COCONVEX APPROXIMATION

D. Leviatan and I. A. Shevchuk[1]

Abstract. Let $f \in \mathbb{C}[-1,1]$ change its convexity finitely many times, in the interval. We are interested in estimating the degree of approximation of f by polynomials which are coconvex with it, namely, polynomials that change their convexity exactly at the points where f does. We discuss some Jackson type estimates where the constants involved depend on the location of the points of change of convexity. We also show that in some cases the constants may be taken independent of the points of change of convexity, but that in other cases this dependence is essential. But mostly we obtain such estimates for functions f that themselves are continuous piecewise polynomials on the Chebyshev partition, which form a single polynomial in a small neighborhood of each point of change of convexity. These estimates involve the k modulus of smoothness of the piecewise polynomials when they themselves are of degree $k-1$.

§1. Introduction

Let $f \in \mathbb{C}[-1,1]$ change its convexity finitely many times, say $s \geq 0$ times, in the interval. We are interested in estimating the degree of approximation of f by polynomials which are coconvex with it, namely, polynomials that change their convexity exactly at the points where f does.

In a recent survey [9] we have collected all known positive and negative results on monotone and comonotone approximation on a finite interval, by algebraic polynomials in the uniform norm (see also [8]). We have established complete truth tables for the validity of Jackson-type estimates, involving the ordinary k-th moduli of smoothness of the r-th derivative of a given monotone or piecewise monotone function, as well as estimates involving the Ditzian-Totik moduli of smoothness. The two main ingredients in the proofs of all positive results in these truth tables were first the approximation of an arbitrary such function by piecewise polynomials with the same changes of monotonicity, and then the approximation of such a piecewise monotone piecewise polynomial, by polynomials with the same changes of monotonicity. See [10] for details.

Our intention in our research program is to construct the corresponding truth table for convex and coconvex polynomial approximation. The main thrust in this paper is to

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obtain Jackson-type estimates for the approximation of a continuous piecewise polynomial which changes convexity finitely many times in the interval, by algebraic polynomials that change convexity at exactly the same points. The main result is Theorem 3 stated below, which is the analogue of [10, Proposition 3]. Our strategy for the future is to approximate an arbitrary continuous function that changes convexity finitely many times in the interval, by an appropriate coconvex piecewise polynomial which in turn, by virtue of Theorem 3, will be approximated by a coconvex polynomial. In order to illustrate the intricacies we begin in Section 3 with some negative results for the coconvex polynomial approximation of more general piecewise convex functions (see Theorem 1 below). Also as a byproduct of Theorem 4 below, we obtain one significant positive result for coconvex polynomial approximation (Theorem 2 below). So the outlay of the paper is the following. We state the main results in Section 2. Section 3 contains the construction of the negative results. Section 4 contains auxiliary lemmas. Section 5 is devoted to the proof of Theorem 4 which is a preliminary step and a special case of Theorem 3, and as a byproduct, its proof yields a proof of Theorem 2. We need some more preparation and lemmas in Sections 6 and 7, and in Section 8 we prove Theorem 5 and with it conclude the proof of Theorem 3. Many of the methods we apply are modifications of similar ones in the papers by DeVore, Dzyubenko, Gilewicz, Kopotun, Mania, Yu and the authors (see the References). Nevertheless, for the sake of completeness proofs are given.

In the sequel we will have positive constants c that depend only on s and k, and we will have positive constants C, which may also depend on $b \in \mathbb{N}$. We will use the notation c and C for such constants which are of no significance to us and may differ on different occurrences, even in the same line. However, we will have constants with indices $c_0, c_1, \ldots, c_5$ and C_0, when we have a reason to keep track of them in the computations that we have to carry out in the proofs.

§2. The main results

Let $I := [-1,1]$ and denote by $\mathbb{C}$ and $\mathbb{C}^r$, respectively the space of continuous functions, and that of r-times continuously differentiable function on I, equipped with the uniform norm

$$\|f\| := \max_{x\in I} |f(x)|.$$

Given $f \in \mathbb{C}$, and $k \in \mathbb{N}$, let

$$\Delta_h^k f(x) := \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x - \frac{k}{2}h + ih),$$

be the symmetric difference of order k, defined for all x and $h \geq 0$, such that $x \pm \frac{k}{2}h \in I$.

The Ditzian-Totik (DT-)moduli of smoothness [3] are defined by

$$\omega_k^\varphi(f,t) := \sup_{0\leq h\leq t} \sup_x |\Delta_{h\varphi(x)}^k f(x)|, \quad t \geq 0,$$

where $\varphi(x)=\sqrt{1-x^2}$, and the inner supremum is taken over all x such that $x\pm\frac{k}{2}h\varphi(x)\in I$. We also deal with the ordinary moduli of smoothness which are given by the above with $\varphi(x)\equiv 1$ replacing the above φ, namely,

$$\omega_k(f,t):=\sup_{0\le h\le t}\sup_x|\Delta_h^k f(x)|,\quad t\ge 0,$$

where the inner supremum is taken over all x such that $x\pm\frac{k}{2}h\in I$.

