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Nikolskii-type estimates for coconvex approximation of functions with one inflection point*

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Abstract

For each $r \in \mathbb{N}$ we prove the Nikolskii type pointwise estimate for coconvex approximation of functions $f \in W^r$, the subspace of all functions $f \in C[-1,1]$, possessing an absolutely continuous $(r-1)$st derivative on $(-1,1)$ and satisfying $f^{(r)} \in L_\infty[-1,1]$, that change their convexity once on $[-1,1]$.

Keywords: Coconvex polynomial approximation, pointwise estimates

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§1. Introduction and main result

Denote by $\mathcal{P}_n$ the space of algebraic polynomials of degree $< n$, $n \in \mathbb{N}$. Set $\|\cdot\| := \|\cdot\|_{L_\infty[-1,1]}$, in particular $\|f\| = \|f\|_{C[-1,1]}$, if $f \in C[-1,1] =: C^0[-1,1]$. For $r \in \mathbb{N}$ let $C^r[-1,1]$ be the space of all r times continuously differentiable functions on $[-1,1]$, and let W^r be the subspace of all functions $f \in C[-1,1]$, possessing an absolutely continuous $(r-1)$st derivative on $(-1,1)$ and satisfying $f^{(r)} \in L_\infty[-1,1]$. Put

$$\varphi(x) := \sqrt{1-x^2}, \quad \varphi_n(x) := \frac{1}{n} + \varphi(x), \quad \text{and} \quad \rho_n(x) := \frac{1}{n}\varphi_n(x).$$

For $f \in C[-1,1]$ denote by

$$E_{n,r}(f) := \inf_{P_n \in \mathcal{P}_n} \left\| \frac{f - P_n}{\varphi_n^r} \right\|,$$

the error of best weighted approximation of f by polynomials $P_n \in \mathcal{P}_n$, with weight φ_n^{-r}.

*Part of this work was done while the first and last authors visited Tel Aviv University

For $f \in W^r$, the Timan estimates

$$E_{n,r}(f) \le c(r)\frac{\|f^{(r)}\|}{n^r}, \quad n \ge r,$$

where $c(r)$ is a constant depending only on r, are well known (see, e.g., [2, p. 381]).

Denote by $\Delta^{(2)}$ the collection of convex functions $f \in C[-1,1]$ and, for $f \in \Delta^{(2)}$, let

$$E^{(2)}_{n,r}(f) := \inf_{P_n \in \mathcal{P}_n \cap \Delta^{(2)}} \left\| \frac{f - P_n}{\varphi_n^r} \right\|,$$

denote the error of best weighted convex approximation. Then, Leviatan [13] (for $r = 1, 2$), and Mania and Shevchuk (see, e.g., [2, Theorem 7.6.5]), (for $r > 2$), have proved that if $f \in W^r \cap \Delta^{(2)}$, then

$$E^{(2)}_{n,r}(f) \le c(r)\frac{\|f^{(r)}\|}{n^r}, \quad n \ge r.$$

That is, the Timan estimates remain valid for convex approximation.

A natural question is what happens for coconvex approximation of piecewise convex functions. We shall discuss here coconvex approximation of continuous functions with one inflection point. To be specific, for a fixed $y \in (-1,1)$ set $Y_1 := \{y\}$. Denote by $\Delta^{(2)}(Y_1)$ the set of continuous on $[-1,1]$ functions f which are convex on $[y,1]$, and concave on $[-1,y]$. In particular, if $f \in C^2[-1,1]$, then $f \in \Delta^{(2)}(Y_1)$, if and only if,

$$f''(x)(x-y) \ge 0, \quad x \in [-1,1].$$

For $f \in \Delta^{(2)}(Y_1)$ denote by

$$E^{(2)}_{n,r}(f, Y_1) := \inf_{P_n \in \mathcal{P}_n \cap \Delta^{(2)}(Y_1)} \left\| \frac{f - P_n}{\varphi_n^r} \right\|,$$

the error of best weighted coconvex approximation. For $r = 1$ or $r = 2$ it was proved by Dzyubenko, Gilewicz, and Shevchuk [4], that if $f \in W^r \cap \Delta^{(2)}(Y_1)$, then

$$E^{(2)}_{n,r}(f, Y_1) \le c(r)\frac{\|f^{(r)}\|}{n^r}, \quad n \ge r. \tag{1}$$

