Comparing the degrees of unconstrained and shape preserving approximation by polynomials [*†]

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Abstract

Let $f\in C[-1,1]$ and denote by $E_n(f)$ its degree of approximation by algebraic polynomials of degree $<n$. Assume that f changes its monotonicity, respectively, its convexity finitely many times, say $s\ge 2$ times, in $(-1,1)$ and we know that for $q=1$ or $q=2$ and some $1<\alpha\le 2$, such that $q\alpha\ne 4$, we have

$$E_n(f)\le n^{-q\alpha},\quad n\ge s+q+1,$$

The purpose of this paper is to prove that the degree of comonotone, respectively, coconvex approximation, of f, by algebraic polynomials of degree $<n$, $n\ge N$, is also $\le c(\alpha,s)n^{-q\alpha}$, where the constant N depends only on the location of the extrema, respectively, inflection points in $(-1,1)$ and on α.

This answers, affirmatively, questions left open by the authors in papers with Kopotun and Vlasiuk (see the list of references).

1 Introduction and main results

Let $C[a,b]$, $-1\le a<b\le 1$, denote the space of continuous functions on $[a,b]$ equipped with the usual uniform norm, $\|f\|_{[a,b]}:=\max_{a\le x\le b}|f(x)|$. When dealing with $[-1,1]$, we suppress referring to the interval, namely, we denote $\|f\|:=\|f\|_{[-1,1]}$. For $\mathbb{P}_n$ is the space of algebraic polynomials of degree $<n$ and $f\in C[-1,1]$, denote by

$$E_n(f):=\inf_{p_n\in\mathbb{P}_n}\|f-p_n\|,$$

the degree of approximation of f by algebraic polynomials of degree $<n$.

Given $s\ge 1$, denote by $\mathbb{Y}_s$, the set of all collections $Y_s=\{y_i\}_{i=1}^s$, of points y_i, such that $y_{s+1}:=-1<y_s<\cdots<y_1<1=:y_0$. For such a collection we write

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$f \in \Delta^{(1)}(Y_s)$ if $f \in C[-1,1]$ and $(-1)^i f$ is nondecreasing on $[y_{i+1}, y_i]$, $0 \le i \le s$. Similarly, we write $f \in \Delta^{(2)}(Y_s)$ if $f \in C[-1,1]$ and $(-1)^i f$ is convex on $[y_{i+1}, y_i]$, $0 \le i \le s$.

For $f \in \Delta^{(q)}(Y_s)$, $q \in \{1,2\}$, we denote by

$$E_n^{(q)}(f, Y_s) := \inf_{P_n \in \mathbb{P}_n \cap \Delta^{(q)}(Y_s)} \|f - P_n\|$$

the degree of best comonotone, respectively, coconvex approximation of f relative to Y_s.

Assuming that for some $\alpha > 0$ and $N \ge 1$,

$$n^\alpha E_n(f) \le 1, \quad n \ge N, \tag{1.1}$$

the answer to the following question was provided (see [3], [4], [5] and [9]).

If (1.1) *holds for an* $f \in \Delta^{(q)}(Y_s)$*, is it possible to have constants* $c(q, \alpha, s, N)$ *and* N^* *such that*

$$n^\alpha E_n^{(q)}(f, Y_s) \le c(q, \alpha, s, N), \quad n \ge N^*? \tag{1.2}$$

Here N^*, if it exists, may depend on q, α, s and N, but may also depend of Y_s or even on f. It turns out that N^* always exists and its dependence on the various parameters, in all cases, but $1 < \alpha \le 2$, $N = s+2$, $s \ge 2$, for the comonotone case ($q = 1$), was given in [5] and [9] and, in all cases, but $2 < \alpha \le 4$, $N = s+3$, $s \ge 3$, for the coconvex case ($q = 2$), was given in [3] and [4].

O. V. Vlasiuk [10], has attempted to close the above gaps, but, regrettably, the proof of the main lemma there is incorrect (see [11]). Our main results are the following.

Theorem 1.1. *Given* $Y_s \in \mathbb{Y}_s$*,* $s \ge 2$*, and* $1 < \alpha \le 2$*. Then, there exist constants* $c(\alpha, s)$ *and* $N^*(\alpha, Y_s)$*, such that for all functions* $f \in \Delta^{(1)}(Y_s)$ *satisfying* (1.1) *with* $N = s+2$*,* (1.2) *with* $q = 1$*, holds.*

And

Theorem 1.2. *Given* $Y_s \in \mathbb{Y}_s$*,* $s \ge 3$*, and* $2 < \alpha < 4$*. Then, there exist constants* $c(\alpha, s)$ *and* $N^*(\alpha, Y_s)$*, such that for all functions* $f \in \Delta^{(2)}(Y_s)$ *satisfying* (1.1) *with* $N = s+3$*,* (1.2) *with* $q = 2$*, holds.*

Remark 1.3. *Note that this leaves open what happens in the coconvex case when* $\alpha = 4 < s+3 = N$*.*

In Section 2 we bring some auxiliary lemmas and in Section 3 we prove Theorems 1.1 and 1.2. Throughout the paper, k, r, s, q, i, j and n, are nonnegative integers, while α, a, b, h, t, u and v, are real numbers.