Denote by $\mathbb{Y}_s$, $s\in\mathbb{N}$, the set of all collections $Y_s:=\{y_i\}_{i=1}^s$, such that $-1<y_s<\cdots<y_1<1$, and for $s=0$, we write $\mathbb{Y}_0:=\{\emptyset\}$. For later reference set $y_0:=1$ and $y_{s+1}:=-1$. Finally, let $\Delta^2(Y_s)$ denote the collection of all functions $f\in\mathbb{C}$ that change convexity at the set Y_s, and are convex in $[y_1,1]$.

Given $n\in\mathbb{N}$, $n>1$, we set $x_j:=x_{j,n}:=\cos(j\pi/n)$, $j=0,\dots,n$, the Chebyshev partition of $[-1,1]$, and we denote $I_j:=I_{j,n}:=[x_j,x_{j-1}]$, $j=1,\dots,n$. Let $\Sigma_{k,n}$ be the collection of all continuous piecewise polynomials of degree $k-1$, on the Chebyshev partition and let $\Sigma^1_{k,n}\subseteq\Sigma_{k,n}$, be the subset of all continuously differentiable such functions. That is, if $S\in\Sigma_{k,n}$, then

$$S|_{I_j}=p_j,\quad j=1,\dots,n,$$

where $p_j\in\Pi_{k-1}$, the collection of polynomials of degree $\le k-1$, and

$$p_j(x_j)=p_{j+1}(x_j),\quad j=1,\dots,n-1,$$

and if $S\in\Sigma^1_{k,n}$, then in addition,

$$p_j'(x_j)=p_{j+1}'(x_j),\quad j=1,\dots,n-1.$$

Given $Y_s\in\mathbb{Y}_s$, let

$$O_i:=O_{i,n}(Y_s):=(x_{j+1},x_{j-2}),\quad\text{if}\quad y_i\in[x_j,x_{j-1}),$$

where $x_{n+1}:=-1$, $x_{-1}:=1$, and denote

$$O=O(n,Y_s):=\bigcup_{i=1}^s O_i,\quad O(n,\emptyset):=\emptyset.$$

Finally, we write $j\in H=H(n,Y_s)$, if $I_j\cap O=\emptyset$.

Denote by $\Sigma_{k,n}(Y_s)\subseteq\Sigma_{k,n}$ and $\Sigma^1_{k,n}(Y_s)\subseteq\Sigma^1_{k,n}$, the subsets of those piecewise polynomials for which

$$p_j\equiv p_{j+1},\quad\text{whenever both}\quad j,(j+1)\notin H.$$

We wish to approximate a general function $f\in\Delta^2(Y_s)$, by means of polynomials which are coconvex with f, that is, which belong to $\Delta^2(Y_s)$. We denote by

$$E_n^{(2)}(f,Y_s):=\inf_{p_n\in\Pi_n\cap\Delta^2(Y_s)}\|f-p_n\|,$$

where Π_n is the set of polynomials of degree not exceeding n.

In a recent paper [7] with Kopotun, we proved that if a function $f \in C[-1,1]$ changes convexity at Y_s, then

$$E_n^{(2)}(f, Y_s) \le c\omega_3^{\varphi}\left(f, \frac{1}{n}\right) \le c\omega_3\left(f, \frac{1}{n}\right), \quad n \ge N, \tag{2.1}$$

where $c = c(s)$, is a constant which depends only on s, and $N = N(Y_s)$, a constant which depends on the location of the points Y_s. On the other hand, Wu and Zhou [14] proved that for $k \ge 4$, estimate (2.1) cannot be had with ω_3 replaced by ω_k, and Pleshakov and Shatalina [11] have just proved, that (2.1) is not valid with $N = N(s)$ replacing $N = N(Y_s)$.

In this paper we will prove that if $s > 1$, then even

$$E_n^{(2)}(f, Y_s) \le c\omega\left(f, \frac{1}{n}\right), \quad n \ge N, \tag{2.2}$$

is not valid with $N = N(s)$ replacing $N = N(Y_s)$. In fact we prove more, namely,

Theorem 1. *For no $k \ge 1$, $r = 0, 1, 2, 3$ and $s \ge 2$, is it possible to have constants $c = c(k, r, s)$ and $N = N(k, r, s)$, depending only on k, r and s, such that the inequality*

$$E_n^{(2)}(f, Y_s) \le \frac{c}{n^r}\omega_k(f^{(r)}, \frac{1}{n}), \tag{2.3}$$

holds for all $n \ge N$, and for all $f \in \mathbb{C}^r \cap \Delta^2(Y_s)$.

On the other hand, we show that if $s = 1$, then (2.2) is valid for $N = 1$, in fact we prove that

Theorem 2. *Let $f \in \mathbb{C} \cap \Delta^2(Y_1)$, that is, changes convexity once on $[-1, 1]$. Then*

$$E_n^{(2)}(f, Y_1) \le c\omega_2^{\varphi}\left(f, \frac{1}{n}\right), \quad n \ge 1. \tag{2.4}$$

As mentioned above, in view of [11], (2.4) is the best that one can expect.
However, our main positive result is

Theorem 3. *For every $k, n \in \mathbb{N}$ and $s \in \mathbb{N}_0$ there are constants $c = c(k, s)$ and $c_* = c_*(k, s)$, such that if $S \in \Sigma_{k,n}(Y_s) \cap \Delta^2(Y_s)$, then there is a polynomial $P_n \in \Delta^2(Y_s)$ of degree $\le c_* n$, satisfying*

$$\|S - P_n\| \le c\omega_k^{\varphi}\left(S, \frac{1}{n}\right). \tag{2.5}$$

Theorem 3 is trivial for $k = 1$, since $\Sigma_{1,n} \subseteq \Pi_0$. On the other hand it is new for $k \ge 4$ even for convex approximation, namely the case $s = 0$. As was proved by Shvedov [13], (2.5) cannot be had for a general convex function f (that is $s = 0$), with $k \ge 4$. The proof for $k \ge 2$ is divided into two stages. First we prove a special case of Theorem 3, which in particular, proves it for the case $k = 2$, namely.