They also proved that if $r > 2$ then (1) is invalid, namely, for each $r > 2$ and $y \in (-1,1)$, there exists a constant $C(Y_1, r) > 0$, such that for every $n \in \mathbb{N}$, there is a function $f \in W^r \cap \Delta^{(2)}(Y_1)$, satisfying

$$E^{(2)}_{n,r}(f, Y_1) > C(Y_1, r)\frac{\|f^{(r)}\|}{n^2}. \tag{2}$$

As it turns out, one may salvage (1) for all $r \in \mathbb{N}$ in a weaker form.

Theorem 1. *For each $r \in \mathbb{N}$, $y \in (-1,1)$ and $f \in W^r \cap \Delta^{(2)}(Y_1)$, there exists an $N = N(f,r,Y_1)$, such that*

$$E_{n,r}^{(2)}(f,Y_1) \le c(r)\frac{\|f^{(r)}\|}{n^r}, \quad n \ge N, \tag{3}$$

where the constant $c(r)$ depends only on r.

Remark 1. For $r=1$ or $r=2$, N in (3) may be taken equal to 1 and 2, respectively (see (1)). If $r>2$, then by virtue of (2), N in (3) may not be taken independent on f.

Remark 2. In this paper we consider just the case of pointwise coconvex approximation, when the function changes its convexity only once, that is, $s=1$. It should be emphasized that the degree of pointwise coconvex approximation of a function that changes its convexity more than once, that is, $s \ge 2$, is investigated in [4] through [7], and that the estimates are quite different for $r>2$. In the latter case, contrary to the case $s=1$, an analog of (3) holds with N independent of f, but with $c=c(r,Y_s)$, where Y_s is the collection of inflection points of f, and it cannot be had with $c=c(r,s)$ and N independent on f. However, note that the arguments of this paper can be easily extended to the case $s>1$, so that the exact analogs of Theorems 1 through 3 hold also for $s>1$.

We are going to discuss here the validity of coconvex analogs of the classical Nikolskii-type pointwise estimates, proved by Timan for $k=1$, Dzyadyk and Freud independently, for $k=2$, and Brudnyi for $k>2$ (for reference see, e.g., [2, p. 381]). Theorem 1 will follow immediately from Theorem 3 below.

§2. Main results and an auxiliary lemma

Although we deal with coconvex polynomial approximation we need first a result about coconvex piecewise polynomial approximation, which will be the main result proved in this paper. In order to formulate it, we need some notations.

We begin by recalling the definition of the kth modulus of smoothness of a function $g \in C[a,b]$,

$$\omega_k(g,t,[a,b]) := \sup_{h\in[0,t]} \left\| \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} g(\cdot + ih) \right\|_{[a,b-kh]}, \quad 0 \le t \le \frac{b-a}{k},$$

where $\|g\|_{[a,b]} := \|g\|_{C[a,b]}$ and, for convenience, we put

$$\omega_k(g,t,[a,b]) \equiv \omega_k\left(g,(b-a)/k,[a,b]\right), \quad t \ge (b-a)/k.$$

Also, we write

$$\omega_k(g,t) := \omega_k(g,t,[-1,1]).$$

A function $\omega : [0,\infty) \mapsto [0,\infty)$, is called a k-majorant, if it is continuous and nondecreasing function on $[0,\infty)$, such that $\omega(0)=0$ and $t^{-k}\omega(t)$ is nonincreasing for

$t>0$. It is well known that each kth modulus of smoothness $\omega_k(g,\cdot)$ has a k-majorant ω, such that $\omega_k(g,t)\le\omega(t)\le 2^k\omega_k(g,t)$, $t\ge 0$.

For each $k\in\mathbb{N}$, $r\in\mathbb{N}$ and a k-majorant ω, let

$$W^rH_k^\omega := \{f:\ f\in C^r[-1,1] \text{ and } \omega_k(f^{(r)},t)\le\omega(t),\ t\ge 0\},$$

be the generalized Hölder class of functions.