In the sequel, constants c will denote constants which may depend on s and, perhaps on α (we will not detail that), and may differ from one occurrence to another, even when they appear in the same line; constants $c_1, c_2, \dots$ will denote specific such constants the values of which remain the same during the paper; constants C will denote constants which, in addition, depend on Y_s, and may differ from one occurrence to another; and constants $C_1, C_2, \dots$, will denote specific such constants the values of which remain the same during the paper.

2 Auxiliary results

For $g \in C[a,b]$, denote by

$$\Delta_h^k(g,x) := \begin{cases} \sum_{i=0}^k (-1)^{k-i}\binom{k}{i} g(x-(k/2-i)h), & \text{if } x \pm kh/2 \in [a,b], \\ 0, & \text{otherwise,} \end{cases}$$

the kth symmetric difference, and define the ordinary kth modulus of smoothness of g by

$$\omega_k(g,t;[a,b]) := \sup_{0<h\le t} \|\Delta_h^k(g,\cdot)\|_{[a,b]}.$$

Let

$$\varphi(x) := \sqrt{1-x^2},$$

and for $\delta > 0$, denote

$$\varphi_\delta(x) := \begin{cases} \sqrt{(1-\delta\varphi(x)/2)^2 - x^2}, & x \pm \delta\varphi(x)/2 \in [-1,1], \\ 0, & \text{otherwise.} \end{cases}$$

The weighted DT modulus of smoothness of a function $f \in C^r(-1,1)$, is defined by

$$\omega_{k,r}^\varphi(f^{(r)},t) := \sup_{0<h\le t} \left\| \varphi_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)},\cdot) \right\|.$$

In particular, if $r = 0$, then

$$\omega_k^\varphi(f,t) := \omega_{k,0}^\varphi(f,t),$$

is the (ordinary) kth DT modulus, [1].

It is known (see, e.g., [2]) that if $r \ge 1$, then $\omega_{k,r}^\varphi(f^{(r)},t) \to 0$, as $t \to 0$, if and only if $\lim_{x\to\pm1} \varphi^r(x) f^{(r)}(x) = 0$. Therefore, we denote $C_\varphi^0 := C[-1,1]$ and, for $r \ge 1$,

$$C_\varphi^r := \{f \in C[-1,1] \cap C^r(-1,1) \mid \lim_{x\to\pm1} \varphi^r(x) f^{(r)}(x) = 0\}.$$

The interrelations between the two moduli are the subject of the following result.

Denote

$$\phi(a,b) := \sqrt{(1+a)(1-b)}.$$

Lemma 2.1. *Let $-1 < a < b < 1$, $k \ge 1$ and $r \ge 1$, be given. If $g \in C_\varphi^r$, then*

$$\omega_k(g^{(r)},t;[a,b]) \le \frac{1}{\phi^r(a,b)} \omega_{k,r}^\varphi\left(g^{(r)}, \frac{t}{\phi(a,b)}\right), \quad t > 0, \tag{2.1}$$

and

$$\omega_k(g^{(r)},t;[a,b]) \le \frac{1}{\phi^r(a,b)} \omega_{k,r}^\varphi(g^{(r)}, \sqrt{2t/k}), \quad t > 0. \tag{2.2}$$

Proof. Let $x \in [a,b]$ and $0 < h \le t$, be such that $x \pm kh/2 \in [a,b]$. Then

$$\begin{aligned} |\Delta_h^k(g^{(r)},x)| &= \frac{\varphi_{kh}^r(x)}{\varphi_{kh}^r(x)} \left|\Delta_{\frac{h}{\varphi(x)}\varphi(x)}^k(g^{(r)},x)\right| \\ &\le \frac{1}{\phi^r(a,b)} \varphi_{kh}^r(x) \left|\Delta_{\frac{h}{\varphi(x)}\varphi(x)}^k(g^{(r)},x)\right| \\ &\le \frac{1}{\phi^r(a,b)} \omega_{k,r}^\varphi\left(g^{(r)}, \frac{h}{\varphi(x)}\right) \\ &\le \frac{1}{\phi^r(a,b)} \omega_{k,r}^\varphi\left(g^{(r)}, \frac{h}{\phi(a,b)}\right), \end{aligned}$$

and (2.1) follows.

Also, since $|x| + kh/2 < 1$, it readily follows that $h < \frac{2}{k}\varphi^2(x)$. Hence $h < \sqrt{\frac{2h}{k}}\varphi(x)$, and (2.2) follows from the above second inequality. □

Next, we quote the following auxiliary lemma, see [2, Theorem 7.1.2] (see also [4, Theorem 3.3]).

Lemma 2.2. *Let $r \ge 0$, $k \ge 1$ and $\alpha > 0$, be such that $r < \alpha < k + r$, and let $f \in C[-1,1]$. If*

$$n^\alpha E_n(f) \le 1, \quad n \ge k + r, \tag{2.3}$$

then $f \in C^r_\varphi$ and

$$\omega^\varphi_{k,r}(f^{(r)}, t) \le A(\alpha, k, r)t^{\alpha - r}, \quad t > 0, \tag{2.4}$$

where $A(\alpha, k, r) = const$, depends only on α, k and r.

