POSITIVE RESULTS AND COUNTEREXAMPLES IN COMONOTONE APPROXIMATION II

D. LEVIATAN, I. A. SHEVCHUK, AND O. V. VLASIUK

Abstract. Let $E_n(f)$ denote the degree of approximation of $f \in C[-1,1]$, by algebraic polynomials of degree $< n$, and assume that we know that for some $\alpha > 0$ and $N \ge 2$,

$$n^\alpha E_n(f) \le 1, \quad n \ge N.$$

Suppose that f changes its monotonicity $s \ge 1$ times in $[-1,1]$. We are interested in what may be said about its degree of approximation by polynomials of degree $< n$ that are comonotone with f. In particular, if f changes its monotonicity at $Y_s := \{y_1, \dots, y_s\}$ and the degree of comonotone approximation is denoted by $E_n(f, Y_s)$, we investigate when can one say that

$$n^\alpha E_n(f, Y_s) \le c(\alpha, s, N), \quad n \ge N^*,$$

for some N^*. Clearly, N^*, if it exists at all (we prove it always does), depends on α, s and N. However, it turns out that for certain values of α, s and N, N^* depends also on Y_s and in some cases even on f itself. The results extend previous results in the case $N = 1$.

1. introduction and the main result

Let $\mathbb{P}_n$ be the space of algebraic polynomials of degree $< n$, and let $C[-1,1]$ be the space of continuous functions on $[-1,1]$ equipped with the uniform norm $\|f\| = \max_{x\in[-1,1]} |f(x)|$. For $f \in C[-1,1]$, denote by

$$E_n(f) = \inf_{P_n \in \mathbb{P}_n} \|f - P_n\|,$$

the degree of approximation of f by algebraic polynomials of degree $< n$.

Given $s \ge 1$, denote by $\mathbb{Y}_s$, the set of all collections $Y_s = \{y_i\}_{i=1}^s$, of points y_i, such that $y_0 := -1 < y_1 < \cdots < y_s < 1 =: y_{s+1}$. For such a collection we write $f \in \Delta^{(1)}(Y_s)$ if $f \in C[-1,1]$ is nondecreasing on $[y_s, 1]$, nonincreasing on $[y_{s-1}, y_s]$ and so forth, so that finally, $(-1)^s f$ nondecreasing on $[-1, y_1]$, in particular, the collection Y_s is the set of extreme points of f. Since $f \in \Delta^{(1)}(Y_s)$ is differentiable a.e. in $(-1,1)$, we have

$$f'(x) \prod_{i=1}^{s} (x - y_i) \ge 0,$$

a.e. there. We say that $f \in \Delta_s^{(1)}$ if there exists $Y_s \in \mathbb{Y}_s$ such that $f \in \Delta^{(1)}(Y_s)$. Note that there may be more than one such $Y_s \in \mathbb{Y}_s$. It is also possible that $f \in \Delta_s^{(1)}$ for different s's.

Key words and phrases. Comonotone polynomial approximation, Degree of approximation, Uniform norm.
Mathematics Subject Classification: 41A10, 41A25, 41A29.
Part of this work was done when the first author was visiting City University of Hong Kong.

For $f \in \Delta^{(1)}(Y_s)$ we denote by

$$E_n^{(1)}(f, Y_s) := \inf_{P_n \in \mathbb{P}_n \cap \Delta^{(1)}(Y_s)} \|f - P_n\|$$

the degree of best comonotone approximation of f relative to Y_s. We will also use the notation

$$E_n^{1,s}(f) := \sup_{Y_s \in \mathbb{Y}_s : f \in \Delta^{(1)}(Y_s)} E_n^{(1)}(f, Y_s), \tag{1.1}$$

for the worst possible degree of best comonotone approximation of $f \in \Delta_s^{(1)}$, for the given s.

Given $f \in \Delta^{(1)}(Y_s)$, satisfying

$$n^\alpha E_n(f) \le 1, \quad n \ge 1, \tag{1.2}$$

for some $\alpha > 0$, the question of the validity of the estimate

$$n^\alpha E_n^{(1)}(f, Y_s) \le c(\alpha, s), \quad n \ge N^*, \tag{1.3}$$

and if valid, how does N^* depend on α, s, Y_s and f, has been investigated in [9, Theorem 2 through 4]. It was shown that for each $s \ge 1$, there is an exceptional discrete set of α's, denoted by A_s, such that, if $f \in \Delta^{(1)}(Y_s)$ satisfies (1.2) and $\alpha \notin A_s$, then (1.3) is valid with $N^* = 1$. For $\alpha \in A_s$, (1.3) holds with $N^* = N^*(Y_s)$ and it is impossible to achieve it with N^* independent of Y_s.

The purpose of the present paper is to investigate a similar question when we have less information about $f \in \Delta^{(1)}(Y_s)$. Namely, if we only know that for some $N \ge 2$,

$$n^\alpha E_n(f) \le 1, \quad n \ge N, \tag{1.4}$$

what may we conclude about

$$n^\alpha E_n^{(1)}(f, Y_s) \le c(\alpha, s, N), \quad n \ge N^*? \tag{1.5}$$

Specifically, we will show that if $f \in \Delta^{(1)}(Y_s)$ satisfies (1.4), then necessarily (1.5) holds with some N^*, which might, in general, depend on f, on Y_s, on α and on N. We will investigate the dependence of N^* on these parameters.

Throughout the paper we will denote by $c(\alpha, s, \dots, N)$ different constants that may depend only on the parameters inside the parentheses. We will use $C(\alpha, Y_s, \dots, N, f)$ for constants that may depend also on sets or functions.

There are three possibilities and in order to describe them we use the following notation.

Definition. The symbol "+" means that (1.5) holds for $N^* = N^*(\alpha, s, N)$;

the symbol "$\oplus$" means that (1.5) holds for $N^* = N^*(\alpha, Y_s, N)$ and does not hold without the dependence on Y_s, that is, for each $A > 0$ and $M > 0$, there are a number $m > M$, a collection $Y_s^* \in \mathbb{Y}_s$ and a function $f \in \Delta^{(1)}(Y_s^*)$ satisfying (1.4), for which $m^\alpha E_m^{(1)}(f, Y_s^*) \ge A$;

and the symbol "$\ominus$" means that (1.5) holds only for $N^* = N^*(\alpha, Y_s, f, N)$, and does not hold without the dependence on f itself, that is, for each $A > 0$ and $M > 0$, and for every $Y_s \in \mathbb{Y}_s$ there are a function $f \in \Delta^{(1)}(Y_s)$, satisfying (1.4), and a number $m > M$, for which $m^\alpha E_m^{(1)}(f, Y_s) \ge A$.

Indeed, we rule out the possibility (usually denoted by "$-$") that for some triple (Y_s, α, N), an $f \in \Delta^{(1)}(Y_s)$ satisfying (1.4), exists, for which there is no N^* at all (see Section 2).

Remark 1. Note that if for some s and α, we have "+" for some N_0, then we also have "+" for all $N \le N_0$. Similarly, if we have "$\oplus$" for some N_0, then for any $N \le N_0$, we cannot have $\ominus$. On the other hand if for some N_0 we have $\ominus$, then we have $\ominus$ for all $N \ge N_0$.

Remark 2. We should emphasize that except for $N \le s+1$, N^* cannot be smaller than N. Indeed, if $N \ge s+2$ and f_s is a polynomial such that $f_s'(x) = A(x+2)^{N-s-2}\prod_{i=1}^{s}(x-y_i)$, with $A > 0$ arbitrary, then clearly $f_s \in \Delta^{(1)}(Y_s)$. Since f_s is a polynomial of degree $N-1$, it follows that $E_n(f_s) = 0$ for all $n \ge N$, thus satisfying (1.4). However, since A is arbitrary, assuming $N^* < N$, (1.5) immediately leads to a contradiction. On the other hand, if $N \le s+1$, then $\mathbb{P}_N \cap \Delta^{(1)}(Y_s) = \mathbb{P}_1 \cap \Delta^{(1)}(Y_s)$, since any polynomial of degree $\le s$ which changes monotonicity s times must be constant. Hence $E_N^{(1)}(f, Y_s) = E_1^{(1)}(f, Y_s) = E_1(f)$, so that if (1.5) holds with $N^* = N$, then it already holds with $N^* = 1$.

Remark 3. We should also emphasize (see Theorem 3 in Section 6) that in all cases where we have " + ", one may take $N^* = N$.

The paper is devoted to proving the following result (the case $N = 1$ was proved in [9]).

Theorem 1. *For every triple (α, s, N), $\alpha > 0$, $s \in \mathbb{N}$, and $N \in \mathbb{N}$, there exists a constant $c(\alpha, s, N)$, satisfying the following properties. If $f \in \Delta^{(1)}(Y_s)$, $Y_s \in \mathbb{Y}_s$, and*

$$n^\alpha E_n(f) \le 1, \quad n \ge N,$$

then

$$n^\alpha E_n^{(1)}(f, Y_s) \le c(\alpha, s, N), \quad n \ge N^*,$$

where
(i) " + ", $N^* \le N$, *if*
(*a*) *α is not an odd integer and $\alpha < s$, or α is not an even integer and $\alpha < 2s$, and $N \le \lceil \alpha/2 \rceil$;*
or
(*b*) *$2s < \alpha \le 2s+2$ and $N \le s+2$;*
or
(*c*) *$\alpha > 2s+2$ and all $N \ge 1$.*
(ii) " $\ominus$ ", $N^* = N^*(\alpha, Y_s, f, N)$, *if*
(*a*) *$\lceil \alpha \rceil = 1$ and $N \ge s+2$;*
or
(*b*) *$\lceil \alpha \rceil = 2$ and $N \ge s+3$.*
(iii) " $\oplus$ ", $N^* = N^*(\alpha, Y_s, N)$, *in all other cases, except, perhaps, the case $\lceil \alpha \rceil = 2 \le s$ and $N = s+2$.*

For the sake of comparison, we mention that in the case of monotone approximation (i.e., $s = 0$, $Y_0 = \emptyset$, and $\Delta^{(1)}(Y_0)$ is the set of nondecreasing functions $f \in C[-1, 1]$), the third possibility, obviously, cannot be present. It is known (see, e.g., [5, Section 11, Table 14]) that we may summarize the results for $s = 0$ in the following table.