Fix $n\in\mathbb{N}$, and denote by $x_{j,n}:=\cos(j\pi/n)$, $\quad j=0,...,n$, the Chebyshev knots. Set $I_{j,n}:=[x_{j,n},x_{j-1,n}]$, $I^0_{j,n}:=[x_{j,n},x_{j-1,n})$, $j=1,...,n$, $|I_{j,n}|=|I^0_{j,n}|:=x_{j-1,n}-x_{j,n}$.

Given $m\in\mathbb{N}$, denote by $\Sigma_{m,n}$, the set of continuous piecewise polynomials s of order $<m$, on $[-1,1]$, with knots $x_{j,n}$, $j=1,...,n-1$. That is, $s\in\Sigma_{m,n}$, if $s\in C[-1,1]$ and

$$s|_{I_{j,n}}=p_{j,n},\quad j=1,...,n,$$

where $p_{j,n}\in\mathcal{P}_m$.

Finally, for $y\in(-1,1)$ and $Y_1:=\{y\}$, denote by $\Sigma_{m,n}(Y_1)$, the collection of all piecewise polynomials in $\Sigma_{m,n}$ satisfying,

$$p_{j_y-1,n}\equiv p_{j_y,n}\equiv p_{j_y+1,n},$$

where $j_y:=j_y(n)$ is the index for which

$$y\in I^0_{j_y,n},$$

and $p_{0,n}:=p_{1,n}$ and $p_{n+1,n}:=p_{n,n}$.

We are ready to state our result for coconvex piecewise polynomial approximation.

Theorem 2. *Let either $r=2$ and $k=1,2,3$, or $r>2$ and $k\in\mathbb{N}$, and assume that ω is a k-majorant. Take $y\in(-1,1)$ and $Y_1=\{y\}$ and denote $m:=k+r$. If $f\in W^rH_k^\omega\cap\Delta^{(2)}(Y_1)$, then there exists an $N=N(f,k,r,Y_1)$, such that for each $n\ge N$, a piecewise polynomial $s\in\Sigma_{m,n}(Y_1)\cap\Delta^{(2)}(Y_1)$ exists, such that*

$$|f(x)-s(x)|\le c(k,r)\,\rho_n^r(x)\,\omega\left(\rho_n(x)\right),\quad x\in[-1,1],$$

where $c(k,r)$ depends only on k and r.

Finally we state our result for coconvex polynomial approximation.

Theorem 3. *Let either $r=2$ and $k=1,2,3$, or $r>2$ and $k\in\mathbb{N}$, and assume that ω is a k-majorant. If $f\in W^rH_k^\omega\cap\Delta^{(2)}(Y_1)$, then there exists an $N=N(f,k,r,Y_1)$, such that for each $n\ge N$ there is a polynomial $P_n\in\mathcal{P}_n\cap\Delta^{(2)}(Y_1)$ satisfying*

$$|f(x)-P_n(x)|\le c(k,r)\,\rho_n^r(x)\,\omega\left(\rho_n(x)\right),\quad x\in[-1,1],$$

where $c(k,r)$ depends only on k and r.

We emphasize that when $r=2$, Theorem 3 is invalid for $k>3$ (see, Gilewicz and Yushchenko [9]), even if we allow both constants c and N to depend on f.

Remark 3. In this paper we are not going to discuss the cases $r = 0$ and $r = 1$. However, we would say a few words about them, without going into proofs. First, if $k + r \le 2$, then it follows from [4] that Theorem 3 is valid with absolute constants c and N. At the other extreme we have the case $k + r \ge 4$, where Theorem 3 is invalid even if we allow both constants c and N to depend on f (see [17] and [18]). So, this leaves us with $k + r = 3$. By (2) we conclude that Theorem 2 (and thus Theorem 3 from which it follows) cannot be had with both constants independent of f. However, it turns out that Theorem 2 is valid with an absolute constant c and $N = N(f)$. The proof follows the lines of the proofs we have here together with arguments from [11] and [12].

The proof of Theorem 3 is long and somewhat tedious. It follows from Theorem 2 by standard techniques in this type of results, (see [1], also see [3], [10], [11], [15], and [16]). Actually, the proof follows the arguments in the long paper (46 pages) [14]. We defer it to a separate paper.

In the sequel the constants c_ν denote fixed positive constants, that may depend only on k and r, which we will keep trace of.