Henceforth, let

$$1 < \alpha \le 2, \quad q \in \{1, 2\}, \quad q\alpha \ne 4,, \quad s \ge 1 \quad \text{and} \quad Y_s \in \mathbb{Y}_s,$$

be given.

The following is needed in dealing with the endpoints.

Lemma 2.3. *There is a constant C_1, such that for any $0 < h < C_1$ and every function $g \in C^q_\varphi$, satisfying*

$$\omega^\varphi_{s+1,q}(g^{(q)}, t) \le t^{q\alpha - q}, \tag{2.5}$$

$$g^{(q)}(y_i) = 0, \quad i = 1, \dots, s, \tag{2.6}$$

and

$$g^{(q)}(1 - h^2) = 0, \tag{2.7}$$

we have,

$$|g^{(q)}(u)| \le c_1 \frac{h^{q(\alpha-1)}}{(1-u)^{q/2}}, \quad 1 - h^2 \le u < 1. \tag{2.8}$$

Proof. Let $h < \frac{1}{2}\phi(y_s, y_1)$, and take $t := \sqrt{1-u} \le h$. Then by virtue of (2.2) and (2.5),

$$\begin{aligned}\omega_{s+1}(g^{(q)}, v; [y_s, 1 - t^2]) &\le \frac{\omega^\varphi_{s+1,q}(g^{(q)}, \sqrt{v})}{(1 + y_s)^{q/2} t^q} \\ &\le \frac{v^{q(\alpha-1)/2}}{(1 + y_s)^{q/2} t^q}.\end{aligned}$$

Hence, by (2.6) and (2.7), Whitney's inequality implies,

$$\|g^{(q)}\|_{[y_s, 1-t^2]} \le C\omega_{s+1}(g^{(q)}, 1; [y_s, 1 - t^2]) \le \frac{C}{t^q}.$$

Similarly, with $y_s^0 := \max\{0, y_s\}$, we get by (2.2) and (2.5),

$$\omega_{s+1}(g^{(q)}, v; [y_s^0, 1-t^2]) \le \frac{\omega^{\varphi}_{s+1,q}(g^{(q)}, \sqrt{v})}{t^q} \le \frac{v^{q(\alpha-1)/2}}{t^q}.$$

By Marchaud's inequality, we obtain for $\tau := h^2 - t^2$,

$$\begin{aligned}
|g^{(q)}(1-t^2)| &= |g^{(q)}(1-t^2) - g^{(q)}(1-h^2)| \le \omega_1(g^{(q)}, \tau; [y_s^0, 1-t^2]) \\
&\le c\tau \int_\tau^1 \frac{\omega_{s+1}(g^{(q)}, v; [y_s^0, 1-t^2])}{v^2} dv + \frac{c}{1-y_s^0} \tau \|g^{(q)}\|_{[y_s, 1-t^2]} \\
&\le c\tau \int_\tau^1 \frac{v^{q(\alpha-1)/2}}{t^q v^2} dv + \frac{C_2 \tau}{t^q} \\
&\le c_2 \frac{\tau^{q(\alpha-1)/2}}{t^q} + \frac{C_2 \tau}{t^q} \\
&\le \frac{c_2 h^{q(\alpha-1)}}{t^q} + \frac{C_2 h^2}{t^q} \le \frac{2c_2 h^{q(\alpha-1)}}{t^q},
\end{aligned}$$

provided h is so small that $C_2 h^{2-q(\alpha-1)} \le c_2$.

This proves (2.8) and completes our proof. □

Remark 2.4. *Clearly, in Lemma* 2.3*, one may replace* (2.7) *by* $g^{(q)}(-1+h^2) = 0$, *and arrive at similar conclusions for* $-1 < u \le -1 + h^2$.

Lemma 2.5. *Let* $g \in C^q_\varphi$ *satisfy* (2.5) *and* (2.6)*, and let* $0 < h < C_1$*. If* $g^{(q)}(1-h^2) \ge 0$*, then there exists a polynomial* $P_+(x) = P_+(x; 1-h^2)$*, of degree* $s+q$*, such that* $P_+^{(q)}(x) \ge 0$, $x \in [1-h^2, 1]$, $P_+^{(j)}(1-h^2) = g^{(j)}(1-h^2)$, $0 \le j \le q-1$, *and*

$$|g(x) - P_+(x)| \le 2c_1 h^{q\alpha}, \quad 1-h^2 \le x \le 1. \tag{2.9}$$

Proof. Let

$$p(x) := g^{(q)}(1-h^2) \prod_{i=1}^{s} \frac{x - y_i}{1 - h^2 - y_i},$$

so that p is nonnegative in $[1-h^2, 1]$.