Theorem 4. *For every $k, n \in \mathbb{N}$ and $s \in \mathbb{N}_0$, if $S \in \Sigma_{k,n}(Y_s) \cap \Delta^2(Y_s)$, then there exists a polynomial $P_n \in \Delta^2(Y_s)$, of degree not exceeding cn, such that*

$$\|S - P_n\| \leq c\omega_2^{\varphi}\left(S, \frac{1}{n}\right). \tag{2.6}$$

Then we note that by virtue of Lemma 1 below, in order to conclude the proof of Theorem 3, it suffices to prove

Theorem 5. *For every $k, n \in \mathbb{N}$ and $s \in \mathbb{N}_0$ there are constants c and c_*, such that if $S \in \Sigma^1_{k,n}(Y_s) \cap \Delta^2(Y_s)$, then there is a polynomial $P_n \cap \Delta^2(Y_s)$ of degree $\leq c_* n$, satisfying (2.5).*

Note that by the above, we have to prove Theorem 5 only for $k \geq 3$, but the cases $k = 1, 2$ are anyway trivial in this setting since $\Sigma^1_{2,n} \subseteq \Pi_1$.

Lemma 1. *Let $k \geq 3$. Then for each $S \in \Sigma_{k,n}(Y_s) \cap \Delta^2(Y_s)$, there is an $\tilde{S} \in \Sigma^1_{k,n}(Y_s) \cap \Delta^2(Y_s)$, such that*

$$\|S - \tilde{S}\| \leq c\omega_k^{\varphi}\left(S, \frac{1}{n}\right). \tag{2.7}$$

In particular

$$\omega_k^{\varphi}\left(\tilde{S}, \frac{1}{n}\right) \leq c\omega_k^{\varphi}\left(S, \frac{1}{n}\right).$$

Proof. For each $2 \leq j \leq n$, set

$$a_j(x) := \frac{1}{2}\frac{x_{j-1} - x_{j-2}}{x_{j-1} - x_j}\frac{p'_{j-1}(x_{j-1}) - p'_j(x_{j-1})}{x_j - x_{j-2}}(x - x_j)^2, \quad \text{if} \quad j, (j-1) \in H,$$

$$a_j(x) := \frac{1}{2}\frac{p'_{j-1}(x_{j-1}) - p'_j(x_{j-1})}{x_{j-1} - x_j}(x - x_j)^2, \quad \text{if} \quad j \in H,\ (j-1) \notin H,$$

$$\text{and} \quad a_j(x) := 0, \quad \text{if} \quad j \notin H.$$

Also for each $1 \leq j \leq n-1$, set

$$b_j(x) := \frac{1}{2}\frac{x_j - x_{j+1}}{x_j - x_{j-1}}\frac{p'_j(x_j) - p'_{j+1}(x_j)}{x_{j+1} - x_{j-1}}(x - x_{j-1})^2, \quad \text{if} \quad j, (j+1) \in H,$$

$$b_j(x) := \frac{1}{2}\frac{p'_j(x_j) - p'_{j+1}(x_j)}{x_{j-1} - x_j}(x - x_{j-1})^2, \quad \text{if} \quad j \in H,\ (j+1) \notin H,$$

$$\text{and} \quad b_j(x) := 0, \quad \text{if} \quad j \notin H.$$

Finally set $a_1(x) := 0$ and $b_n(x) := 0$. Then

$$\tilde{S}(x) = p_j(x) + a_j(x) + b_j(x) + J(x), \quad x \in I_j,$$

is the required function, where J is a piecewise constant function with jumps in at most the $2s$ points x_j near the y_i's, explicitly, the jumps at these x_j's are

$$J(x_j+)-J(x_j-):=\begin{cases}\frac{1}{2}[p_j'(x_j)-p_{j+1}'(x_j)](x_j-x_{j+1}) & \text{if} \quad j\notin H,\ (j+1)\in H\\ \frac{1}{2}[p_j'(x_j)-p_{j+1}'(x_j)](x_j-x_{j-1}) & \text{if} \quad j\in H,\ (j+1)\notin H.\end{cases}$$

Indeed, straightforward computations show that $\tilde{S}\in\Sigma^1_{k,n}(Y_s)\cap\Delta^2(Y_s)$, and by Markov's inequality

$$|p_j'(x_j)-p_{j+1}'(x_j)|\le\frac{2k^2}{x_{j-1}-x_j}\|p_j-p_{j+1}\|_{I_j}.$$

Thus (2.7) readily follows by the inequality

$$\|p_j-p_{j+1}\|_{I_j}\le c\omega_k^\varphi\left(S,\frac{1}{n}\right),$$

which is an immediate consequence of [10, Lemma 9] (see more details at the beginning of Section 6). $\square$