Lemma 1. *Assume that either $r = 2$ and $k \le 3$, or $r \ge 3$. Given $y \in (-1, 1)$, let $[a, b] \subset [-1, 1]$ be an interval such that either $y \in [a + (b - a)/10, b - (b - a)/10]$, or $y \notin [a, b]$. Finally, let ω be a k-majorant, and let $f \in W^r H^\omega_k$, be such that*

$$f''(x)(x - y) \ge 0, \quad x \in [-1, 1].$$

Then, there exists a polynomial $P(\cdot, f, [a, b]) \in \mathcal{P}_m$, $m = k+r$, such that $P(a, f, [a, b]) = f(a)$, satisfying

$$(P'(a, f, [a, b]) - f'(a))(a - y) \ge 0, \tag{4}$$

$$(P'(b, f, [a, b]) - f'(b))(b - y) \le 0, \tag{5}$$

$$P''(x, f, [a, b])(x - y) \ge 0, \quad x \in [a, b], \tag{6}$$

and

$$\|f - P(\cdot, f, [a, b])\|_{[a,b]} \le c_1\, (b - a)^r\, \omega(b - a). \tag{7}$$

Moreover, if $y \notin [a, b]$, then also $P(b, f, [a, b]) = f(b)$.

Proof. If $y \notin [a, b]$, then Lemma 1 is [15, Corollary 2.4], and if $y \in [a + (b - a)/10, b - (b - a)/10]$, then Lemma 1 readily follows from [15, Corollary 2.6] and [8, Corollary 3.1]. Note that the latter is actually a “copositive” Whitney inequality, and that the former reduces the “coconvex” Whitney inequality to the copositive one. □

We require a couple of inequalities relating the lengths of the intervals $I_{j,n}$ and $\rho_n(x)$, $x \in I_{j,n}$. Namely, for each $j = 1, ..., n$, we have

$$|I_{j\pm1,n}| < 3|I_{j,n}|, \tag{8}$$

where for convenience we write $|I_{0,n}| = |I_{n+1,n}| := |I_{1,n}|$, and

$$\rho_n(x) < |I_{j,n}| < 5\rho_n(x), \quad x \in I_{j,n}, \quad n > 1. \tag{9}$$

Inequalities (8) and (9) have appeared, without a proof, in many of the papers in the References (see, e.g., [2, (7.4.1) and (7.4.2)]. As illustration, we prove them for $j \le \frac{n}{2}$. Now

$$\begin{aligned}
1 \le \frac{\sin(j+\frac{1}{2})\frac{\pi}{n}}{\sin(j-\frac{1}{2})\frac{\pi}{n}} &= \frac{|I_{j+1,n}|}{|I_{j,n}|} \\
&= \cos\frac{\pi}{n} + \sin\frac{\pi}{n}\cot\left(j-\frac{1}{2}\right)\frac{\pi}{n} \\
&< 1 + \frac{1}{j-1/2} \le 3,
\end{aligned}$$

$$\begin{aligned}
\frac{|I_{j,n}|}{\rho_n(x)} &\le \frac{|I_{j,n}|}{\rho_n(x_{j-1,n})} \\
&= \frac{2\sin^2\frac{\pi}{2n}\cos(j-1)\frac{\pi}{n} + \sin\frac{\pi}{n}\sin(j-1)\frac{\pi}{n}}{\frac{1}{n^2} + \frac{1}{n}\sin(j-1)\frac{\pi}{n}} \\
&< \frac{\pi^2}{2} < 5,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\rho_n(x)}{|I_{j,n}|} \le \frac{\rho_n(x_{j,n})}{|I_{j,n}|} &= \frac{1}{n^2|I_{j,n}|} + \frac{1}{2n}\cot\frac{\pi}{2n} + \frac{1}{2n}\cot\left(j-\frac{1}{2}\right)\frac{\pi}{n} \\
&\le \frac{1}{n^2|I_{j,n}|} + \frac{1}{n}\cot\frac{\pi}{2n} \\
&< \frac{1}{n^2|I_{1,n}|} + \frac{2}{\pi} \le \frac{1}{4} + \frac{2}{\pi} < 1.
\end{aligned}$$