Denote

$$P_+(x) := g(1-h^2) + (q-1)g'(1-h^2)(x-1+h^2) + \int_{1-h^2}^{x} (x-u)^{q-1} p(u) du.$$

Then $P_+^{(q)}(x) \ge 0$, $x \in [1-h^2, 1]$, $P_+^{(j)}(1-h^2) = g^{(j)}(1-h^2)$, $0 \le j \le q-1$, and if we set $G(x) := g(x) - P(x)$, we observe that G satisfies (2.5), (2.6) and (2.7). It follows from Lemma 2.3 that

$$|g^{(q)}(u) - P_+^{(q)}(u)| \le c_1 \frac{h^{q(\alpha-1)}}{(1-u)^{q/2}}, \quad 1-h^2 \le u < 1.$$

Hence,

$$\begin{aligned}
|g(x) - P_+(x)| &\le c_1 h^{q(\alpha-1)} \int_{1-h^2}^{x} \frac{(x-u)^{q-1}}{(1-u)^{q/2}} du \\
&\le c_1 h^{q(\alpha-1)} \int_{1-h^2}^{1} (1-u)^{q/2-1} du = \frac{2c_1}{q} h^{q\alpha},
\end{aligned}$$

and (2.9) is proved. □

Remark 2.6. *Clearly, in Lemma 2.5, one may replace* $g^{(q)}(1-h^2)\ge 0$ *by* $g^{(q)}(-1+h^2)\ge 0$*, and arrive at similar conclusions for a polynomial* $P_-(x)=P_-(x;-1+h^2)$*. Similarly, if one replaces* $g^{(q)}(1-h^2)\ge 0$ *by* $g^{(q)}(-1+h^2)\le 0$*, then one arrives at analogous, modified, conclusions.*

The next two lemmas are applied in the neighborhoods of the points y_i, $1\le i\le s$.

Denote

$$d:=\frac{1}{2}\min_{1\le i\le s+1}(y_{i-1}-y_i),$$

and put

$$y_1^*:=y_1+d \quad \text{and} \quad y_s^*:=y_s-d.$$

Lemma 2.7. *There is a constant* $C_3\le d$*, such that for any* $0<h\le C_3$ *and every function* $g\in C_\varphi^q$*, satisfying* (2.5) *and* (2.6)*, if for some* $1\le i^*\le s$*,*

$$g^{(q)}(y_{i^*}+h)=0, \tag{2.10}$$

then

$$\|g^{(q)}\|_{[y_s^*,y_1^*]}\le C_4\int_h^2 t^{q\alpha-q-2}dt, \tag{2.11}$$

and

$$\|g^{(q)}\|_{[y_{i^*}-h,y_{i^*}-h]}\le \frac{c_5h^{q\alpha-q}}{\varphi(y_{i^*})^{q\alpha}}. \tag{2.12}$$

Proof. Note that (2.1) and (2.5) imply that

$$\omega_{s+1}(g^{(q)},t;[y_s^*,y_1^*])\le Ct^{q\alpha-q}. \tag{2.13}$$

Denote by $L_s(x):=L_s(x;g^{(q)};y_1,\dots,y_s,y_1^*)$, the Lagrange polynomial, of degree s, that interpolates $g^{(q)}$ at the points y_i, $i=1,\dots,s$ and at y_1^*, and note that by virtue of (2.6),

$$L_s(x)=g^{(q)}(y_1^*)\prod_{i=1}^s\frac{x-y_i}{y_1^*-y_i},$$

whence

$$\|L_s\|_{[y_s^*,y_1^*]}\le C|g^{(q)}(y_1^*)|. \tag{2.14}$$

Set

$$G(x):=g^{(q)}(x)-L_s(x),$$

so that $G(y_1^*)=G(y_i)=0$, $1\le i\le s$.

Evidently,

$$\omega_{s+1}(G,t;[y_s^*,y_1^*])=\omega_{s+1}(g^{(q)},t;[y_s^*,y_1^*]),\quad t>0.$$

Therefore, by (2.13) and Whitney's inequality,

$$\|G\|_{[y_s^*,y_1^*]}\le C\omega_{s+1}(G,1;[y_s^*,y_1^*])\le C. \tag{2.15}$$

Thus, by (2.14),

$$\|g^{(q)}\|_{[y_s^*,y_1^*]}\le\|G\|_{[y_s^*,y_1^*]}+\|L_s\|_{[y_s^*,y_1^*]}\le C+C|g^{(q)}(y_1^*)|. \tag{2.16}$$

so that we need to estimate $|g^{(q)}(y_1^*)|$.

To this end, denote $y^* := y_{i^*} + h$. Then,

$$\begin{aligned}|g^{(q)}(y_1^*)| =&|(y_1^* - y_1)\cdots(y_1^* - y_s)(y_1^* - y^*)|[g^{(q)}; y_1^*, y_1, \ldots, y_s, y^*]|\\ \le&2^s|(y_1^* - y^*)[g^{(q)}; y_1^*, y_1, \ldots, y_s, y^*]|\\ =&2^s|(y_1^* - y^*)[G; y_1^*, y_1, \cdots, y_s, y^*]|\\ =&2^s|(y^* - y_1)\ldots(y^* - y_s)|^{-1}|G(y^*)|\\ \le&\frac{C}{h}|G(y^*)| = \frac{C}{h}|G(y^*) - G(y_{i^*})|.\end{aligned}$$

Hence, Marchaud's inequality and (2.15) imply

$$\begin{aligned}|g^{(q)}(y_1^*)| \le&\frac{C}{h}\omega_1(G, h; [y_s^*, y_1^*])\\ \le&C\int_h^{y_1^*-y_s^*}\frac{t^{q\alpha-q}}{t^2}dt + \frac{c}{y_1^* - y_s^*}\|G\|_{[y_s^*,y_1^*]}\\ \le&C\int_h^2 t^{q\alpha-q-2}dt + C.\end{aligned}$$

Thus, by (2.16), we conclude that

$$\|g^{(q)}\|_{[y_s^*,y_1^*]} \le C\int_h^2 t^{q\alpha-q-2}dt + C \le C\int_h^2 t^{q\alpha-q-2}dt,$$

and (2.11) is proved.