§3. Negative results

Given $0<b<1$, set

$$g_b''(x):=\begin{cases}-b^{-4}(x^2-b^2)^2, & |x|<b,\\ 0, & \text{elsewhere,}\end{cases}$$

and let

$$g_b(x):=\int_0^x(x-u)g_b''(u)\,du.$$

Then clearly $g_b\in\mathbb{C}^3$, and it is readily seen that

$$\begin{aligned}\|g_b\|&=\frac{8b}{15}-\frac{b^2}{6}\le\frac{2b}{3},\quad \|g_b'\|=\frac{8b}{15},\\ \|g_b''\|&=1,\quad\text{and}\quad \|g_b^{(3)}\|=\frac{8}{3\sqrt{3}}b^{-1}\le 2b^{-1}.\end{aligned}\tag{3.1}$$

Lemma 2. *Given $n\ge1$, for each polynomial p_n of degree $\le n$, and satisfying*

$$(x^2-b^2)p_n''(x)\ge0,\quad x\in[-\frac{1}{2},\frac{1}{2}],$$

with $b=\frac{1}{2}n^{-\frac{4}{3}}$, we have

$$\|g_b-p_n\|>\frac{b}{40}.$$

Proof. First we observe that $p_n''(\pm b) = 0$, and that $p_n''(x) \le 0$, for $-b < x < b$. Assume that for some $-b < x_0 < b$, $p_n''(x_0) < -\frac{1}{4}$. Then

$$|[p_n''; -b, x_0, b]| = \frac{|p_n''(x_0)|}{(b - x_0)(b + x_0)} > \frac{1}{4b^2}.$$

Since

$$[p_n''; -b, x_0, b] = \frac{1}{2} p_n^{(4)}(\theta),$$

for some $-b < \theta < b(\le \frac{1}{12})$, it follows by Bernstein's inequality that

$$n^4 \|p_n\| \ge \frac{1}{2} |p_n^{(4)}(\theta)| > \frac{1}{4b^2}.$$

Now by (3.1) and the prescribed value of b,

$$\|g_b - p_n\| \ge \|p_n\| - \|g_b\| > \frac{1}{4n^4 b^2} - \frac{2b}{3} = \frac{4b}{3}. \tag{3.2}$$

If on the other hand, $p_n''(x) \ge -\frac{1}{4}$, for all $-b < x < b$, then we represent p_n in the form

$$p_n(x) = p_n(0) + x p_n'(0) + \int_0^x (x - u) p_n''(u)\, du.$$

Since $p_n''(x) \ge 0$ for $b \le |x| \le \frac{1}{2}$, it follows that

$$\begin{aligned} p_n(-\frac{1}{2}) - 2p_n(0) + p_n(\frac{1}{2}) &= \int_0^{\frac{1}{2}} (\frac{1}{2} - u) p_n''(u)\, du + \int_0^{-\frac{1}{2}} (-\frac{1}{2} - u) p_n''(u)\, du \\ &\ge \int_0^b (\frac{1}{2} - u) p_n''(u)\, du + \int_0^b (\frac{1}{2} - u) p_n''(-u)\, du \ge -\frac{b}{4}. \end{aligned}$$

Similarly,

$$\begin{aligned} g_b(-\frac{1}{2}) - 2g_b(0) + g_b(\frac{1}{2}) &= 2\int_0^b (\frac{1}{2} - u) g''(u)\, du \\ &= -\frac{8b}{15} + \frac{b^2}{3}. \end{aligned}$$

Therefore

$$\begin{aligned} 4\|g_b - p_n\| &\ge (p_n(-\frac{1}{2}) - g_b(-\frac{1}{2})) - 2(p_n(0) - g_b(0)) + (p_n(\frac{1}{2}) - g_b(\frac{1}{2})) \\ &\ge -\frac{b}{4} + \frac{8b}{15} - \frac{b^2}{3} \ge \frac{b}{10}. \end{aligned}$$

Thus together with (3.2), this concludes the proof of Lemma 2. $\square$

As an immediate consequence we get

Corollary 1. *For every constant $A > 1$ there exists an $N(A)$ sufficiently large such that if $n > N(A)$, then for any $s \geq 2$, there is a function $g = g_n \in C^3[-1,1]$, which changes convexity s times in $[-1,1]$, and such that any polynomial p_n of degree $\leq n$ which is coconvex with it, satisfies*

$$\|g - p_n\| > \frac{A\|g^{(3)}\|}{n^3},$$

$$\|g - p_n\| > \frac{A\|g''\|}{n^2},$$

and

$$\|g - p_n\| > \frac{A\|g'\|}{n}.$$

Proof. Let $N(A) = (80A)^3$ and let $s \geq 2$. We take $b = b_n$, $n > N(A)$, as in Lemma 2, and let $g = g_b$. The function g changes convexity at $y_2 = -b$ and $y_1 = b$, it is convex in $[y_1, 1]$, and if $s > 2$, then we take $s - 2$ arbitrary points satisfying $-1 < y_s < \cdots < y_3 < -\frac{1}{2}$, and regard g as changing convexity at these points too, hence $g \in \Delta^2(Y_s)$. If the polynomial p_n is coconvex with g, then it satisfies the requirements of Lemma 2. Therefore, by Lemma 2 we have

$$\|g - p_n\| > \frac{b}{40} \geq \frac{\|g^{(3)}\|b^2}{80} > \frac{A\|g^{(3)}\|}{n^3},$$

$$\|g - p_n\| > \frac{b}{40} = \frac{\|g''\|b}{40} > \frac{A\|g''\|}{n^2},$$

and

$$\|g - p_n\| > \frac{b}{40} = \frac{3n\|g'\|}{64n} > \frac{A\|g'\|}{n}. \quad \square$$