$\lceil \alpha/2 \rceil$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
2	+	+	+	$\cdots$
1	+	+	$\ominus$	$\cdots$
	1	2	3	N

Fig. 1, $s = 0$

The results for $s \geq 2$ presented in this paper may be illustrated in the following tables, where we require two more symbols, namely,

$$\overset{\oplus}{+} := \begin{cases} \oplus, \text{ if either } \alpha \text{ is odd and } < s, \text{ or } \alpha \text{ is even and } \leq 2s, \\ +, \text{ otherwise,} \end{cases}$$

and

$$\overset{?}{\ominus} := \begin{cases} \ominus \text{ when } \alpha \leq 1, \\ ? \text{ when } 1 < \alpha \leq 2. \end{cases}$$

A table for $s \geq 4$.

$\lceil \alpha/2 \rceil$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
$s+2$	$+$	$+$	$+$	$\cdots$	$+$	$+$	$+$	$+$	$+$	$\cdots$
$s+1$	$+$	$+$	$+$	$\cdots$	$+$	$+$	$+$	$+$	$\oplus$	$\cdots$
s	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\cdots$	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
$s-1$	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\cdots$	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
1	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\oplus$	$\overset{?}{\ominus}$	$\ominus$	$\cdots$
	1	2	3	$\cdots$	$s-1$	s	$s+1$	$s+2$	$s+3$	N

Fig. 2, $s \geq 4$

The question mark in entry $(1, s+2)$ indicates that when $N = s+2$, we do not know for any given $1 < \alpha \leq 2$, whether the correct symbol should be $\oplus$ or $\ominus$.

For the benefit of the reader we present the tables for $s = 2$ and $s = 3$, with the vertical axis of $\lceil \alpha \rceil$ rather than $\lceil \alpha/2 \rceil$. It demonstrates the above pattern, but for clarity we separate the rows for $\alpha \leq 1$ and $1 < \alpha \leq 2$.

$\lceil \alpha \rceil$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
7	$+$	$+$	$+$	$+$	$+$	$\cdots$
6	$+$	$+$	$+$	$+$	$\oplus$	$\cdots$
5	$+$	$+$	$+$	$+$	$\oplus$	$\cdots$
4	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$+$	$+$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$?$	$\ominus$	$\cdots$
1	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	N

Fig. 3, $s = 2$

$\lceil\alpha\rceil$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
9	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
8	$+$	$+$	$+$	$+$	$+$	$\oplus$	$\cdots$
7	$+$	$+$	$+$	$+$	$+$	$\oplus$	$\cdots$
6	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
5	$+$	$+$	$+$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
4	$\overset{\oplus}{+}$	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$+$	$+$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\oplus$	?	$\ominus$	$\cdots$
1	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	6	N

Fig. 4, $s = 3$

The table for $s = 1$ is somewhat different and again we present it with the vertical axis of $\lceil\alpha\rceil$ rather than $\lceil\alpha/2\rceil$, since for $s = 1$ and $N = 3$, we do know that for $1 < \alpha \le 2$ we have $\oplus$.

$\lceil\alpha\rceil$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$ $\iddots$
5	$+$	$+$	$+$	$+$	$\cdots$
4	$+$	$+$	$+$	$\oplus$	$\cdots$
3	$+$	$+$	$+$	$\oplus$	$\cdots$
2	$\overset{\oplus}{+}$	$\oplus$	$\oplus$	$\ominus$	$\cdots$
1	$+$	$\oplus$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	N

Fig. 5, $s = 1$

2. Auxiliary results

We begin with a few notions. Let $g \in C[a, b]$, the space of continuous functions on $[a, b]$, with the uniform norm. Denote by

$$\Delta_h^k(g, x) := \begin{cases} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} g(x - (k/2 - i)h), & \text{if } x \pm kh/2 \in [a, b], \\ 0, & \text{otherwise}, \end{cases}$$

the kth symmetric difference, and define the ordinary kth modulus of smoothness of g by

$$\omega_k(g, t; [a, b]) := \sup_{0<h\le t} \|\Delta_h^k(g, \cdot)\|_{[a,b]}.$$

If $[a, b] = [-1, 1]$, we suppress the reference to the interval, that is, we write $\omega_k(g, t) := \omega_k(g, t; [-1, 1])$.

Let $\varphi := \sqrt{1 - x^2}$ and denote

$$\varphi_\delta(x) := \sqrt{(1 - x - \delta\varphi(x)/2)\,(1 + x - \delta\varphi(x)/2)} = \sqrt{(1 - \delta\varphi(x)/2)^2 - x^2}.$$

The weighted D-T modulus of smoothness of a function $f \in C^r(-1,1)$, is defined by

$$\omega_{k,r}^{\varphi}(f^{(r)},t) := \sup_{0<h\le t} \left\| \varphi_{kh}^r(\cdot)\Delta_{h\varphi(\cdot)}^k(f^{(r)},\cdot) \right\|.$$

In particular, if $r=0$, then

$$\omega_k^{\varphi}(f,t) := \omega_{k,0}^{\varphi}(f,t),$$

is the (usual) kth D-T modulus.

It is known (see, e.g., [1, 11]) that $\omega_{k,r}^{\varphi}(f^{(r)},t)$ is bounded for all $t>0$, if and only if $f \in \mathbb{B}^r$, the Babenko class, that is, f possesses a locally absolutely continuous $(r-1)$st derivative in $(-1,1)$ and $\varphi^r f^{(r)} \in L_\infty[-1,1]$. The following result is well known.

Lemma 1. *If $f \in \mathbb{B}^r$, then*

$$E_n(f) \le \frac{c(r)}{n^r}\|\varphi^r f^{(r)}\|_{L_\infty[-1,1]}, \quad n \ge r.$$

Also, if $r \ge 1$, then $\omega_{k,r}^{\varphi}(f^{(r)},t) \to 0$, as $t \to 0$, if and only if $\lim_{x\to\pm1}\varphi^r(x)f^{(r)}(x)=0$. Therefore, we denote $C_\varphi^0 := C[-1,1]$ and, for $r \ge 1$,

$$C_\varphi^r := \{f \in C^r(-1,1)\cap C[-1,1] \mid \lim_{x\to\pm1}\varphi^r(x)f^{(r)}(x)=0\}.$$

First, we wish to rule out the possibility that for some triple (Y_s,α,N), $\alpha \ne 3$, there exists an $f \in \Delta^{(1)}(Y_s)$ satisfying (1.4), for which there is no N^* at all. The case $\alpha=3$ is deferred to Section 6, Proposition 5. We need the following result of [1, Theorem 7.1.2] (see also [7, Theorem 3.3]).

Lemma 2. *Let $r \in \mathbb{N}_0, k \in \mathbb{N}$, and $\alpha>0$, be such that $r<\alpha<k+r$, and let $f \in C[-1,1]$. If*

$$n^\alpha E_n(f) \le 1 \quad \forall n \ge N,$$

where $N \ge k+r$, then $f \in C_\varphi^r$ and

$$\omega_{k,r}^{\varphi}(f^{(r)},t) \le c(\alpha,k,r)t^{\alpha-r} + c(N,k,r)t^k E_{k+r}(f), \quad t>0. \tag{2.1}$$

In case $N=k+r$, (2.1) takes the form

$$\omega_{k,r}^{\varphi}(f^{(r)},t) \le c(\alpha,k,r)t^{\alpha-r}, \quad t>0. \tag{2.2}$$

We also need the comonotone approximation estimates of $f \in \Delta^{(1)}(Y_s)\cap C_\varphi^r$, [11], [12] (see also [5, §13.2, Statement 11, Tables 19-20]). Namely,

$$E_n^{(1)}(f,Y_s) \le \frac{c(r,s)}{n^r}\omega_{1,r}^{\varphi}(f^{(r)},n^{-1}), \quad n \ge N(Y_s) \tag{2.3}$$

and, for $r \ne 2$,

$$E_n^{(1)}(f,Y_s) \le \frac{c(r,s)}{n^r}\omega_{2,r}^{\varphi}(f^{(r)},n^{-1}), \quad n \ge N(Y_s). \tag{2.4}$$

Note that (2.4) is, in general, invalid for $r=2$ (see Nesterenko and Petrova [13]).

Now, if (1.4) holds, then, obviously, it holds for some $N \ge [\alpha]+1 =: r+1$, so we may assume so. Hence, by (2.1), for $\alpha \notin \mathbb{N}$,

$$\omega_{1,r}^{\varphi}(f^{(r)},n^{-1}) \le c(\alpha)n^{r-\alpha} + c(N,r)n^{-1}E_{r+1}(f),$$

and for $\alpha = r \in \mathbb{N}$,

$$\omega^{\varphi}_{2,r-1}(f^{(r-1)}, n^{-1}) \le c(r)n^{-1} + c(N,r)n^{-2}E_{r+1}(f).$$

Hence, for $\alpha \notin \mathbb{N}$, by (2.3)

$$\begin{aligned} n^{\alpha}E^{(1)}_n(f, Y_s) &\le c(\alpha, s)n^{\alpha-r}\omega^{\varphi}_{1,r}(f^{(r)}, n^{-1}) \\ &\le c(\alpha, s) + c(N, \alpha, s)n^{\alpha-r-1}E_{r+1}(f), \quad n \ge N(Y_s), \end{aligned}$$

which, in turn, implies

$$n^{\alpha}E^{(1)}_n(f, Y_s) \le c(\alpha, s), \quad n \ge N(f). \tag{2.5}$$

Also for $\alpha = r \in \mathbb{N}$, provided $\alpha \ne 3$, we obtain by (2.4)

$$\begin{aligned} n^{r}E^{(1)}_n(f, Y_s) &\le c(r, s)n\omega^{\varphi}_{2,r-1}(f^{(r-1)}, n^{-1}) \\ &\le c(r, s) + c(N, r, s)n^{-1}E_{r+1}(f), \quad n \ge N(Y_s), \end{aligned}$$

and (2.5) follows for integer $\alpha \ne 3$.