Now denote $y_{i^*}^{\pm} := y_{i^*} \pm d$ and note that (2.1) and (2.5) yield,

$$\omega_{s+1}(g^{(q)}, t; [y_{i^*}^-, y_{i^*}^+]) \le \frac{c}{\varphi^q(y_{i^*})}\omega_{s+1,q}\left(g^{(q)}, \frac{ct}{\varphi(y_{i^*})}\right) \le c\frac{t^{q\alpha-q}}{\varphi^{q\alpha}(y_{i^*})},$$

where we used the fact that $\varphi(y_{i^*}) \le c\phi(y_{i^*}^-, y_{i^*}^+)$.

It follows by Marchaud's inequality that,

$$\begin{aligned}\omega_2(g^{(q)}, t; [y_{i^*}^-, y_{i^*}^+]) \le& ct^2\int_t^{2d}\frac{\omega_{s+1}(g^{(q)}, u; [y_{i^*}^-, y_{i^*}^+])}{u^3}du + c\frac{t^2}{d}\|g^{(q)}\|_{[[y_{i^*}^-,y_{i^*}^+]}\\ \le& \frac{c}{\varphi^{q\alpha}(y_{i^*})}t^2\int_t^{\infty}u^{q\alpha-q-3}du + c\frac{t^2}{d}\|g^{(q)}\|_{[y_s^*,y_1^*]}\\ =& \frac{c_6}{\varphi^{q\alpha}(y_{i^*})}t^{q\alpha-q} + c\frac{t^2}{d}\|g^{(q)}\|_{[y_s^*,y_1^*]}.\end{aligned}$$

In particular,

$$\begin{aligned}\omega_2(g^{(q)}, h; [y_{i^*} - h, y_{i^*} + h]) \le&\omega_2(g^{(q)}, h; [y_{i^*} - d, y_{i^*} + d])\\ \le&\frac{c_6}{\varphi^{q\alpha}(y_{i^*})}h^{q\alpha-q} + C_5h^2\int_h^2 t^{q\alpha-q-2}dt\\ \le&\frac{2c_6}{\varphi^{q\alpha}(y_{i^*})}h^{q\alpha-q},\end{aligned}$$

where for the middle inequality we applied (2.11), and provided we take C_3 small enough.

Since $g^{(q)}(y_{i^*}) = g^{(q)}(y_{i^*} + h) = 0$, (2.12) now follows by Whitney's inequality. □

Lemma 2.8. *Let $0 < h_1 \le C_3$, $0 < h_2 \le C_3$ and $y_{i^*} \in Y_s$, be given. If a function $g \in C^q_\varphi$ satisfies (2.6) and (2.5), and $g^{(q)}(x)(x - y_{i^*}) \ge 0$, $x \in [y_{i^*} - d, y_{i^*} + d]$, then there exists a polynomial $P_*(x) = P_*(x; y_{i^*} - h_1, y_{i^*} + h_2; y_{i^*})$, of degree $s+q$, such that $P_*^{(q)}(x)(x - y_{i^*}) \ge 0$, $x \in [y_{i^*} - d, y_{i^*} + d]$, satisfying*

$$\|g - P_*\|_{[y_{i^*}-h_1, y_{i^*}+h_2]} \le \frac{c_7 h^{q\alpha}}{\varphi^{q\alpha}(y_{i^*})}, \tag{2.17}$$

where $h := \max\{h_1, h_2\}$, and

$$P_*(y_{i^*} - h_1) = g(y_{i^*} - h_1). \tag{2.18}$$

If $q = 2$, then, in addition, either

$$P_*'(y_{i^*} - h_1) = g'(y_{i^*} - h_1) \quad \text{and} \quad P_*'(y_{i^*} + h_2) \le g'(y_{i^*} + h_2), \tag{2.19}$$

or

$$P_*'(y_{i^*} - h_1) \le g'(y_{i^*} - h_1) \quad \text{and} \quad P_*'(y_{i^*} + h_2) = g'(y_{i^*} + h_2). \tag{2.20}$$

Proof. Set

$$p_{i^*}(x) := g^{(q)}(y_{i^*} + h) \prod_{i=1}^{s} \frac{x - y_i}{y_{i^*} + h - y_i}.$$

Evidently, since $h \le d$, p_{i^*} is nonpositive in $[y_{i^*} - d, y_{i^*}]$ and nonnegative in $[y_{i^*}, y_{i^*} + d]$.