Remark. It should be noted that the function g_b above is independent of A.

We are ready to prove Theorem 1.

Proof of Theorem 1. The proof readily follows from the observation that for all $k \geq 1$,

$$\omega_k(f,t) \leq 2^{k-1}\omega(f,t) \leq 2^{k-1}t\|f'\|,$$

which by Corollary 1 does not allow the case $r = 0$ in (2.3) and

$$\omega_k(f,t) \leq 2^k\|f\|,$$

which takes care of the other cases. $\square$

§4. Some auxiliary lemmas

We begin with two lemmas of independent interest which are needed only in the proof of Theorem 4. We need the notation $[f; z_1, z_2, z_3]$ for the second divided difference of $f \in \mathbb{C}$ at the points z_1, z_2 and z_3.

Lemma 3. *Let $E := [a, b] \subset [0, 1]$ and set $X_E'' := \chi_E$, where χ_E is the characteristic function of E. Then for every $x_0 \in (0, 1)$, we have*

$$\frac{(b-a)^2}{2} < [X_E; 0, x_0, 1] < b - a,$$

Proof. Recall that if a function $f \in C^1[0, 1]$ has an absolutely continuous first derivative, then its second divided difference possesses the well known representation,

$$[f; 0, x_0, 1] = \int_0^1 \int_0^x f''(x - (1 - x_0)y)\, dy\, dx.$$

Hence,

$$\Delta := [X_E; 0, x_0, 1] = \int_0^1 \int_0^x \chi_E(x - (1 - x_0)y)\, dy\, dx,$$

and we observe that, putting $\lambda := (1 - x_0)^{-1}$, Δ is the area of the set

$$A := \{(x, y) : a \le x - \lambda^{-1} y \le b\} \cap \{(x, y) : 0 \le y \le x \le 1\}.$$

Note that A is readily seen to be the intersection of the right-angle triangle bounded by the x–axis and the lines $y = x$ and $x = 1$, with the parallelogram in the first quadrant, the basis of which is $[a, b]$, the height 1, and the sides of which are the lines $y = \lambda(x - a)$ and $y = \lambda(x - b)$. The area of the parallelogram is $b - a$, hence the upper estimate.

As for the lower bound, we observe that since $\lambda > 1$, it follows that A contains the right-angle triangle which is bounded by the x-axis and the lines $x = b$ and $y = x - a$, the area of which is exactly $\frac{1}{2}(b - a)^2$. The proof of the lower estimate is therefore concluded. $\square$

Corollary 2. *If $E \subseteq [0, 1]$ is a finite union of intervals, then*

$$[X_E; 0, x_0, 1] < \operatorname{meas} E =: |E|.$$

The second result is

Lemma 4. *Let p_k be a polynomial of degree not exceeding k and let $a < b$. If*

$$\operatorname{meas}\{x \in [a, b] : p_k''(x) \le 0\} < \frac{b - a}{16k^3},$$

then for every $x_0 \in (a, b)$,

$$[p_k; a, x_0, b] \ge 0.$$

Proof. Without loss of generality assume that $a = 0$ and $b = 1$. If $p_k'' \equiv 0$, then there is nothing to prove, so we may assume that $\|p_k''\|_{[0,1]} := \max\{|p_k''(x)| : 0 \le x \le 1\} = 1$. Write

$$E_2 := \{x \in [0,1] : p_k''(x) \le 0\},$$

so that E_2 is a finite union of intervals, and let $x \in E_2$ be arbitrary. Then there is an $x_2 \in E_2$ such that $|x - x_2| \le |E_2|$ and $p_k''(x_2) = 0$. By Markov's inequality,

$$|p_k''(x)| = |(x - x_2)p_k^{(3)}(\theta)| \le |E_2| 2k^2 \|p_k''\|_{[0,1]} < \frac{1}{8k},$$

so that

$$p_k''(x) > -\frac{1}{8k}, \quad x \in E_2. \tag{4.1}$$

Since we have assumed that $\|p_k''\|_{[0,1]} = 1$, this implies that there exists $x_1 \in [0,1]$ such that $p_k''(x_1) = 1$. We take an interval $E_1 \subset [0,1]$ of length $|E_1| = \frac{1}{4k^2}$ which contains x_1. Then for each $x \in E_1$, it follows again by Markov's inequality that

$$|p_k''(x) - p_k''(x_1)| = |(x - x_1)p_k^{(3)}(\theta)| \le |E_1| 2k^2 \|p_k''\|_{[0,1]} = \frac{1}{2},$$

which in turn implies that

$$p_k''(x) \ge \frac{1}{2}, \quad x \in E_1. \tag{4.2}$$

Combining (4.1) and (4.2) we get,

$$p_k''(x) \ge \frac{1}{2}\chi_{E_1} - \frac{1}{8k}\chi_{E_2}, \quad x \in [0,1].$$