We also need the following well known result (see, e.g., [7, Theorem 3.1]).

Lemma 3. *Let $2r < \alpha < 2k + 2r$ and $f \in C[-1, 1]$. If*

$$n^{\alpha}E_n(f) \le 1, \qquad n \ge k + r,$$

then $f \in C^r[-1, 1]$ and

$$\omega_k(f^{(r)}, t) \le c(\alpha, k, r)t^{\alpha/2-r}.$$

In Section 6 we have to consider subintervals $[a, b]$ of $[-1, 1]$. Thus we need the notation $\Delta^{(1)}_s[a, b]$ for the set of all continuous functions in $[a, b]$ that have $s \in \mathbb{N}_0$ changes of monotonicity there, and for $g \in \Delta^{(1)}_s[a, b]$, we denote by $E^{1,s}_n(g)_{[a,b]}$, the worst possible degree of best comonotone polynomial approximation of g in $[a, b]$ (see (1.1)). We state two lemmas which follow from [2, Corollary 3.1] (see also [9, Corollaries 1 and 2], respectively).

Lemma 4. *If $g \in \Delta^{(1)}_s[a, b] \cap C^r[a, b]$, then*

$$E^{1,s}_{r+1}(g)_{[a,b]} \le (b - a)^r\omega(g^{(r)}, b - a, [a, b]). \tag{2.6}$$

And

Lemma 5. *If $g \in \Delta^{(1)}_s[a, b] \cap C^r[a, b]$ and $r \ge s$, then*

$$E^{1,s}_{r+2}(g)_{[a,b]} \le c(r)(b - a)^r\omega_2(g^{(r)}, b - a, [a, b]). \tag{2.7}$$

3. Negative results

In order to establish the various negative conclusions, we need a few lemmas. We begin with a sharper version of [10, Theorem 2]. We denote by B^r the set of all functions $f \in \mathbb{B}^r$ such that $|\varphi^r(x)f^{(r)}(x)| \le 1$ a.e. in $x \in (-1, 1)$.

Lemma 6. *Let $r \in \mathbb{N}$, $r > 2$, and denote $\rho := [\frac{r+1}{2}]$. Assume that $s \ge \rho$ and $m \ge 1$. Then for each $A > 0$, there are a collection Y_s and a function $F \in \Delta^{(1)}(Y_s) \cap B^r$, such that*

$$E_{\rho+1}(F) \le 1$$

and

$$E_m^{(1)}(F, Y_s) > A.$$

Proof. Let

$$g_r(x) := \begin{cases} (x+1)^{r/2}\ln(x+1), & \text{if } r \text{ is even}, \\ -(x+1)^{r/2}, & \text{if } r \text{ is odd}, \end{cases} \quad x \in (-1,1],$$

and

$$g_r(-1) := 0.$$

Simple calculations show that

$$g_r^{(\rho)}(x) = \begin{cases} \rho!\ln(x+1) + c(r), & \text{if } r \text{ is even}, \\ -\frac{c(r)}{(x+1)^{1/2}}, & \text{if } r \text{ is odd}, \end{cases} \quad x \in (-1,1],$$

so that, in particular, $\lim_{x\to-1}|g_r^{(\rho)}(x)| = \infty$, but $|(1+x)g_r^{(\rho)}(x)| \le c(r,0)$, $x \in (-1,1]$. Denote $M_i(r) := \|g_r^{(i)}\|$, $1 \le i < \rho$. Also, note that $|g_r^{(\rho+i)}(x)|$, $i \ge 1$, is decreasing and $g_r^{(\rho+1)}(x) > 0$ for $x \in (-1,1]$, and

$$|(1+x)^{\rho+i-r/2}g_r^{(\rho+i)}(x)| \equiv c(r,i), \quad x \in (-1,1]. \tag{3.1}$$

Let $S \in C^\infty(\mathbb{R})$ be a monotone function, such that

$$S(x) := \begin{cases} 0, & x \le 0, \\ 1, & x \ge 1, \end{cases}$$

and denote $C_j := \|S^{(j)}\|$, $j \ge 0$.

For $b \in (0,1)$ to be prescribed, write $a := b-1$, and let

$$l_a(x) := g_r'(a) + \frac{g_r''(a)}{1!}(x-a) + \dots + \frac{g_r^{(\rho)}(a)}{(\rho-1)!}(x-a)^{\rho-1},$$

be the Taylor polynomial of the derivative g_r'.

Set

$$h_a(x) := g_r'(x) - l_a(x) = \frac{1}{(\rho-1)!}\int_a^x (x-u)^{\rho-1}g_r^{(\rho+1)}(u)du, \quad x \in [-1,1],$$

and

$$f_a(x) = S\left(\frac{x-a}{b}\right)h_a(x),$$

and denote

$$F_a(x) := \int_{-1}^x f_a(t)dt \quad \text{and} \quad L_a(x) := \int_{-1}^x l_a(t)dt.$$

First we show that for some constant $c_1 = c_1(r)$,

$$\frac{1}{c_1}F_a \in B^r. \tag{3.2}$$

Indeed, we observe that

$$(1+x)^{r/2}F_a^{(r)}(x) = (1+x)^{r/2}f_a^{(r-1)}(x),$$

so that we should prove that the latter is bounded in $(-1,1]$.

To this end, if $x \in [-1,-1+b]$, then $f_a(x) \equiv 0$ and there is nothing to prove, and if $x \in [-1+2b,1]$, then $|(1+x)^{r/2} f_a^{(r-1)}(x)| = |(1+x)^{r/2} h_a^{(r-1)}(x)| = |(1+x)^{r/2} g_r^{(r)}(x)| \equiv c(r, r-\rho)$.

Since $|g_r^{(\rho+i)}(x)|$, $i \ge 1$, is decreasing in $(-1,1]$, we conclude that

$$|h_a^{(i)}(x)| \le b^{\rho-i} g_r^{(\rho+1)}(a)/(\rho-i-1)!\,, \quad 0 \le i < \rho, \quad x \in (-1+b,-1+2b). \tag{3.3}$$

Also, since l_a is of degree $\rho - 1$, we have

$$|h_a^{(i)}(x)| = |g_r^{(i+1)}(x)| \le |g_r^{(i+1)}(a)|, \quad \rho \le i < r, \quad x \in (-1+b,-1+2b). \tag{3.4}$$

Thus, for $x \in (-1+b,-1+2b)$,

$$\begin{aligned}
|(1+x)^{r/2} f_a^{(r-1)}(x)| &\le (1+x)^{r/2} \sum_{i=0}^{r-1} \binom{r-1}{i} \frac{C_{r-1-i}}{b^{r-1-i}} |h_a^{(i)}(x)| \\
&= (1+x)^{r/2} \sum_{i=0}^{\rho-1} \binom{r-1}{i} \frac{C_{r-1-i}}{b^{r-1-i}} |h_a^{(i)}(x)| \\
&\qquad + (1+x)^{r/2} \sum_{i=\rho}^{r-1} \binom{r-1}{i} \frac{C_{r-1-i}}{b^{r-1-i}} |h_a^{(i)}(x)| \\
&\le c(r) b^{\rho-r/2+1} |g_r^{(\rho+1)}(a)| + c(r) \sum_{i=\rho}^{r-1} b^{-r/2+1+i} |g_r^{(i+1)}(a)| \\
&\equiv c_1(r) =: c_1,
\end{aligned}$$

where in the second inequality we applied (3.3) and (3.4), and for the last equality, we used (3.1). This proves (3.2).

We claim that

$$E_{\rho+1}(F_a) \le \|F_a + L_a\| \le 2\|f_a + l_a\| \le c_2(r) =: c_2. \tag{3.5}$$

Indeed, if $x \in [-1+2b,1]$, then

$$|f_a(x) + l_a(x)| = |h_a(x) + l_a(x)| = |g_r'(x)| \le M_1(r).$$

If $x \in [-1,-1+2b)$, then,

$$\begin{aligned}
|l_a(x)| &\le M_1(r) + \frac{1}{1!} M_2(r) + \cdots + \frac{1}{(\rho-2)!} M_{\rho-1}(r) + \frac{g_r^{(\rho)}(a)}{(\rho-1)!} b^{\rho-1}. \\
&\le c_3(r) + c(r,0) =: c_4(r).
\end{aligned}$$

Therefore, if $x \in [-1,-1+b)$, then,

$$|f_a(x) + l_a(x)| = |l_a(x)| \le c_4(r),$$

and if $x \in [-1+b,-1+2b)$, then,

$$|f_a(x) + l_a(x)| \le |h_a(x)| + |l_a(x)| \le 2|l_a(x)| + |g_r'(x)| \le 2c_4(r) + M_1(r) =: \frac{1}{2} c_2(r).$$

Hence, (3.5) is proved.