Now, let

$$G(x) := \int_0^x (x-t)^{q-1}\big(g^{(q)}(t) - p_{i^*}(t)\big)dt.$$

Then, G satisfies the assumptions of Lemma 2.7. Hence, by (2.12),

$$\begin{aligned} \|g^{(q)} - p_{i^*}\|_{[y_{i^*}-h_1, y_{i^*}+h_2]} &\le \|g^{(q)} - p_{i^*}\|_{[y_{i^*}-h, y_{i^*}+h]} \\ &= \|G^{(q)}\|_{[y_{i^*}-h, y_{i^*}+h]} \le \frac{c_4 h^{q\alpha - q}}{\varphi^{q\alpha}(y_{i^*})}. \end{aligned} \tag{2.21}$$

For $q = 1$, let

$$P_*(x) := g(y_{i^*} - h_1) + \int_{y_{i^*}-h_1}^{x} p_{i^*}(u)du.$$

Then (2.18) holds, and P_* is comonotone with g on $[y_{i^*} - h_1, y_{i^*} + h_2]$. Finally, by (2.21),

$$\|g - P_*\|_{[y_{i^*}-h_1, y_{i^*}+h_2]} \le \frac{2c_4 h^{\alpha}}{\varphi^{\alpha}(y_{i^*})}.$$

For $q = 2$, we apply [8, Corollary 2.6] with g instead of f, $\beta = y_{j_{i^*}}$ and $P_{k-1} = p_{i^*}$, and conclude that there exists a polynomial P_* of degree $s+2$ such that it satisfies (2.18), and (2.19) or (2.20), and by (2.21),

$$\begin{aligned} \|g - P_*\|_{[y_{i^*}-h_1, y_{i^*}+h_2]} &\le ch^2 \|g'' - p_{i^*}\|_{[y_{i^*}-h_1, y_{i^*}+h_2]} \\ &\le \frac{ch^{2\alpha}}{\varphi^{2\alpha}(y_{i^*})}. \end{aligned}$$

This completes the proof. □

Remark 2.9. *Note that if* $g^{(q)}(x)(x-y_{i^*})\le 0$, $x\in[y_{i^*}-h_1,y_{i^*}+h_2]$, *then the same proof yields a polynomial* P_*, *of degree* $s+q$, *satisfying* (2.17) *and* (2.18) *and, if* $q=2$, *then, in addition, either*

$$P'_*(y_{i^*}-h_1)=g'(y_{i^*}-h_1)\quad\text{and}\quad P'_*(y_{i^*}+h_2)\ge g'(y_{i^*}+h_2), \tag{2.22}$$

or

$$P'_*(y_{i^*}-h_1)\ge g'(y_{i^*}-h_1)\quad\text{and}\quad P'_*(y_{i^*}+h_2)= g'(y_{i^*}+h_2). \tag{2.23}$$

The following lemma is an immediate consequence of [6, p. 125, Lemma 2], for the monotone case, and of [8, Corollary 2.4], for the convex case. We will give a few details.

Lemma 2.10. *Let* $g\in C^q_\varphi$ *be such that* (2.5) *holds. Let* $-1<a<a+\frac{4}{3}h<1$, *and assume that* $g^{(q)}(x)\ge 0$, $x\in[a,a+h]$. *Then, there exists a polynomial* $P(x)=P(x;a,a+h)$, *of degree* $s+q$, *such that* $P^{(q)}(x)\ge 0$, $x\in[a,a+h]$, *satisfying*

$$\|g-P\|_{[a,a+h]}\le \frac{c_8h^{q\alpha}}{\varphi^{q\alpha}(a)}, \tag{2.24}$$

and

$$P(a)=g(a)\quad\text{and}\quad P(a+h)=g(a+h). \tag{2.25}$$

If $q=2$, *then, in addition, either*

$$P'(a)=g'(a)\quad\text{and}\quad P'(a+h)\le g'(a+h), \tag{2.26}$$

or

$$P'(a)\ge g'(a)\quad\text{and}\quad P'(a+h)= g'(a+h). \tag{2.27}$$

Proof. First we note that since $a+\frac{4}{3}h<1$, we have $\phi(a,a+h)\le\varphi(a)\le 2\phi(a,a+h)$.

For $q=1$, our lemma follows directly from [6, Lemma 2] with $k=s+1$ and $r=1$, when we apply the estimate (2.5).

For $q=2$, we obtain by virtue of Lemma 2.1 and (2.5),

$$\omega_{s+1}(g'',t,[a,a+h])\le\frac{1}{\phi^2(a,a+h)}\left(\frac{t}{\phi(a,a+h)}\right)^{2\alpha-2}\le\frac{ct^{2\alpha-2}}{\varphi^{2\alpha}(a)}.$$

Hence, by [8, Corollary 2.4], there exists a convex polynomial p_j, of degree $s+2$, such that (2.25), and either (2.26), or (2.27) hold. Moreover,

$$\|g-P\|_{[a,a+h]}\le ch^2\omega_{s+1}(g'',h,[a,a+h])\le\frac{ch^{2\alpha}}{\varphi^{2\alpha}(a)}.$$