By virtue of Lemma 3 and its corollary we obtain

$$[p_k; 0, x_0, 1] \ge \frac{1}{2}\frac{1}{2}|E_1|^2 - \frac{1}{8k}|E_2| \ge \frac{1}{2^6 k^4} - \frac{1}{8k}\frac{1}{16k^3} > 0. \quad \square$$

Now denote

$$\rho_n(x) := \frac{1}{n^2} + \frac{1}{n}\sqrt{1 - x^2} = \frac{1}{n^2} + \frac{1}{n}\varphi(x).$$

Throughout the paper we will have x and n as the generic variables, so whenever it will be clear that we deal with them, then we will write ρ for $\rho_n(x)$. For each $j = 1, \dots, n$ set $h_j = h_{j,n} := |I_j| = x_{j-1} - x_j$, where we recall that $x_j := x_{j,n} := \cos \pi j/n$ are the Chebyshev nodes. Then the following inequalities are well-known (see, e.g., [10]).

$$\begin{gathered}
\rho < h_j < 5\rho, \quad x \in I_j, \\
h_{j\pm 1} < 3h_j, \\
\rho_n^2(y) < 4\rho(|x-y| + \rho), \quad x, y \in I, \\
(|x-y| + \rho)/2 < |x-y| + \rho_n(y) < 2(|x-y| + \rho), \quad x, y \in I.
\end{gathered} \tag{4.3}$$

In particular,

$$(|x - x_j| + h_j)/10 < |x - x_j| + \rho < 2(|x - x_j| + h_j), \quad x \in I, \quad j = 0, \dots, n. \tag{4.4}$$

The next two lemmas are needed in the proofs of both Theorems 4 and 5.

Lemma 5. *If* $0 \le j \le i < J \le n$, *then*

$$\frac{1}{2}(J-j) \le \frac{x_j - x_J}{x_i - x_{i+1}} \le (J-j)^2. \tag{4.5}$$

Furthermore, if either $J \le 3j$ *or* $n-j \le 3(n-J)$, *then*

$$\frac{1}{2}(J-j) \le \frac{x_j - x_J}{x_i - x_{i+1}} \le 2(J-j). \tag{4.6}$$

Proof. Let $t := \frac{\pi}{2n}$. We begin with the upper bound and first assume that $2i+1 \le J+j$. Then

$$\begin{aligned}\frac{x_j - x_J}{x_i - x_{i+1}} &= \frac{\sin{(J+j)t}\sin{(J-j)t}}{\sin{(2i+1)t}\sin t} \\ &\le \frac{J+j}{2i+1}(J-j) \\ &\le \frac{J+j}{2j+1}(J-j) \le (J-j)^2,\end{aligned}$$

where we have used the fact that $\sin u/u$ is decreasing for $0 < u < \pi$. If on the other hand $2i+1 > J+j$, then we observe that $x_j - x_J = x_{n-J} - x_{n-j}$ and $x_i - x_{i+1} = x_{n-i-1} - x_{n-i}$, and $2(n-i-1)+1 < (n-J)+(n-j)$. Thus we obtain the same bound. This proves the upper bound in (4.5). Further, if $J \le 3j$, then clearly $\frac{J+j}{2j+1} \le 2$, so that the upper bound in (4.6) follows. Similar considerations yield the upper bound in (4.6) when $n-j \le 3(n-J)$.

As for the lower bound, we first assume that $J \le \frac{n}{2}$. Then

$$\begin{aligned}\frac{x_j - x_J}{x_i - x_{i+1}} &\ge \frac{x_j - x_J}{x_{J-1} - x_J} \\ &= \frac{\sin 2Jt + \sin 2jt}{2\sin{(2J-1)t}}\frac{\tan{(J-j)t}}{\sin t} \\ &\ge \frac{1}{2}(J-j).\end{aligned}$$

If $j \ge \frac{n}{2}$, then we have the symmetric situation and the proof is the same. We are left with the case $j < \frac{n}{2} < J$. To this end we observe that if n is even, then $x_{\frac{n}{2}} - x_{\frac{n}{2}+1} = x_{\frac{n}{2}-1} - x_{\frac{n}{2}} \ge x_i - x_{i+1}$, $j \le i < J$. Hence by the above inequalities

$$\begin{aligned}\frac{x_j - x_J}{x_i - x_{i+1}} &= \frac{(x_j - x_{\frac{n}{2}}) + (x_{\frac{n}{2}} - x_J)}{x_i - x_{i+1}} \\ &\ge \frac{x_j - x_{\frac{n}{2}}}{x_{\frac{n}{2}-1} - x_{\frac{n}{2}}} + \frac{x_{\frac{n}{2}} - x_J}{x_{\frac{n}{2}} - x_{\frac{n}{2}+1}} \\ &\ge \frac{1}{2}\big((\frac{n}{2} - j) + (J - \frac{n}{2})\big) = \frac{1}{2}(J-j).\end{aligned}$$