Finally, for $c_5 := c_1 + c_2$, take b so small that,

$$\frac{1}{\max\{\rho, m\}^{2\rho}} |g_r^{(\rho)}(a)| > c_2 + c_5 A. \tag{3.6}$$

Now let

$$F(x) := \frac{1}{c_5} F_a(x).$$

Then, it follows by (3.2), that $F \in B^r$, and (3.5) implies that $E_{\rho+1}(F) \le 1$. Also, since $f_a \equiv 0$ on $[-1, a]$ and $f_a > 0$ on $(a, 1]$, taking $Y_s := \{y_1, \dots, y_s\}$, with $-1 < y_1 < \dots < y_s \le a$, we, evidently have, $F \in \Delta^{(1)}(Y_s)$. Assume, to the contrary, that there exists a polynomial $P_m \in \mathbb{P}_m \cap \Delta^{(1)}(Y_s)$, such that

$$\|F - P_m\| \le A,$$

equivalently, that

$$\|F_a - Q_m\| \le c_5 A,$$

where $Q_m = c_5 P_m$.

Then, since $s \ge \rho$, it follows that there exists a point $\theta \in (-1, a)$ where $Q_m^{(\rho)}(\theta) = 0$. We also observe that $L_a^{(\rho)}(x) = l_a^{(\rho-1)}(x) \equiv g_r^{(\rho)}(a)$. Hence, by Markov's inequality,

$$\begin{aligned}\frac{1}{\max\{\rho, m\}^{2\rho}} |g_r^{(\rho)}(a)| &= \frac{1}{\max\{\rho, m\}^{2\rho}} |L_a^{(\rho)}(\theta)| = \frac{1}{\max\{\rho, m\}^{2\rho}} |L_a^{(\rho)}(\theta) + Q_m^{(\rho)}(\theta)| \\ &\le \|L_a + Q_m\| \le \|L_a + F_a\| + \|Q_m - F_a\| \\ &\le c_2 + c_5 A,\end{aligned}$$

a contradiction to (3.6). This completes the proof. □

As a consequence we have,

Corollary 1. Let $r \in \mathbb{N}$, $r > 2$, and denote $\rho := [\frac{r+1}{2}]$. Assume that $s \ge \rho$ and $m \ge 1$. Then for each $A > 0$, there are a collection Y_s and a function $f \in \Delta^{(1)}(Y_s)$, such that

$$n^r E_n(f) \le 1, \quad n \ge \rho + 1,$$

and

$$E_m^{(1)}(f, Y_s) > A.$$

Proof. Take F of Lemma 6. Since $F \in B^r$, it follows by Lemma 1, that

$$n^r E_n(F) \le c(r) \|\varphi^r F^{(r)}\| \le c(r), \quad n \ge r.$$

At the same time, for $\rho < n < r$, we have $n^r E_n(F) \le r^r E_{\rho+1}(F) \le r^r$. Therefore, let $f := \frac{F}{\max\{c(r), r^r\}}$, and we have the desired function. □

Further, following [3], let $r \in \mathbb{N}$ and $G_r(x) := (x+1)^r \ln(x+1)$, $G_r(-1) := 0$. Since $G_r \in \mathbb{B}^{2r}$ and $\|\varphi^{2r} G_r^{(2r)}\| < +\infty$, we have

$$E_n(G_r) \le c(r) n^{-2r}, \qquad n \in \mathbb{N}. \tag{3.7}$$

We need this estimate for the proof of an important case, similar to [7, Lemma 2.3].

Lemma 7. *Let $s \in \mathbb{N}_0$ and let $Y_s \in \mathbb{Y}_s$. For each $A > 0$ and every $m \in \mathbb{N}$, there is a function $f = f_{A,m} \in \Delta^{(1)}(Y_s)$, such that*

$$n^2 E_n(f) \le 1, \quad n \ge s + 3,$$

and

$$E_m^{(1)}(f, Y_s) \ge A.$$

Proof. For $b \in (-1, 0)$, let

$$f_b(x) := \int_0^x \Pi(t) \left(\int_b^t \frac{t-u}{(u+1)^2} du \right) dt,$$

where $\Pi(t) := \prod_{i=1}^s (t - y_i)$.

Clearly, $f_b'(x)\Pi(x) \ge 0$, $x \in (-1, 1)$, hence $f_b \in \Delta^{(1)}(Y_s)$. Substituting the Taylor expansion of $\Pi(x)$ about $x = -1$, yields

$$f_b = P_{s+3} - \sum_{r=0}^{s} \frac{\Pi^{(r)}(-1)}{(r+1)!} G_{r+1},$$

where $P_{s+3} \in \mathbb{P}_{s+3}$. By virtue of (3.7), we have

$$n^2 E_n(f_b) \le c(s), \qquad n \ge s+3, \tag{3.8}$$

as $\|\Pi^{(r)}(\cdot)\| \le c(s)$, $0 \le r \le s$. We also have that polynomial

$$p_{s+3} := \int_0^x \Pi(t) \left(\int_b^1 \frac{t-u}{(u+1)^2} du \right) dt$$

belongs to $\mathbb{P}_{s+3}$ and satisfies

$$\Pi(-1) p_{s+3}'(-1) = \Pi^2(-1) \ln \frac{b+1}{2}.$$

Therefore, for every $P_m \in \mathbb{P}_m \cap \Delta^{(1)}(Y_s)$, $m \ge s+3$, we get

$$\begin{aligned} -\Pi^2(-1) \ln \frac{b+1}{2} &= -\Pi(-1) p_{s+3}'(-1) \\ &\le \Pi(-1)(P_m'(-1) - p_{s+3}'(-1)) \\ &\le m^2 |\Pi(-1)| \|P_m - p_{s+3}\|, \end{aligned} \tag{3.9}$$

where for the last inequality we have applied Markov's inequality. At the same time,

$$p_{s+3}(x) - f_b(x) = \int_0^x \Pi(t) \left(\int_t^1 \frac{t-u}{(u+1)^2} du \right) dt$$

does not depend on b. Hence, it follows by (3.9) that

$$m^{-2} |\Pi(-1)| \ln \frac{2}{b+1} \le \|P_m - f_b\| + \|f_b - p_{s+3}\| \le \|P_m - f_b\| + c(s). \tag{3.10}$$

Thus,

$$E_m^{(1)}(f; Y_s) \ge m^{-2} |\Pi(-1)| \ln \frac{2}{b+1} - c(s),$$

which for $f := c f_b$ with suitable b and $c = c(s)$ completes the proof. □

Finally, we adapt the ideas of the proof of [7, Lemma 2.4] to obtain the following.

Lemma 8. *Let $s \in \mathbb{N}$ and let $Y_s \in \mathbb{Y}_s$. For each $A > 0$ and every $m \in \mathbb{N}$, there is a function $f = f_{A,m} \in \Delta^{(1)}(Y_s)$, such that*

$$n E_n(f) \le 1, \quad n \ge s+2,$$

and

$$E_m^{(1)}(f, Y_s) \ge A.$$

Proof. Denote $D_j(x) := x^j \ln|x|$, $j \ge 1$ ($D_j(0) := 0$). It is well known that if $D_{j,\gamma}(x) := D_j(x+\gamma)$, $-1 < \gamma < 1$, then

$$nE_n(D_{j,\gamma}) \le c(j), \quad n \ge 1. \tag{3.11}$$

Let $0 < b < \frac{1}{2}\min\{1-y_s, y_s - y_{s-1}\}$ and denote $\tilde{l}_b(x) := \frac{x}{b} - 1 + \ln b$, the tangent to $\ln x$ at $x = b$. Further, let b^* be the negative root of the equation $\tilde{l}_b(x) = \ln|x|$.Then, clearly $|b^*| = -b^* < b$, and $(x - b^*)(\tilde{l}_b(x) - \ln|x|) \ge 0$, $x \ne 0$. Hence, for $l_b(x) := \tilde{l}_b(x + b^*)$, we have

$$x(l_b(x) - \ln|x + b^*|) \ge 0, \quad x \ne -b^*. \tag{3.12}$$

Write $\tilde{\Pi}(x) := \prod_{i=1}^{s-1}(x - y_i)$ ($\tilde{\Pi} \equiv 1$ if $s = 1$), and let

$$L_b(x) := \int_0^x \tilde{\Pi}(u) l_b(u - y_s)\, du,$$

and

$$g_b(x) := \int_0^x \tilde{\Pi}(u) \ln|u + b^* - y_s|\, du.$$

Finally, denote

$$f_b := L_b - g_b,$$

and observe that (3.12) implies that $f_b \in \Delta^{(1)}(Y_s)$.

Integration by parts and induction readily show that

$$g_b(x) = \sum_{r=0}^{s-1} \frac{\tilde{\Pi}^{(r)}(y_s - b^*)}{(r+1)!} D_{r+1}(x + b^* - y_s) + p_{s+1}(x),$$

where $p_{s+1} \in \mathbb{P}_{s+1}$, and since $L_b \in \mathbb{P}_{s+2}$, it follows by (3.11) that

$$nE_n(f_b) \le c(s), \quad n \ge s+2. \tag{3.13}$$

Now, given any $P_m \in \mathbb{P}_m \cap \Delta^{(1)}(Y_s)$, we see that

$$\begin{aligned} 0 < \tilde{\Pi}(y_s) \ln\frac{1}{b} &< \tilde{\Pi}(y_s)\left(\ln\frac{1}{b} + 1 - \frac{b^*}{b}\right) \\ &= -L_b'(y_s) = P_m'(y_s) - L_b'(y_s) \le C(s, y_s) m \|P_m - L_b\|, \end{aligned}$$

where we have applied the Bernstein inequality. Hence

$$\begin{aligned} 0 < \tilde{\Pi}(y_s) \ln\frac{1}{b} &< C(s, y_s) m (\|P_m - f_b\| + \|g_b\|) \\ &\le C(Y_s) m (\|P_m - f_b\| + 1), \end{aligned}$$

since $\|g_b\| \le 2^s \ln 2$.