This completes the proof. □

Remark 2.11. *Note that if* $g^{(q)}(x)\le 0$, $x\in[a,a+h]$, *then the same proof yields a polynomial* P, *of degree* $s+q$, *such that* $P^{(q)}(x)\le 0$, $x\in[a,a+h]$, *interpolates* g *at both ends of the interval and satisfies* (2.24) *and, if* $q=2$, *then, in addition, it satisfies either*

$$P'(a)=g'(a)\quad\text{and}\quad P'(a+h)\ge g'(a+h), \tag{2.28}$$

or

$$P'(a)\le g'(a)\quad\text{and}\quad P'(a+h)= g'(a+h). \tag{2.29}$$

Denote by $x_{n,j} := \cos(j\pi/n)$, $j = 0,\dots,n$, the Chebyshev partition of order n. Further, denote $I_{n,j} := [x_{n,j}, x_{n,j-1}]$, $j = 1,\dots,n$ and let $|I_{n,j}| := x_{n,j-1} - x_{n,j}$.

Given $g \in C^q_\varphi \cap \Delta^{(q)}$, we now construct a continuous piecewise polynomial S_n on the Chebyshev partition, that is,

$$S_n|_{I_{n,j}} = P_j \quad j = 1,\dots n,$$

where P_j are algebraic polynomials, so that S_n is comonotone, respectively, coconvex with g, and is sufficiently close to it. We take $N = N(Y_s)$ so big that $\frac{2\pi}{N} \le \min\{C_1^2, C_3\}$, thus $|I_{n,j}| \le \frac{1}{2}\min\{C_1^2, C_3\}$ for all $n \ge N$ and $j = 1,\dots,n$.

Lemma 2.12. *If a function $g \in C^q_\varphi \cap \Delta^{(q)}$ satisfies (2.5), then for each $n \ge N(Y_s)$ there is a piecewise polynomial, $S_n(x) = S_n(x;g)$, on the Chebyshev partition, such that*

$$S_n|_{I_{n,j}} = P_j \in \mathbb{P}_{q+s+1}, \quad j = 1,\dots n, \tag{2.30}$$

$$P_{j\pm1} \equiv P_j, \quad \text{if} \quad y_i \in [x_{n,j}, x_{n,j-1}), \quad i = 1\dots,s, \tag{2.31}$$

$$S_n \in \Delta^{(q)}(Y_s), \tag{2.32}$$

and

$$\|g - S_n\| \le \frac{c}{n^{q\alpha}}. \tag{2.33}$$

Proof. Fix $n \ge N$ and, for simplicity set $x_j := x_{n,j}$, $I_j := I_{n,j}$ and $|I_j| := |I_{n,j}|$. For each $i = 1,\dots,s$ denote by j_i the index for which $y_i \in [x_{j_i}, x_{j_i-1})$.

If $2 \le j \le n-1$, $j \ne j_i, j_i\pm1$, $1 \le i \le s$, then we denote $P_j(x) := P(x; x_j, x_{j-1})$, with P from Lemma 2.10, or Remark 2.11, as the case may be. Then P_j is of degree $s+q$, satisfies $\operatorname{sgn} P_j^{(q)}(x) = \operatorname{sgn} g^{(q)}(x)$, $x \in I_j$, interpolates g at both x_j and x_{j-1} and, if $q = 2$, such that either (2.26) or (2.27) holds. Also, by virtue of (2.24),

$$\|g - P_j\|_{I_j} \le \frac{c_4|I_j|^{q\alpha}}{\varphi(x_j)^{q\alpha}} \le \frac{c}{n^{q\alpha}}, \tag{2.34}$$

where we used the inequality $\frac{|I_j|}{\varphi(x_j)} \le \frac{c}{n}$.

Next, we denote $P_{j_i\pm1}(x) = P_{j_i}(x) := P_*(x; x_{j_i+1}, x_{j_i-2}; y_{j_i})$, where P_* is defined in Lemma 2.8. Then P_{j_i} is a polynomial of degree $s+q$, comonotone, respectively, coconvex with g on $[x_{j_i+1}, x_{j_i-2}]$, with $P_{j_i}(x_{j_i+1}) = g(x_{j_i+1})$ and, if $q = 2$, such that either (2.19) or (2.20) holds with P_{j_i} instead of P_*. Also, in view of (2.17), P_{j_i} satisfies

$$\|g - P_{j_i}\|_{[x_{j_i+1}, x_{j_i-2}]} \le \frac{c|I_{j_i}|^{q\alpha}}{\varphi(y_{j_i})^{q\alpha}} \le \frac{c}{n^{q\alpha}}, \tag{2.35}$$

where we used the fact that $\max\{y_{j_i} - x_{j_i+1}, x_{j_i-2} - y_{j_i}\} \le c|I_{j_i}|$, and $\frac{|I_{j_i}|}{\varphi(y_{j_i})} \le \frac{c}{n}$.

Note that it follows by (2.35) that,

$$|\delta_i := g(x_{j_i-2}) - P_{j_i}(x_{j_i-2})| \le \frac{c}{n^{q\alpha}}, \quad 1 \le i \le s. \tag{2.36}$$

Finally, we have to define P_1 and P_n. We Denote $P_1(x) := P_+(x; x_1)$, with P_+ from Lemma 2.9. The polynomial $P_n(x) := P_-(x; x_{n-1})$, is obtained in the same way, applying Remark 2.6.