If on the other hand n is odd, then the biggest denominator is $x_{\frac{n-1}{2}} - x_{\frac{n+1}{2}}$. Observe that $x_{i,n} = x_{2i,2n}$ so that by the inequality for the even case we have

$$\begin{aligned}\frac{x_j - x_J}{x_i - x_{i+1}} &\geq \frac{x_j - x_J}{x_{\frac{n-1}{2}} - x_{\frac{n+1}{2}}} \\ &= \frac{x_{2j,2n} - x_{2J,2n}}{x_{n-1,2n} - x_{n+1,2n}} \\ &= \left(\frac{(x_{n-1,2n} - x_{n,2n}) + (x_{n,2n} - x_{n+1,2n})}{x_{2j,2n} - x_{2J,2n}}\right)^{-1} \\ &\geq \left(\frac{2}{2J-2j} + \frac{2}{2J-2j}\right)^{-1} = \frac{1}{2}(J-j). \quad \square\end{aligned}$$

Given Y_s, $s > 0$, set

$$\Pi(x) := \prod_{i=1}^{s}(x - y_i) \quad \text{and} \quad \delta(x) := \operatorname{sgn}\Pi(x), \quad x \in I. \tag{4.7}$$

Let

$$\pi(x) := \prod_{i=1}^{s} \frac{|x - y_i|}{|x - y_i| + \rho}, \tag{4.8}$$

then it follows immediately from (4.3) that

$$\pi(x) > 2^{-s}, \quad x \in (-1,1) \setminus O. \tag{4.9}$$

Now, by virtue of (4.4)

$$|x - y_i| + \rho < 2|x - x_j| + |x_j - y_i| + 2h_j,$$

and if $j \in H$, then $3|x_j - y_i| \geq h_j$. Hence

$$\frac{h_j}{(|x - x_j| + h_j)|x_j - y_i|} \leq \frac{7}{|x - y_i| + \rho}, \quad j \in H,$$

which in turn implies

$$\left(\frac{h_j}{|x - x_j| + h_j}\right)^s \frac{|\Pi(x)|}{|\Pi(x_j)|} \leq 7^s \pi(x), \quad x \in I, \quad j \in H. \tag{4.10}$$

Similarly,

$$\left(\frac{|x - x_j| + \rho}{\rho}\right)^s \frac{|\Pi(x)|}{|\Pi(x_j)|} \geq \pi(x), \quad x \in I, \quad j = 0, \ldots, n. \tag{4.11}$$

Following [12], let

$$t_j(x) := t_{j,n}(x) := \frac{\cos^2 2n\arccos x}{(x-x_j^0)^2} + \frac{\sin^2 2n\arccos x}{(x-\bar{x}_j)^2}, \tag{4.12}$$

where $\bar{x}_j = \cos(j-\frac{1}{2})\pi/n$ and $x_j^0 = \cos\beta_j^0$ with $\beta_j^0 = (j-\frac{1}{4})\pi/n$, $j \le n/2$, and $\beta_j^0 = (j-\frac{3}{4})\pi/n$, $j > n/2$. Note that $\bar{x}_j$ and x_j^0 are zeros of the respective numerators which are contained in $\mathring{I}_j$ (the interior of I_j), and that the t_j are algebraic polynomials of degree $4n-2$. Recall, that

$$t_j(x) \le \frac{c}{(|x-x_j|+h_j)^2} \le ct_j(x), \quad x \in I. \tag{4.13}$$

With $j \in H$ and an integer $b \ge 6(s+1)$, we associate the polynomial of degree $\le Cn$,

$$T_j(x) = T_{j,n}(x;b;Y_s) := \frac{1}{d_j}\int_{-1}^{x} t_j^b(u)\Pi(u)du, \tag{4.14}$$

where

$$d_j = \int_{-1}^{1} t_j^b(u)\Pi(u)du.$$

It follows by [5, Lemma 5.3] that

$$Ch_j^{2b-1} \le \frac{\Pi(x_j)}{d_j} \le Ch_j^{2b-1}, \tag{4.15}$$

which clearly yields

$$T_j'(x)\Pi(x)\Pi(x_j) \ge 0, \quad x \in I. \tag{4.16}$$

Denoting

$$\Gamma_j(x) := \frac{h_j}{|x-x_j|+h_j},$$

we obtain by (4.13) and (4.15),

$$|T_j'(x)| \le \frac{C}{h_j}\Gamma_j^{2b}(x)\frac{|\Pi(x)|}{|\Pi(x_j)|} \le C|T_j'(x)|, \quad x \in I. \tag{4.17}$$