§3. Proof of Theorem 2

Set

$$J_y := \left(y - \frac{1}{2}(1-|y|), y + \frac{1}{2}(1-|y|)\right),$$

and

$$h_n := \rho_n^r(y)\,\omega(\rho_n(y)), \quad n \in \mathbb{N}.$$

Since for all $x \in J_y$, $\frac{1}{2}\varphi(x) < \varphi(y) < 2\varphi(x)$, we obtain that,

$$c_2^{-1}h_n \le \rho_n^r(x)\,\omega(\rho_n(x)) \le c_2\,h_n, \quad x \in J_y. \tag{10}$$

First, suppose that $f''(x) \equiv 0$ for $x \in J_y$, so that $f(x) =: \ell(x)$, is a linear function there. Recall that $j_y = j_y(n)$ is such that $y \in I^0_{j_y,n}$ and take N so big that $I_{j_y \pm 2,n} \subset J_y$, for all $n \geq N$. Then we may take

$$P(\cdot, f, I_{j_y+1,n}) = P(\cdot, f, I_{j_y,n}) = P(\cdot, f, I_{j_y-1,n}) = \ell.$$

Hence, by virtue of Lemma 1 and (9) the required piecewise polynomial s may be defined by

$$s|_{I_{j,n}} = p_{j,n} := P(\cdot, f, I_{j,n}), \quad j = 1, ..., n.$$

Otherwise, we may assume that there is an $y_0 \in J_y$, such that $f''(y_0) \neq 0$. Subtracting, if necessary, a linear polynomial, we may assume that $f(y_0) = 0$ and $f'(y_0) = 0$, and without loss of generality $f''(y_0) > 0$, so that $y < y_0 < y^0 := y + \frac{1}{2}(1 - |y|)$. Now, since f is convex on $[y, y^0]$, we conclude that

$$\frac{1}{3} \min\{f(y), f(y^0)\} =: \delta > 0.$$

For each $n \in \mathbb{N}$, let $j_0 = j_0(n)$ and $j^0 = j^0(n)$ be the indices such that

$$y_0 \in I^0_{j_0,n} \quad \text{and} \quad y^0 \in I^0_{j^0,n},$$

respectively. We take N so big that for all $n \geq N$, we have $y < x_{j_0,n}$ and

$$f(x_{j_y-2,n}) > 2\delta, \quad f(x_{j_0,n}) < \delta, \quad f(x_{j_0-1,n}) < \delta, \quad f(x_{j^0,n}) > 2\delta,$$

and

$$c_3 h_n < \delta$$

where $c_3 := c_1(35)^m$.

Evidently, f is nonincreasing in $[y, y_0]$, whence the above inequalities imply that $j_y - 2 \geq j_0 + 1$.

Fix $n \geq N$, so that we may drop it from the notation in the index of $x_{j,n}$, $p_{j,n}$, $I_{j,n}$, and $I^0_{j,n}$. As first step in the construction of the required piecewise polynomial s, we put, by virtue of Lemma 1,

$$s|_{[x_{j_y+1}, x_{j_y-2}]} := P(\cdot, f, [x_{j_y+1}, x_{j_y-2}]),$$

and denote

$$d := s(x_{j_y-2}) - f(x_{j_y-2}).$$

Since $x_{j_y-2} - x_{j_y+1} < 7|I_{j_y}|$, it follows by (7) that,

$$|d| \leq c_1 \, (x_{j_y-2} - x_{j_y+1})^r \omega(x_{j_y-2} - x_{j_y+1}) \leq c_3 h_n,$$

whence $|d| < |\delta|$, and (10) yields

$$|d| \leq c_4 \, \rho_n^r(x) \omega(\rho_n(x)), \quad x \in J. \tag{11}$$

If $d = 0$, then we take $p_j := P(\cdot, f, I_j)$, for all $j = 1, ..., j_y - 2$ and $j = j_y + 2, ..., n$, and again Lemma 1 and (9) imply that we have constructed the required piecewise polynomial s.

Therefore we assume that $d \neq 0$.