This, in turn, implies

$$E_m^{(1)}(f_b, Y_s) > \frac{C(Y_s)}{m} \ln\frac{1}{b} - 1,$$

which, together with (3.13), yields a function $f = cf_b$, fulfilling the statements of the lemma for $c = c(s)$ and sufficiently small b. This completes the proof. □

4. Positive results

In order to establish the various positive conclusions, we also need a few lemmas.

The following lemma can be derived from [2, Theorem 1.3']. For the benefit of the readers we provide a short proof.

Lemma 9. *If* $f \in \Delta^{(1)}(Y_s)$, $\quad s \geq 1$, *then*

$$E_1(f) \leq C(Y_s)E_{s+1}(f)$$

Proof. Denote by L the Lagrange polynomial of degree $\leq s$, that interpolates the function $g(x) := f(x) - f(y_s)$ at the points $y_1, \dots, y_s$ and at the point -1. Since $g \in \Delta^{(1)}(Y_s)$, the polynomial L has $s-1$ extreme points in $(-1, y_s)$, therefore the polynomial L' of degree $\leq s-1$ has all its $(s-1)$ zeroes in $(-1, y_s)$. Since $L'(y_s) < 0$, it follows that $L'(x) < 0$ for all $x \in [y_s, 1]$. Hence $L(x) \leq 0$, $x \in [y_s, 1]$, which, in turn, implies

$$0 \leq g(x) \leq g(x) - L(x), \quad x \in [y_s, 1]. \tag{4.1}$$

Evidently,

$$\begin{aligned} \max_{x\in[y_s,1]} |g(x) - L(x)| \leq \|g - L\| &\leq C(Y_s)E_{s+1}(g) \\ &= C(Y_s)E_{s+1}(f). \end{aligned} \tag{4.2}$$

Also, if l denotes the Lagrange polynomial that interpolates g at $s+1$ equidistant points in $[y_s, 1]$, including y_s and 1, then we have

$$\begin{aligned} E_1(f) \leq \|g\| \leq \|g - l\| + \|l\| &\leq C(y_s)\big(E_{s+1}(g) + \max_{x\in[y_s,1]} |g(x)|\big) \\ &= C(y_s)\big(E_{s+1}(f) + \max_{x\in[y_s,1]} |g(x)|\big). \end{aligned}$$

Hence, substituting it together with (4.2) into (4.1) yields

$$E_1(f) \leq C(Y_s)E_{s+1}(f).$$

This completes the proof. □

For the proof of the positive results, we shall first establish certain approximation rate for piecewise polynomials and then extend it to polynomials. To this end we first introduce some notation.

For a fixed $n \geq 1$, denote $x_j := x_{j,n} := \cos(j\pi/n)$, $j = 0, \dots, n$. Then $-1 = x_{n,n} < \cdots < x_{0,n}$ is the Chebyshev partition. Further, denote $I_j := I_{j,n} := [x_{j,n}, x_{j-1,n}]$, $j = 1, \dots, n$ and let $|I_j| := x_{j-1,n} - x_{j,n}$.

For a given Y_s, let

$$O_i := O_{i,n}(Y_s) := (x_{j+1,n}, x_{j-2,n}), \quad \text{if} \quad y_i \in [x_{j,n}, x_{j-1,n}),$$

where $x_{n+1,n} := -1$ and $x_{-1,n} := 1$. Finally, define

$$O := O(Y_s, n) := \bigcup_{i=1}^{s} O_i,$$

let (a_q, b_q), $q = 1, \dots, l \leq s$, be the connected components of $O(Y_s, n)$ and denote $\tilde{O}_q := [a_q, b_q],$.

We take a subset of the set of continuous piecewise polynomials of degree $< r$, with extreme points at Y_s, that may be well approximated by comonotone polynomials. Let $\Sigma_{r,n} := \Sigma_{r,n}(Y_s)$ be the set of continuous piecewise polynomials S, on the Chebyshev partition $x_{0,n} \dots, x_{n,n}$ composed of polynomial pieces of degree $< r$, with the additional restriction that S is a single polynomial on each $\tilde{O}_q$, $\quad q = 1, \dots, l$. By [11, Proposition 3] we know the following.

Lemma 10. *If $S \in \Sigma_{r,n}(Y_s) \cap \Delta^{(1)}(Y_s)$, then*

$$E^{(1)}_{c_1 n}(S, Y_s) \le c_2 \omega^{\varphi}_r(S, \frac{1}{n}), \tag{4.3}$$

where $c_1 = c_1(r, s)$ and $c_2 = c_2(r, s)$.

Another necessary result is an estimate of the ordinary Ditzian-Totik moduli of smoothness by the weighted moduli. By virtue of [6, (3.4)] and [6, (3.5)], we have for $f \in C^r_\varphi$, $1 < j < n$, $k \ge 1$ and $0 \le l \le r$,

$$|I_j|^l \omega_{k+r-l}(f^{(l)}, |I_j|; I_j) \le c(k, r) n^{-r} \omega^{\varphi}_{k,r}(f^{(r)}, 1/n). \tag{4.4}$$

Moreover, if $0 \le l < r/2$, then (4.4) is valid also for $j = 1$ and $j = n$.

5. Negative conclusions

In this section we collect all the information on what cannot be achieved in the various cases.

(1) It follows from Lemma 7, that if $\alpha \le 2$, then in Fig. 2, there can be neither "+" nor "⊕" in position (N, α) for any $N \ge s + 3$. Moreover, it follows from Lemma 8 that if $\alpha \le 1$, then the same is true for $N = s + 2$.

(2) By virtue of Corollary 1, we obtain

Proposition 1. *Given $s \ge 2$ and $\alpha \in (2, 2s]$. In* Figs. 2 *through* 4*, there can be no " + " in position (N, α) for any $N \ge \lceil \alpha/2 \rceil + 1$.*

Proof. We may apply Corollary 1 with $r = \lceil \alpha \rceil > 2$, as we observe that $\rho = \left[\frac{r+1}{2}\right] \le s$. Therefore, there can be no " + " for $N \ge \left[\frac{\lceil \alpha \rceil + 1}{2}\right] + 1 = \lceil \alpha/2 \rceil + 1$. □

(3) By [10, Theorem 2], for every constant $A > 0$, $s \ge 1$ and $2 \le r \le 2s + 2$, excluding $r - 2 = 1 = s$, and any $m \ge 1$, there is a function $f = f_{A,s,r,m} \in \Delta^{(1)}_s \cap \mathbb{B}^r_\varphi$ such that

$$E^{1,s}_m(f) \ge A \|\varphi^r f^{(r)}\| > 0. \tag{5.1}$$

Hence, by virtue of Lemma 1, we may conclude the following.

Proposition 2. *If $0 < \alpha \le 2$, then there is no " + " in position (N, α), for all $N \ge 2$. If $s = 1$ and $2 < \alpha \le 4$, then there is no " + " in position (N, α), for all $N \ge 4$*

Proof. Taking (5.1) with $r = 2$, yields the assertion for $0 < \alpha \le 2$, and taking it with $r = 4$ yields the other. □

(4) A closer look at the proof of [10, Theorem 2] sharpens the conclusions we may draw from the statement in (3) above. Namely,

Proposition 3. *If $4 \le 2s < \alpha \le 2s+2$, then there is no "$+$" in position (N,α) for $N \ge s+3$.*

(5) Given $s \ge 1$, we follow [9] and define the sets $A_1 := \{2\}$, and for $s \ge 2$,

$$A_s := \{j \mid 1 \le j \le s-1 \text{ or } j = 2i, 1 \le i \le s\}.$$

For $\alpha \in A_s$ and each $m \ge 1$, a function $f_{s,\alpha,m}$, was constructed in [9], such that on the one hand

$$n^\alpha E_n(f_{s,\alpha,m}) \le 1, \quad n \ge 1,$$

while at the same time,

$$m^\alpha E_m^{1,s}(f_{s,\alpha,m}) \ge c(s) \ln m.$$

Clearly, this very function proves that for $\alpha \in A_s$, in position (N,α) in Fig. 2, there can be no "+" for any $N \ge 1$.

6. Positive conclusions

(1) By virtue of [11, Theorem 4], we have either "+" or "$\oplus$" in Fig. 2, in all positions (N,α), where $N \le [\alpha]+1$ and $\alpha \ne 2$.
Also, [11, Theorem 3] implies that we have "+" in Fig. 2, in all positions (N,α) where $N \le [\alpha]+1$ and either $0 < \alpha < 1$, or $s = 1$ and $2 < \alpha < 3$, or $\alpha > 2s+2$.

At the same time [12, Theorem 1] implies that we have $\oplus$ in Fig. 5 (the table for the case $s = 1$), in position $(3,\alpha)$ for $1 < \alpha \le 2$. The proof is similar to that of [12, Corollary 2].

(2) In [5, Tables 19-20], there are truth tables depending on (k,r,s) for the validity of the estimate

$$E_n^{(1)}(f,Y_s) \le \frac{c(k,r,s)}{n^r}\omega_{k,r}^\varphi(f^{(r)},n^{-1}), \quad n \ge \mathcal{N}, \tag{6.1}$$

for various integers $\mathcal{N}$.

By virtue of [5, Tables 19-20] (relating α and r by $r = \lceil\alpha\rceil - 1$), applying Lemma 2 and (6.1), we conclude the following.

Proposition 4. *Let $\alpha \in (3, 2s+3]$, $N \ge \lceil\alpha\rceil$. Then we have either "$+$" or "$\oplus$" in position (N,α) in* Fig. 2.
Further, let $\alpha > 2s+3$, and N as above. Then we have "$+$" in position (N,α) in Fig. 2.
Consequently, if $\alpha > 2s+3$, then we have "$+$" in all positions (N,α).