We are ready to define, S_n. Denote $j_0 := 3$ and $j_{s+1} := n + 2$, and set

$$S_n|_{[x_j, x_{j-1}]} := P_j + \sum_{k=1}^{i} \delta_k, \quad j_i - 2 \le j \le j_{i+1} - 2, \quad 0 \le i \le s.$$

It follows by our construction that S_n is continuous, comonotone, respectively, coconvex with g, and since by (2.36),

$$\sum_{i=1}^{s} |\delta_i| \le \frac{c}{n^{q\alpha}},$$

S_n satisfies (2.33). This completes our construction. □

3 Proof of the theorems

We summarize the lemmas in the following theorem from which both Theorems 1.1 and 1.2 follow. We devote this section to proving it.

Theorem 3.1. *For each* $1 < \alpha \le 2$, $q \in \{1, 2\}$, $q\alpha \ne 4$, *and* $s \ge 1$, *there is a constant* $c = c(\alpha, s)$ *and for each* $Y_s \in \mathbf{Y}_s$ *there is a constant* $N^* = N^*(\alpha, Y_s)$, *such that for every function* $f \in \Delta^{(q)}(Y_s)$, *satisfying*

$$E_n(f) \le n^{-q\alpha}, \quad n \ge s + q + 1, \tag{3.1}$$

we have

$$E_n^{(q)}(f, Y_s) \le cn^{-q\alpha}, \quad n \ge N^*.$$

Proof. First we observe that by Lemma 2.2, (3.1) implies that

$$f \in C_\varphi^q \quad \text{and} \quad \omega_{s+1,q}(f^{(q)}, t) \le ct^{q\alpha - q}, \quad t > 0. \tag{3.2}$$

Therefore we may apply Lemma 2.12 with $g = cf$. Denote by $S_n(x) = S_n(x; cf)$, the piecewise polynomial, guaranteed by Lemma 2.12.

Also, by Lemma 2.2, (3.1) implies that

$$\omega_{s+1+q}^\varphi(f, t) \le ct^{q\alpha}.$$

Hence, by (2.33)

$$\omega_{s+1+q}^\varphi(S_n, 1/n) \le \omega_{s+1+q}^\varphi(f, 1/n) + c\|f - S_n\| \le \frac{c}{n^{q\alpha}}. \tag{3.3}$$

Observe that (2.30) and (2.31) imply that, if $q = 1$, then, in the notation of [6], $S_n \in \Sigma_{s+2,O(Y_s,n)}$ and, if $q = 2$, then, in the notation of [7], $S_n \in \Sigma_{s+3,n}(Y_s)$. Therefore, by virtue of (2.32), [6, p. 137, Proposition 3], for $q = 1$, and [7, p. 24, Theorem 3], for $q = 2$, we conclude by (3.3) that there exists a polynomial $Q_n \in \Delta^q(Y_s)$ of degree $\le cn$, such that

$$\|S_n - Q_n\| \le c\omega_{s+1+q}^\varphi(S_n, \frac{1}{n}) \le \frac{c}{n^{q\alpha}},$$

which, in turn, by (2.33) yields

$$\|f - Q_n\| \le \|f - S_n\| + \|S_n - Q_n\| \le \frac{c}{n^{2\alpha}}.$$

This completes our proof. □

References

[1] Z. Ditzian and V. Totik, *Moduli of smoothness*, Springer Series in Computational Mathematics, vol. 9, Springer-Verlag, New York, 1987.

[2] V. K. Dzyadyk and I. A. Shevchuk, *Theory of Uniform Approximation of Functions by Polynomials*, Walter de Gruyter, Berlin, 2008.

[3] K. A. Kopotun, D. Leviatan, and I. A. Shevchuk, *Are the degrees of best (co)convex and unconstrained polynomial approximation the same?*, Acta Math. Hungar. **123** (2009), 273–290.

[4] ———, *Are the degrees of the best (co)convex and unconstrained polynomial approximations the same? II*, Ukrainian Math. J. **62** (2010), 369–386.

[5] D. Leviatan, D. V. Radchenko, and I. A. Shevchuk, *Positive results and counterexamples in comonotone approximation*, Constr. Approx. **36** (2012), 243–266.

[6] D. Leviatan and I. A. Shevchuk, *Some positive results and counterexamples in comonotone approximation II*, J. Approx. Theory **100** (1999), 113–143.

[7] ———, *Coconvex approximation*, J. Approx. Theory **118** (2002), 20–65.

[8] ———, *Coconvex polynomial approximation*, J. Approx. Theory **121** (2003), 100–118.

[9] D. Leviatan, I. A. Shevchuk, and O. V. Vlasiuk, *Positive results and counterexamples in comonotone approximation II*, J. Approx. Theory **179** (2014), 1–23.

[10] O. V. Vlasiuk, *On the degree of piecewise shape-preserving approximation by polynomials*, J. Approx. Theory **189** (2015), 67–75.

[11] ———, *Corrigendum to "On the degree of piecewise shape-preserving approximation by polynomials" [J. Approx. Theory 189 (2015), 67–75]*, J. Approx. Theory **?** (2015), ?.