 0$ Theorem 1 is close to Dzjadyk [9, Chapter IX] classical direct theorem and is an analog of the pointwise estimates for $[-1,1]$ by Teljakovski [21], Gopengauz [13] and DeVore [4, 5]. (See also Gonska and Hinnemann [11, 14], and Gonska, Leviatan, Shevchuk and Wenz [12].) Finally, recall that a corresponding "β-bridge" for $[-1,1]$, was proved by Ditzian and Jiang [6].

Theorem 2. *Let $r \in \mathbb{N}$ and $0 \le \beta \le r$. If $f : G \mapsto \mathbb{C}$, then for each sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \mathbb{P}_n$ we have*

$$\left\|f^{(r)}\varphi^{\beta}\right\|_G \le \liminf_{n\to\infty}\left(r!\, n^r \left\|\frac{(f-P_n)\varphi^{\beta}}{\varphi^r}\right\|_G + \left\|P_n^{(r)}\varphi^r\right\|_G\right).$$

Theorems 1 and 2 readily imply:

Let $r \in \mathbb{N}$ and $0 \le \beta \le r$. If f is an analytic function in G, then there is sequence $\{P_n\}_{n=1}^{\infty}$ of polynomials $P_n \in \mathbb{P}_n$ such that,

$$\exists \lim_{n\to\infty}\left(r!\, n^r \left\|\frac{(f-P_n)\varphi^{\beta}}{\varphi^r}\right\|_G + \left\|P_n^{(r)}\varphi^{\beta}\right\|_G\right) = c(r,G)\left\|f^{(r)}\varphi^{\beta}\right\|_G.$$

6. Shape preserving approximation

Leviatan, Radchenko and Shevchuk[19] (2012)considered functions $f \in C[0,1] \cap C^1(-1,1)$ that changes monotonicity finitely many times in $[0,1]$. Let

$$E_n^*(f) := \inf \|f - P_n\|_{C[-1,1]},$$

where the infimum is taken on all $P_n \in \mathbb{P}_n$ such that $P_n'(x)f'(x) \ge 0$, $x \in (-1,1)$, be the error of the best comonotone approximation.

We write $f \in \Delta_s$, if the function changes its monotonicity exactly $s \ge 1$ times in the interval.

Set $A_1 := \{2\}$, and for each $s \ge 2$, let

$$A_s := \{j \mid 1 \le j \le 2\left[\frac{s}{2}\right],\ \text{or}\ j = 2i, 1 \le i \le s\},$$

e.g.,

$$A_2 = \{1,2,4\}, \quad A_3 = \{1,2,4,6\}, \quad A_4 = \{1,2,3,4,6,8\}, \text{ etc.}.$$

Theorem 3. *Given $s \in \mathbf{N}$, let $\alpha > 0$ be such that $\alpha \notin A_s$. If a function $f \in \Delta_s$ satisfies*

$$n^{\alpha}E_n(f) \le 1, \quad n \ge 1, \tag{10}$$

then

$$n^{\alpha}E_n^*(f) \le c(\alpha,s), \quad n \ge 1.$$

Theorem 4. *Given $s \in \mathbf{N}$, there is a constant $c = c(s) > 0$ such that if $\alpha \in A_s$, then for each $m \in \mathbf{N}$, there exists a function $f = f_m \in \Delta_s$, satisfying* (10)*, while*

$$m^{\alpha}E_m^*(f) \ge c(s)\ln m.$$

However, we still have a positive result for $\alpha \in A_s$, namely,

Theorem 5. *Given $s \in \mathbf{N}$, let $\alpha \in A_s$. Then there exist constants $c(s)$ and $N(Y_s)$, such that for each function $f \in \Delta^1(Y_s)$, satisfying* (10)*, we have*

$$n^{\alpha}E_n^{(1)}(f,Y_s) \le c(s), \quad n \ge N(Y_s).$$

In the proofs we applied properties of Babenko classes, the DT-moduli of smoothness and the relations between B_2 and the Zygmund class, all of which, as mentioned above, emanated from the work of I. I. Ibragimov. Specifically, in order to prove Theorem 4, we had to construct a function that is well approximated by algebraic polynomials when no constraints are imposed on the polynomials, but if certain derivatives of these polynomials have to vanish, then they yield weaker approximation rate. Then by adding an oscillating polynomial to the function we have guaranteed that we have an element with s changes of monotonicity without destroying the above two properties. To this end, for even $\alpha \in A_s$, we have constructed such a function, f_α, on $[0, 1]$, in an appropriate Babenko class, more precisely $f_\alpha \in B^\alpha$. For odd $\alpha \in A_s$, we have constructed a continuous odd function, $f_\alpha \in C^{\alpha-1}[-1, 1]$, which is separately in an appropriate Babenko class on $[0, 1]$ and on $[-1, 0]$, more precisely, $f_\alpha^{(\alpha-1)} \in B^2[0, 1]$ and $f_\alpha^{(\alpha-1)} \in B^2[-1, 0]$. Therefore, while we could not have an additional derivative at $x = 0$, we could conclude that $f_\alpha^{(\alpha-1)} \in Z[-1, 1]$ and proceed from there. On the other hand, in proving Theorems 3 and 5, we relied heavily on properties of the Ditzian-Totik moduli of smoothness and, in fact, on their extensions.

It is worth mentioning that similar investigation for coconvex approximation was done by the authors and K. Kopotun [17]. However, in coconvex approximation, there are no results analogous to those of Theorem 3, for $s \geq 2$. Namely, the analogous $A_s = \{\alpha : \alpha > 0\}$, for all $s \geq 2$. Interestingly for $s = 1$, we do have an analog of Theorem 3. Namely, suppose that $f \in C^2[-1, 1]$ changes convexity once in the interval, and define

$$E_n^{**}(f) := \inf \|f - P_n\|_{C[-1,1]},$$

where the infimum is taken on all $P_n \in \mathbb{P}_n$ such that $P_n''(x)f''(x) \geq 0$, $x \in (-1, 1)$, to be the error of the best coconvex approximation.

Theorem 6. *let $\alpha > 0$ be such that $\alpha \neq 4$. If a function $f \in C[-1, 1]$ changes its convexity once in $[-1, 1]$ and satisfies (10), then*

$$n^\alpha E_n^{**}(f) \leq c(\alpha, s), \quad n \geq 1.$$

On the other hand, if $\alpha = 4$, then for any f which changes its convexity once, say, at y_1, and satisfies (10), we have

$$n^4 E_n^{**}(f) \leq c, \quad n \geq \frac{1}{\sqrt{1 - y_1^2}}.$$

However, there is a constant $c > 0$ such that for every $y_1 \in (-1, 1)$, there exists an f which changes convexity at y_1, and satisfies

$$\sup_{n\geq 1} n^4 E_n(f) = 1,$$

such that for each $m \in \mathbb{N}$,

$$m^4 E_m^{**}(f) \geq \left(c \ln \frac{m}{1 + m^2\sqrt{1 - y_1^2}} - 1 \right),$$

and

$$\sup_{n\geq 1} n^4 E_n^{**}(f) \geq c \ln \sqrt{1-y_1^2}.$$

See the recent survey [16] for more results in the shape preserving approximation.

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