(3) We can show that for $0 < \alpha \le 2$ and $2 \le N \le s+1$ we have "$\oplus$" in the position (N,α) in Fig. 2. Indeed, if $0 < \alpha < 2$, then by Lemma 2,

$$\omega_2^\varphi(f,t) \le c(\alpha,s)t^\alpha + c(N,s)t^2 E_2(f), \tag{6.2}$$

and if $\alpha = 2$, then by Lemma 2, applied to $r = 1 < \alpha = 2 < 3 = r+k$, $f \in C_\varphi^1$ and,

$$\omega_{2,1}^\varphi(f',t) \le c(s)t + c(N,s)t^2 E_3(f). \tag{6.3}$$

It follows from [4, Theorem 1] (see also [8, Theorem 1']) that,

$$E_n^{(1)}(f,Y_s) \le c(s)\omega_2^\varphi(f,1/n), \quad n \ge N(Y_s).$$

Hence, for $0 < \alpha < 2$, we conclude by (6.2), that

$$E_n^{(1)}(f,Y_s) \le c(\alpha,s)n^{-\alpha} + c(N,s)n^{-2}E_2(f), \quad n \ge N(Y_s). \tag{6.4}$$

On the other hand by virtue of Lemma 9, we obtain

$$E_2(f) \le E_1(f) \le C(Y_s)E_{s+1}(f) \le C(\alpha, Y_s),$$

where for the last inequality we applied the assumption that $(s+1)^\alpha E_{s+1}(f) \le 1$. Thus, substituting in (6.4), we have

$$n^\alpha E_n^{(1)}(f, Y_s) \le c(\alpha, s) + C(N, \alpha, Y_s)n^{-2+\alpha}, \quad n \ge N(Y_s),$$

which in turn implies

$$n^\alpha E_n^{(1)}(f, Y_s) \le c(\alpha, s), \quad n \ge N^*(N, \alpha, Y_s).$$

If $\alpha = 2$, then by [10] and [12] (see also [5, paragraph 13.2, Statement 11, Tables 19-20]),

$$E_n^{(1)}(f, Y_s) \le \frac{c(s)}{n}\omega_{2,1}^\varphi(f', 1/n), \quad n \ge N(Y_s).$$

Hence, by (6.3), we obtain for $\alpha = 2$, that

$$n^2 E_n^{(1)}(f, Y_s) \le c(s) + c(N, s)n^{-1}E_3(f), \quad n \ge N(Y_s).$$

We proceed as above and conclude that

$$n^2 E_n^{(1)}(f, Y_s) \le c(s), \quad n \ge N^*(N, Y_s).$$

In view of Propositions 1 and 2, we conclude that for all (N, α) under consideration, we have "$\oplus$".

(4) What follows is a proof of a theorem that, in essence, appears in the appendix of [9] (see [9, Theorem 5]). We give a simpler and more transparent proof, and also rectify an inadvertent minor omission in that proof.

Theorem 2. *Assume $s \in \mathbb{N}$, $\alpha > 1$, $\alpha \notin A_s$ and $N \le \lceil \alpha/2 \rceil$. Then there are constants $N^*(\alpha, s, N)$ and $c(\alpha, s)$, such that for every function $f \in \Delta_s^{(1)}$ satisfying*

$$n^\alpha E_n(f) \le 1, \quad n \ge N, \tag{6.5}$$

we have

$$n^\alpha E_n^{1,s}(f) \le c(\alpha, s), \quad n \ge N^*(\alpha, s, N). \tag{6.6}$$

Proof. As in [9, proof of Theorem 3], we first have to establish the inequalities

$$E_{r+1}^{1,\sigma}(f)_{J_{j,n}} \le \frac{c(\alpha, s)}{n^\alpha}, \quad j = 1, \dots, n-1, \tag{6.7}$$

where $J_{j,n} := [x_{j+1}, x_{j-1}]$ and σ is defined by $f \in \Delta_\sigma^{(1)}[x_{j+1}, x_{j-1}]$, and where $r := [\alpha] \ge 1$. But unlike [9], where (6.5) held with $N = 1$, here we have to take into consideration N. Thus, if $\alpha \notin \mathbb{N}$, then by virtue of Lemma 2, with $k = 1$ and $r = [\alpha]$, (6.5) and (4.4) imply that $f \in C^r(-1, 1)$ and

$$\omega(f^{(r)}, |J_{j,n}|, J_{j,n}) \le \frac{c(\alpha)}{|J_{j,n}|^r n^\alpha}, \quad 2 \le j \le n-2,$$

and if $\alpha \in \mathbb{N}$, then observing that $r-1<\alpha=r<r+1$, Lemma 2, with $k=2$, combined with (6.5) and (4.4), yields

$$\omega_2(f^{(r-1)}, |J_{j,n}|, J_{j,n}) \le \frac{c(\alpha)}{|J_{j,n}|^{r-1} n^\alpha}, \quad 2 \le j \le n-2.$$

Note that in order to apply Lemma 2, we have to have $N \le r+1 = [\alpha]+1$, but we actually have to restrict N even further as is seen below.

In order to obtain (6.7) for the end intervals $J_{1,n}$ and $J_{n-1,n}$, we need the inequality

$$\omega(f^{([\frac{r}{2}])}, |J_{j,n}|, J_{j,n}) \le \frac{c(\alpha)}{|J_{j,n}|^{[\frac{r}{2}]} n^\alpha}, \quad j=1, n-1,$$

for α that is not an even integer. This follows by virtue of Lemma 3, when we observe that with $r=[\alpha]$, we have $2[\frac{r}{2}] < \alpha < 2[\frac{r}{2}]+2$, and that $|J_{j,n}| \sim n^{-2}$, $j=1,n-1$. Again, note that in order to apply Lemma 3 we must have $N \le [\frac{r}{2}]+1 = \lceil \alpha/2 \rceil$.

Finally, when α is an even integer, we have $2(\frac{r}{2}-1) < r = \alpha < 2(\frac{r}{2}-1)+4$, so we apply Lemma 3 with $k=2$ to obtain that $f \in C^{\frac{r}{2}-1}[-1,1]$,

$$\omega_2(f^{(\frac{r}{2}-1)}, |J_{j,n}|, J_{j,n}) \le \frac{c(\alpha)}{|J_{j,n}|^{\frac{r}{2}-1} n^\alpha}, \quad j=1, n-1.$$

And again, note that in order to apply Lemma 3 we must have $N \le (\frac{r}{2}-1)+2 = \alpha/2+1$.

Summarizing, we see that taking $N \le \lceil \alpha/2 \rceil$, yields all the above estimates. We observe that if α is an even integer, then we actually have the estimates also for $N=\alpha/2+1$.

The above four inequalities, combined with (2.6) and (2.7) yield (6.7). Note that for α an even integer, Lemma 5 is applicable only when $\alpha \ge 2s+2$. This is related to why we have to exclude the even α's in A_s.

Clearly, it suffices to prove (6.6) for $E_n^{(1)}(f, Y_s)$, where $Y_s \in \mathbb{Y}_s$ is an arbitrary collection such that $f \in \Delta^{(1)}(Y_s)$. So denote

$$\sigma^{(1)}_{r+1,n}(f, Y_s) := \inf\{\|f-S\| : S \in \Sigma_{r+1,n}(Y_s) \cap \Delta^{(1)}(Y_s)\},$$

the degree of best approximation by the specific class of splines as defined in Section 2. We will show the piecewise polynomial analogue of (6.6), namely,

$$n^\alpha \sigma^{(1)}_{r+1,n}(f, Y_s) \le \tilde{c}(\alpha, s), \quad n \ge \tilde{N}(\alpha, s), \tag{6.8}$$

and that it suffices to take $\tilde{N}(\alpha, s) = s(2s+1)$.

To this end, let $m \ge s$, and take two Chebyshev partitions, $\{x_{0,m}, \dots, x_{m,m}\}$ and $\{x_{0,m(2s+1)} \dots, x_{m(2s+1),m(2s+1)}\}$. Let $\tilde{O}_q$ be the connected components of $O(Y_s, m(2s+1))$ (see Section 2). Clearly, every $\tilde{O}_q$ is contained in an interval $J_{j,m} := I_{j,m} \cup I_{j+1,m}$, for some j, $2 \le j \le m$, since each interval $I_{j,m}$ contains $(2s+1)$ adjacent intervals of the type $I_{i,m(2s+1)}$, and $\tilde{O}_q$ contains *at most* $(2s+1)$ of the latter intervals.

Now, for each $\tilde{O}_q$, define S on the corresponding $J_{j,m}$ to be a polynomial of degree $\le r$, comonotone with f on $J_{j,m}$, that yields (6.7), i.e., satisfies

$$E^{1,\sigma}_{r+1}(f)_{J_{j,m}} \le \frac{c(\alpha, s)}{m^\alpha} = \frac{c(\alpha, s)(2s+1)^\alpha}{m^\alpha (2s+1)^\alpha}. \tag{6.9}$$

Evidently, doubling the constant on the right hand side of (6.9), we can guarantee that, in addition, $S(x_{j+1,m}) = f(x_{j+1,m})$.

Let $I_{i,m(2s+1)}$ be an interval where S has yet not been defined that is, a partition interval not contained in any of the above $J_{j,m}$. In particular, f is monotone on $I_{i,m(2s+1)}$, and without loss we may assume that it is nondecreasing there. By virtue of (6.7) there exists a nondecreasing polynomial $P_{r+1,i}$ of degree $\le r$, on $I_{i,m(2s+1)}$, satisfying

$$\|f - P_{r+1,i}\|_{I_{i,m(2s+1)}} = E^{1,\sigma}_{r+1}(f)_{I_{i,m(2s+1)}} \le \frac{c(\alpha, s)}{(m(2s+1))^\alpha}. \tag{6.10}$$

To simplify notation, we may assume that $I_{i,m(2s+1)} = [a, b]$, $b > a$, that $f(a) = 0$ and, by adding a constant to the polynomial, that $f(a) = P_{r+1,i}(a)$ (again doubling the constant on the right hand side of (6.10)).