Case I: $d > 0$. Let ℓ be the tangent to f at $(x_{j_0}, f(x_{j_0}))$. Since f is convex in $[x_{j_0}, x_{j^0}]$ and $\ell'(x_{j_0}) = f'(x_{j_0}) < 0$, and we recall that $f(x_{j_0}) < \delta$, $f(x_{j^0}) > 2\delta$, and $d < \delta$, it follows that there exists a unique point $x_* \in (x_{j_0}, x_{j^0})$, such that

$$\ell(x_*) + d = f(x_*).$$

Denote by j_* the index for which $x_* \in I^0_{j_*}$. Clearly $j_0 \geq j_*$. Let s on the interval $[x_{j_y+1}, x_{j_y-2}]$ be as above. By virtue of Lemma 1, we put for all $j = 1, \ldots, j_* - 1$ and for all $j = j_y + 2, \ldots, n$,

$$p_j := P(\cdot, f, I_j),$$

while for all $j = j_0 + 1, \ldots, j_y - 2$, we put

$$p_j := P(\cdot, f, I_j) + d.$$

Thus we have constructed s on the intervals $[-1, \alpha]$ and $[\beta, 1]$, where

$$\alpha := x_{j_0} \quad \text{and} \quad \beta := x_{j_*-1}.$$

By Lemma 1, (9) and (11) the piecewise polynomial s satisfies all conditions required by Theorem 2 on these two intervals. Moreover,

$$s'(\alpha-) \leq f'(\alpha) \quad \text{and} \quad s'(\beta+) \geq f'(\beta).$$

In order to complete the construction of s and hence the proof of Theorem 2 in Case I, we will construct a polynomial $p_* \in \mathcal{P}_m$, satisfying

$$p_*(\alpha) = f(\alpha) + d, \quad p_*(\beta) = f(\beta), \tag{12}$$

$$p'_*(\alpha) \geq f'(\alpha), \quad p'_*(\beta) \leq f'(\beta), \tag{13}$$

$$p''_*(x) \geq 0, \quad x \in [\alpha, \beta], \tag{14}$$

and

$$|f(x) - p_*(x)| \leq c_5 \rho^r_n(x) \omega(\rho_n(x)), \quad x \in [\alpha, \beta]. \tag{15}$$

If one set

$$g(x) := f(x) - \ell(x) \quad \text{and} \quad q(x) := p_*(x) - \ell(x),$$

then (12) through (15) are equivalent to

$$q(\alpha) = d, \quad q(\beta) = g(\beta), \tag{16}$$

$$q'(\alpha) \ge 0, \quad q'(\beta) \le g'(\beta), \tag{17}$$

$$q''(x) \ge 0, \quad x \in [\alpha, \beta], \tag{18}$$

and

$$|g(x) - q(x)| \le c_5 \rho_n^r(x) \omega(\rho_n(x)), \quad x \in [\alpha, \beta]. \tag{19}$$

We will show that

$$q(x) := d + \lambda P(x),$$

where

$$P(x) := \begin{cases} P(x, g, [\alpha, \beta]), & \text{if} \quad j_* = j_0, \\ g(\beta)\frac{x-\alpha}{\beta-\alpha}, & \text{otherwise}, \end{cases}$$

and λ is taken so that

$$q(\beta) = g(\beta),$$

satisfies (16) through (19). We first note that the latter is possible with $0 < \lambda < 1$. Indeed, this follows since g is convex in $[\alpha, \beta]$, $g(\alpha) = g'(\alpha) = 0$, so that g is increasing there, and $x_* < \beta$. Hence

$$d = g(x_*) < \|g\|_{[\alpha,\beta]} = g(\beta).$$

Now $P(\beta) = g(\beta)$, thus

$$\lambda = 1 - \frac{d}{g(\beta)}. \tag{20}$$

The other part of (16) readily follows from the equality $P(\alpha) = g(\alpha) = 0$, while (18) follows by (6). If $j_0 = j_*$, then (17) follows from (4) and (5). If $j_0 \ne j_*$, then we have

$$g'(\alpha) = 0 < \lambda \frac{g(\beta)}{\beta - \alpha} = q'(\alpha) = \lambda \frac{g(\beta) - g(\alpha)}{\beta - \alpha} < \lambda g'(\beta) < g'(\beta),$$

where in the second inequality we applied the convexity of g.

Finally we establish (19). To this end, since

$$g - q = \lambda(g - P) + (1 - \lambda)g - d,$$

it follows by (20) that

$$\begin{aligned} \|g - q\|_{[\alpha,\beta]} &\le \|g - P\|_{[\alpha,\beta]} + (1 - \lambda)\|g\|_{[\alpha,\beta]} + d \\ &= \|g - P\|_{[\alpha,\beta]} + 2d. \end{aligned}$$

If $j_* = j_0$, then $[\alpha, \beta] = I_{j_0}$, so that (19) follows from (7), (9), and (11).