We will further modify $P_{r+1,i}$, to ensure that

$$f(b) = P_{r+1,i}(b). \tag{6.11}$$

If (6.11) does not hold, then we distinguish between two cases.

First, assume that $P_{r+1,i}(b) > f(b)$. Then, let $0 < \theta := \frac{f(b)}{P_{r+1,i}(b)} < 1$. Obviously, $Q_{r+1,i}(x) := \theta P_{r+1,i}(x)$ is nondecreasing and $Q_{r+1,i}(b) = f(b)$. Also, by (6.10),

$$\begin{aligned}
\|Q_{r+1,i} - f\| &\le \|Q_{r+1,i} - P_{r+1,i}\| + \|P_{r+1,i} - f\| \\
&\le (1-\theta)\|P_{r+1,i}\| + 2\frac{c(\alpha, s)}{(m(2s+1))^\alpha} \\
&= (1-\theta)|P_{r+1,i}(b)| + 2\frac{c(\alpha, s)}{(m(2s+1))^\alpha} \\
&= |P_{r+1,i}(b) - f(b)| + 2\frac{c(\alpha, s)}{(m(2s+1))^\alpha} \\
&\le 4\frac{c(\alpha, s)}{(m(2s+1))^\alpha}.
\end{aligned}$$

Otherwise $P_{r+1,i}(b) < f(b)$. Let the linear function l be defined by

$$l(x) := (f(b) - P_{r+1,i}(b))\,\frac{x-a}{b-a}.$$

Then, clearly, l is nondecreasing on $I_{i,m(2s+1)}$ and so is $Q_{r+1,i} := P_{r+1,i} + l$. Also $Q_{r+1,i}(b) = f(b)$ and

$$\|Q_{r+1,i} - f\| \le |f(b) - P_{r+1,i}(b)| + \|P_{r+1,i} - f\| \le 4\frac{c(\alpha, s)}{(m(2s+1))^\alpha}.$$

Thus, in both cases we set $S := Q_{r+1,i}$ on $I_{i,m(2s+1)}$.

To summarize, we have constructed a piecewise polynomial S of degree $\le r$, which is comonotone with f in $[-1, 1]$ and,

$$\|f - S\| \le \frac{4c(\alpha, s)(2s+1)^\alpha}{(m(2s+1))^\alpha} =: \hat{c}(\alpha, s).$$

However, S may not be continuous at the right hand ends of the intervals $J_{j,m}$ corresponding to the $\tilde{O}_q$'s. So we change S going from left to right adding constants to the left part to match it to the right part to obtain $\tilde{S}$. This increases the error $\|f - \tilde{S}\|$ by at most an additional $s\hat{c}(\alpha, s)$.

In conclusion, we have established (6.8) with $\tilde{c}(\alpha, s) = 4(s+1)c(\alpha, s)(2s+1)^\alpha$, where $c(\alpha, s)$ as in (6.7), for every $n = m(2s+1)$, $m \geq s$. This, in turn, trivially implies (6.8) with $\tilde{N}(\alpha, s) = s(2s+1)$.

Now, $\tilde{S} \in \Sigma_{r+1,n}(Y_s) \cap \Delta^{(1)}(Y_s)$, so that by Lemma 10, we have

$$\begin{aligned} E_n^{(1)}(f, Y_s) &\leq E_n^{(1)}(\tilde{S}, Y_s) + \|\tilde{S} - f\| \\ &\leq c_2(r+1, s)\omega_{r+1}^\varphi(\tilde{S}, 1/n) + \tilde{c}(\alpha, s)n^{-\alpha} \\ &\leq c_2(r+1, s)\omega_{r+1}^\varphi(f, 1/n) + c_3\tilde{c}(\alpha, s)n^{-\alpha}, \end{aligned}$$

for all $n \geq \max\{c_1(r+1, s), \tilde{N}(\alpha, s)\}$.

Finally, combining (6.5) and Lemma 2, yields

$$\omega_{r+1}^\varphi(f, 1/n) \leq c_4(\alpha)n^{-\alpha}.$$

This completes the proof. □

Combining Theorem 1 with the positive result for $0 < \alpha < 1$ and $N = 1$ from [9], we conclude,

Corollary 2. For $\alpha \notin A_s$ and $N \leq \lceil \frac{\alpha}{2} \rceil$, there is "+" in Fig 2. in position (N, α).

(5) We may apply the ideas of the proof of Theorem 2 in order to obtain new information about some of the outstanding cases. We begin with $\alpha \in (2, 3]$. In particular, the following proposition will show that, for $\alpha = 3$, we have "$\oplus$". Recall that earlier we could not even guarantee that we have no "$-$" at some positions of the form $(N, 3)$.

Proposition 5. *For $\alpha \in (2, 3]$, $N \geq 3$ we have either "$+$" or "$\oplus$".*

Proof. Since we wish to prove at least "$\oplus$", we may limit ourselves to $n \geq N^*$, where $N^* = N^*(Y_s)$ is so big that any interval $J_{j,n}$, $1 < j < n-1$, contains at most one of the extreme points y_i, and both $J_{1,n}$ and $J_{n-1,n}$ contain no extreme point, that is, f is monotone there. Following the lines of proof of Theorem 2, it suffices to establish the inequalities

$$E_{N+1}^{1,\sigma}(f)_{J_{j,n}} \leq \frac{c(\alpha, s, N)}{n^\alpha}, \quad j = 1, \dots, n-1, \quad n \geq N^*. \tag{6.12}$$

Note that by our choice of N^*, $\sigma = 0$ or $\sigma = 1$.

Observing that $2 < \alpha < N+1 = 2+N-1$, we conclude by Lemma 2 that $f \in C_\varphi^2$, and

$$\omega_{N-1,2}^\varphi(f'', t) \leq c(\alpha, N)t^{\alpha-2}, \quad t > 0.$$

Hence, combining with (4.4), we obtain

$$|J_{j,n}|^2\omega_{N-1}(f'', |J_{j,n}|; J_{j,n}) \leq cn^{-2}\omega_{N-1,2}^\varphi(f'', 1/n) \leq c(\alpha, N)n^{-\alpha}, \quad 1 < j < n-1. \tag{6.13}$$

At the same time, by [5, Tables 2 and 16], we know that if a function $g \in C^{(2)}[-1, 1]$ is monotone in $[-1, 1]$, or if it changes its monotonicity once there, say, at $\tilde{Y}_1 = \{\tilde{y}_1\}$, then, respectively,

$$E_{N+1}^{(1)}(g) \leq c(N)\omega_{N-1}(g'', 1),$$

and

$$E_{N+1}^{(1)}(g, \tilde{Y}_1) \leq c(N)\omega_{N-1}(g'', 1).$$

Translating the last two inequalities to an interval $[a, b]$, (by the linear transformation $2y = (b-a)x + b + a$), we readily see that if g is monotone in $[a, b]$, or if g changes monotonicity once there, say, at $\tilde{Y}_1 = \{\tilde{y}_1\}$, then, respectively,

$$E^{(1)}_{N+1}(g)_{[a,b]} \le c(N)(b-a)^2\omega_{N-1}(g'', b-a; [a, b]),$$

and

$$E^{(1)}_{N+1}(g, \tilde{Y}_1)_{[a,b]} \le c(N)(b-a)^2\omega_{N-1}(g'', b-a; [a, b]).$$

Therefore, for each $J_{j,n}$, $1 < j < n-1$, we obtain by (6.13), that

$$E^{1,\sigma}_{N+1}(f)_{J_{j,n}} \le c(\alpha, N)|J_{j,n}|^2\omega_{N-1}(f'', |J_{j,n}|; J_{j,n}) \le c(\alpha, N)n^{-\alpha}.$$

This proves (6.12) for $1 < j < n-1$.

For the intervals $J_{1,n}$ and $J_{n-1,n}$, we observe that $2 < \alpha < 2 + 2N$, and apply Lemma 3 to obtain that f is continuously differentiable in $[-1, 1]$ and,

$$\omega_N(f', t) \le c(\alpha, N)t^{\frac{\alpha}{2}-1}, \quad t > 0.$$

As above, by [5, Table 2], we end up having

$$E^{1,0}_{N+1}(f)_{J_{1,n}} \le c(\alpha, N)|J_{1,n}|\omega_N(f', |J_{1,n}|; J_{1,n}) \le \frac{c(\alpha, N)}{n^2}\omega_N(f', 1/n^2) \le c(\alpha, N)n^{-\alpha}.$$

The case of $J_{n-1,n}$ is the same. This completes the proof of (6.12).

We proceed with the proof as in the proof of Theorem 2, except that the piecewise polynomials are of degree $< N + 1$ instead of being of degree $< r + 1$. □

(6) We still need to close some gaps for $\alpha \in (2s, 2s + 3]$.

Proposition 6. *For $\alpha \in (2s, 2s + 2]$ and $N = s + 2$, there is "$+$" in position (N, α), and for $\alpha \in (2s + 2, 2s + 3]$ and $N \ge s + 3$, there is "$+$" in position (N, α).*

Proof. We apply the same strategy as in the proof of Proposition 5. Assume first that $2s < \alpha \le 2s + 2$. Then from $2s < \alpha < 2s + 3$, we conclude by Lemma 2 that $f \in C^{2s}_\varphi$ and,

$$\omega^\varphi_{3,2s}(f^{(2s)}, t) \le c(\alpha, s, N)t^{\alpha-2s}, \quad t > 0.$$

Combining with (4.4), we obtain

$$|J_{j,n}|^{2s}\omega_3(f^{(2s)}, |J_{j,n}|; J_{j,n}) \le cn^{-2s}\omega^\varphi_{3,2s}(f^{(2s)}, 1/n) \le c(\alpha, s, N)n^{-\alpha}, \quad 1 < j < n-1.$$

As explained in the proof of Proposition 5, [5, Tables 16, 17 and 18] show us that

$$E^{1,\sigma}_{2s+3}(f)_{J_{j,n}} \le c(\alpha, s, N)|J_{j,n}|^{2s}\omega_3(f^{(2s)}, |J_{j,n}|; J_{j,n}) \le c(\alpha, s, N)n^{-\alpha}, \quad 1 < j < n-1.$$

In order to deal with $J_{1,n}$ and $J_{n-1,n}$, we use the fact that $2s < \alpha < 2s+4$, so that by Lemma 3, $f \in C^s[-1, 1]$ and,

$$\omega_2(f^{(s)}, t) \le c(\alpha, s, N)t^{\frac{\alpha}{2}-s}.$$

Again, we get by [5, Tables 16, 17 and 18] that

$$E^{1,\sigma}_{s+1}(f)_{J_{1,n}} \le c(\alpha, s, N)|J_{1,n}|^s\omega_2(f^{(s)}, |J_{1,n}|; J_{1,n}) \le c(\alpha, s, N)n^{-\alpha},$$

and similarly for $J_{n-1,n}$. We complete the proof as before.

Now, if $2s + 2 < \alpha \le 2s + 3$ and $N \ge s + 3$, then we write $2s + 2 < \alpha < s + N + 1 = 2s + 2 + N - s - 1$, and we conclude from Lemma 2, that $f \in C^{2s+2}_\varphi$ and,

$$\omega^\varphi_{N-s-1,2s+2}(f^{(2s+2)}, t) \le c(\alpha, s, N)t^{\alpha-2s-2}.$$

We may also write $2s+2<\alpha<2N=2s+2+2(N-s-1)$, and apply Lemma 3 to obtain that $f\in C^{s+1}[-1,1]$ and

$$\omega_{N-s-1}(f^{(s+1)},t)\le c(\alpha,s,N)t^{\frac{\alpha}{2}-s-1}.$$

Thus, we proceed as before. We leave the details to the reader. □

Finally, we summarize the " + " cases in the following statement (thus proving the last statement in Remark 3 in the introduction).

Theorem 3. *In all cases where we have " + ", we may take $N^*=N$.*

Proof. We are going to apply the generalized Whitney inequality for comonotone polynomial approximation (see [2, Corollary 3.1] and Pleshakov and Shatalina [14, Theorem 2]). The following Whitney inequality holds for comonotone polynomial approximation for $f\in C^r[-1,1]\cap\Delta^{(1)}(Y_s)$.

$$E^{(1)}_{k+r}(f,Y_s)\le c(k,r,s)\omega_k(f^{(r)},1), \tag{6.14}$$

when either $k=1$ and $r\ge 0$, or $k=2$ and $r=s$, or $r\ge s+1$ and $k\ge 1$.

Thus assume that for the triple (α,s,N) satisfying (1.2), there exists $N^*=N^*(\alpha,s,N)$ for which (1.3) is valid. Obviously, we may assume that $N^*\ge N$ for otherwise there is nothing to prove.

First, assume $N\le\lceil\alpha/2\rceil$. Take $r=N-1$ and observe that $\alpha>2r$. Let $2r<\beta<\min\{2r+2,\alpha\}$ and write $k=1$ so that $N=k+r$. Then by (1.2),

$$n^\beta E_n(f)\le 1,\quad n\ge k+r,$$

which, by virtue of Lemma 3, yields that $f\in C^r[-1,1]$ and

$$\omega_1(f^{(r)},1)\le c(\alpha)1^{\beta/2-r}=c(\alpha).$$

Hence, we apply (6.14) and conclude that

$$E^{(1)}_N(f,Y_s)\le c(\alpha,s). \tag{6.15}$$

In order to complete the proof in this case we have to prove that (1.3) is valid for $N\le n<N^*$. Indeed, noting that $N^*=N^*(\alpha,s,N)$, we obtain by (6.15),

$$n^\alpha E^{(1)}_n(f,Y_s)\le (N^*)^\alpha E^{(1)}_N(f,Y_s)\le (N^*(\alpha,s,N))^\alpha c(\alpha,s)=:c(\alpha,s,N),$$

and the proof is complete.

Second, assume that $\lceil\alpha/2\rceil=s+1$ and $N=s+2$. Then we take $r=s$ and $k=2$, so that $N=k+r$ and $2r=2s<\alpha<2r+2k$. Since (1.2) is satisfied, it follows by Lemma 3 that $f\in C^r[-1,1]$ and

$$\omega_2(f^{(r)},1)\le c(\alpha,s)1^{\alpha/2-r}=c(\alpha,s).$$

Hence, we again apply (6.14) and obtain (6.15). We complete the proof in this case as above.

Finally, if $N>\lceil\alpha/2\rceil\ge s+2$, then we take $r=s+1$ and $k=N-r$. Once more $2r=2s+2<\alpha<2N=2r+2k$, and (1.2) is satisfied. Hence, by virtue of Lemma 3, $f\in C^r[-1,1]$ and

$$\omega_k(f^{(r)},1)\le c(\alpha,s,N),$$

which, in turn, by virtue of (6.14), implies

$$E^{(1)}_N(f,Y_s)\le c(\alpha,s,N).$$

The proof of this case now follows as above. This completes the proof of our theorem. □

7. Proof of Theorem 1

(i) First, it follows by Theorem 3 that in all " + " cases, $N^* \le N$ (see Remarks 2 and 3 in the introduction).

(a) That one has " + ", in these α's, for $N \le \lceil \alpha/2 \rceil$, follows from Theorem 2.

(b) This case follows from Proposition 6.

(c) That one has " + " for $\alpha > 2s+2$ for all $N \ge 2$, follows from Propositions 4 and 6.

(ii) Both cases are the conclusion of Section 5(1).

(iii) For $\alpha \in A_s$ it follows from Section 5(5). For $s \ge 2$ and $2 \le \lceil \alpha/2 \rceil \le s$ we have $\oplus$ for all $N \ge \lceil \alpha/2 \rceil + 1$, by virtue of Proposition 1. For $s \ge 2$ and $\lceil \alpha/2 \rceil = s+1$ we obtain $\oplus$ for $N \ge s+3$ by combining Propositions 2 and 4, bearing in mind Remark 1. For $\lceil \alpha/2 \rceil = 1$, it follows by Section 6(3) that we have $\oplus$ for $2 \le N \le s+1$. Finally, for $s = 1$ if $1 < \alpha \le 2$, see Section 6(1), and if $2 < \alpha \le 4$, then we combine Propositions 2 through 5 to obtain $\oplus$ for $N \ge 4$. This completes the proof. □

References

[1] V. K. Dzyadyk and I. A. Shevchuk, *Theory of Uniform Approximation of Functions by Polynomials*, Walter de Gruyter, Berlin (2008).

[2] Ya. Gilewicz and I. A. Shevchuk, *Comonotone approximation*, Fundam. Prikl. Mat. 2 (1996), 319–363 (in Russian, with English and Russian summaries).

[3] I. I. Ibragimov, "Sur la valeur asymptotique de la meilleure approximation d'une fonction ayant un point singulier réel" (in Russian, French summary), Izv. Akad. Nauk SSSR, Ser. Mat. 10 (1946), 429–460.

[4] K.A. Kopotun, and D. Leviatan, *Comonotone polynomial approximation in* $L_p[-1,1]$, $0 < p \le \infty$, Acta Math. Hungarica **77** (1997), 301–310.

[5] K. A. Kopotun, D. Leviatan, A. Prymak, and I. A. Shevchuk, *Uniform and pointwise shape preserving approximation by algebraic polynomials*, Surveys in Approximation Theory 6 (2011), 24–74.

[6] K. A. Kopotun, D. Leviatan and I. A. Shevchuk, *Convex Polynomial Approximation in the Uniform Norm: Conclusion*, Canad. J. Math. **57**(6), (2005), 1224–1248.

[7] K. Kopotun, D. Leviatan, and I. A. Shevchuk, *Are the degrees of the best (co)convex and unconstrained polynomial approximations the same? II*, Ukrainian Math. J. **62** (2010), 369–386.

[8] D. Leviatan, *Monotone and comonotone polynomial approximation revisited* J. Approx. Theory **53** (1988), 1–16.

[9] D. Leviatan, D. V. Radchenko and I. A. Shevchuk, *Positive Results and Counterexamples in Comonotone Approximation*, Constr. Approx. **36** (2012), 243–266.

[10] D. Leviatan and I. A. Shevchuk, *Some positive results and conterexamples in comonotone approximation*, J. Approx. Theory **89** (1997), 195–206.

[11] D. Leviatan and I. A. Shevchuk, *Some positive results and counterexamples in comonotone approximation II*, J. Approx. Theory **100** (1999), 113–143.

[12] D. Leviatan and I. A. Shevchuk, *More on comonotone polynomial approximation*, Constr. Approx. **16** (2000), 475–486.

[13] O. N. Nesterenko and T. O. Petrova, *On one problem for comonotone approximation*, Ukrainian Math. J. **57** (2005), 1667–1673.

[14] M. G. Pleshakov and A. V. Shatalina, *Piecewise Coapproximation and the Whitney Inequality*, J. Approx. Theory **105** (2000), 189–210.

Raymond and Beverly Sackler School of Mathematics, Tel Aviv University, 69978 Tel Aviv, Israel
E-mail address: leviatan@post.tau.ac.il

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
E-mail address: shevchuk@univ.kiev.ua

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
E-mail address: an7onlooker@gmail.com