If $j_0 \ne j_*$, then evidently, $I_{j_*+1} \subset [\alpha, x_*]$. Hence

$$\|g\|_{I_{j_*+1}} \le \|g\|_{[\alpha,x_*]} = d \le c_6 |I_{j_*+1}|^r \omega(|I_{j_*+1}|).$$

Since by (8), $x_{j_*-1}-x_{j_*+1} < 4|I_{j_*+1}|$, and $g \in W^r H_k^\omega$ and $\|g\|_{I_{j_*+1}} \le c_6 |I_{j_*+1}|^r \omega(|I_{j_*+1}|)$, it follows that (see, e.g., [2, Lemma 4.1.1])

$$\|g\|_{[x_{j_*+1}, x_{j_*-1}]} \le c_7 (x_{j_*-1} - x_{j_*+1})^r \omega(x_{j_*-1} - x_{j_*+1}).$$

This, in turn, together with (9), implies

$$\begin{aligned} \|g-q\|_{[\alpha,\beta]} &\le \|g-P\|_{[\alpha,\beta]} + 2d \\ &\le \|g\|_{[\alpha,\beta]} + \|P\|_{[\alpha,\beta]} + 2d = 2g(\beta) + 2d \\ &= 2\|g\|_{[x_{j_*+1}, x_{j_*-1}]} + 2d \\ &\le c_8 (x_{j_*-1} - x_{j_*+1})^r \omega(x_{j_*-1} - x_{j_*+1}), \end{aligned}$$

which yields (19), and Theorem 2 is proved when $d > 0$.

Case II: $d < 0$. Let ℓ be the tangent to f at the point $(x_{j_0-1}, f(x_{j_0-1}))$. Since f is convex in $[x_{j_y-2}, x_{j_0-1}]$, $\ell'(x_{j_0-1}) = f'(x_{j_0-1}) > 0$, $f(x_{j_0-1}) < \delta$, $|d| < \delta$, and $f(x_{j_y-2}) > 2\delta$, we conclude that there exists a unique point $x_* \in (x_{j_y-2}, x_{j_0} - 1)$, such that

$$\ell(x_*) - d = f(x_*).$$

Denote by j_* the index for which $x_* \in I^0_{j_*}$. Again, let s be defined on $[x_{j_y+1}, x_{j_y-2}]$ as above, and by virtue of Lemma 1, we put for all $j = 1, \ldots, j_0 - 1$ and for all $j = j_y + 2, \ldots, n$,

$$p_j := P(\cdot, f, I_j),$$

while for all $j = j_* + 1, \ldots, j_y - 2$, we set

$$p_j := d + P(\cdot, f, I_j).$$

Thus we have constructed s on the intervals $[-1, \alpha]$ and $[\beta, 1]$, where

$$\alpha := x_{j_*} \quad \text{and} \quad \beta := x_{j_0-1}.$$

By Lemma 1, (9) and (11) the piecewise polynomial s satisfies all conditions required by Theorem 2 on these two intervals. Moreover,

$$s'(\alpha-) \le f'(\alpha) \quad \text{and} \quad s'(\beta+) \ge f'(\beta).$$

So again, in order to complete the construction of s and hence the proof of Theorem 2 in Case II, we will construct a polynomial $p_* \in \mathcal{P}_m$, satisfying (12) through (15).

If one set

$$g(x) := f(x) - \ell(x) \quad \text{and} \quad q(x) := p_*(x) - \ell(x),$$

then (12) through (15) are equivalent to (16) through (19).

Just as in Case I, one can show that the polynomial satisfying (16) through (19), may be taken in the form

$$q(x) := \lambda P(x),$$

where

$$P(x) := \begin{cases} P(x, g, [\alpha, \beta]), & \text{if} \quad j_* = j_0, \\ \\ g(\alpha)\frac{x-\beta}{\alpha-\beta}, & \text{otherwise,} \end{cases}$$

and λ is taken so that

$$q(\alpha) = g(\alpha) + d.$$

This completes the proof of Theorem 2. □

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