Also by [5, Lemma 5.3], if

$$\chi_j(x) := \chi_{(x_j,1]}(x),$$

is the characteristic function of $(x_j, 1]$, then for $j \in H$,

$$|\chi_j(x) - T_j(x)| \le C\Gamma_j^{2b-s-1}(x), \quad x \in I. \tag{4.18}$$

Similarly, the polynomials of degree $\le Cn$,

$$\bar{T}_j(x) := \frac{1}{\bar{d}_j}\int_{-1}^{x}(u-x_j)(x_{j-1}-u)t_j^{b+1}(u)\Pi(u)du,$$

so that $\bar{T}_j(1)=1$, satisfy

$$\bar{T}_j'(x)\Pi(x)\Pi(x_j)\le 0, \quad x\in I\setminus I_j,$$

and, in addition, they satisfy inequalities similar to (4.17), (4.18), namely,

$$|\bar{T}_j'(x)|\le \frac{C}{h_j}\Gamma_j^{2b}(x)\frac{|\Pi(x)|}{|\Pi(x_j)|}, \quad x\in I,$$

and

$$|\chi_j(x)-\bar{T}_j(x)|\le C\Gamma_j^{2b-s-1}(x), \quad x\in I.$$

Then we obtain

Lemma 6. *Let $b=6(s+1)$. Then for each $j\in H$ there exist polynomials τ_j and $\bar{\tau}_j$ of degree $\le cn$, satisfying*

$$\begin{aligned}\tau_j''(x)\Pi(x)\Pi(x_j)&\ge 0, \quad x\in I,\\ \bar{\tau}_j''(x)\Pi(x)\Pi(x_j)&\le 0, \quad x\in I\setminus I_j,\end{aligned} \tag{4.19}$$

$$|\bar{\tau}_j''(x)|\le \frac{c}{h_j}\Gamma_j^{2b}(x)\frac{|\Pi(x)|}{|\Pi(x_j)|}\le c|\tau_j''(x)|, \quad x\in I, \tag{4.20}$$

and

$$\begin{aligned}|(x-x_j)_+-\tau_j(x)|&\le ch_j\Gamma_j^{2b-s-2}(x),\\ |(x-x_j)_+-\bar{\tau}_j(x)|&\le ch_j\Gamma_j^{2b-s-2}(x), \quad x\in I.\end{aligned} \tag{4.21}$$

Proof. We will prove only the existence of the polynomials τ_j, the other case being completely analogous. For every $j\in H$ let T_j be defined by (4.14). We use it to construct τ_j. By virtue of (4.18)

$$\int_{-1}^{1}|\chi_j(x)-T_j(x)|\,dx\le c\int_{-1}^{1}\Gamma_j^2(x)\,dx \le: c_0h_j, \quad j\in H. \tag{4.22}$$

If for $r:=\lceil 6c_0\rceil$ (where $\lceil a\rceil$ denotes the ceiling of a), both $j-r\ge 0$ and $j+r\le n$, and if for all $j-r\le i\le j+r$, we have $i\in H$, then by Lemma 5 we have

$$x_{j-r}-x_j\ge 3c_0h_{j-r-1}\ge c_0h_{j-r}, \quad \text{and} \quad x_j-x_{j+r}\ge c_0h_{j+r},$$

so that it follows from (4.22) that,

$$\int_{-1}^{1}\big(T_{j-r}(x)-\chi_j(x)\big)\,dx=\int_{-1}^{1}\big(T_{j-r}(x)-\chi_{j-r}(x)\big)\,dx-(x_{j-r}-x_j)\le 0,$$

and

$$\int_{-1}^{1}\big(T_{j+r}(x)-\chi_j(x)\big)\,dx=\int_{-1}^{1}\big(T_{j+r}(x)-\chi_{j+r}(x)\big)\,dx+(x_j-x_{j+r})\ge 0.$$

Hence for some $0\le\alpha\le 1$, we have

$$\alpha\int_{-1}^{1}\big(T_{j-r}(x)-\chi_j(x)\big)\,dx+(1-\alpha)\int_{-1}^{1}\big(T_{j+r}(x)-\chi_j(x)\big)\,dx=0.$$

We set

$$\tau_{j,n}:=\tau_j(x):=\alpha\int_{-1}^{x}T_{j-r}(u)\,du+(1-\alpha)\int_{-1}^{x}T_{j+r}(u)\,du,$$

so that

$$\tau_j(1)=1-x_j,$$

which by (4.18), in turn implies (4.21). Now (4.19) follows from (4.16) and (4.20) follows from (4.17) since by our assumption $\operatorname{sgn}\Pi(x_{j-r})=\operatorname{sgn}\Pi(x_{j+r})=\operatorname{sgn}\Pi(x_j)$.

If $j-r<0$, then it suffices to take

$$\tau_j(x):=\int_{-1}^{x}T_j(u)\,du,$$

and if $j+r>n$, then it suffices to take

$$\tau_j(x):=1-x_j-\int_{x}^{1}T_j(u)\,du.$$

We are left with the case where there is an $i\notin H$, such that $0\le j-r\le i<j+r\le n$. In this case we take the Chebyshev partition of order $2rn$, so that we have $x_j=x_{2rj,2rn}$ and $i\in H(Y_s,2rn)$, for all $2rj-r\le i\le 2rj+r$. Thus we set

$$\tau_j(x):=\tau_{2rj,2rn}(x),$$

and we observe that by the above construction this τ_j satisfies (4.19) through (4.21), since by virtue of (4.5),

$$h_{2rj,2rn}\le h_j\le 4r^2h_{2rj,2rn}.\quad\square$$

Remark. One should note that by going from n to $2rn$, we may reduce all cases save $j=0$ and $j=n$, to the first situation.

The last four lemmas of this section are required in the proof of Theorem 5. Combining Lemma 6 with (4.3), (4.10) and (4.11), readily yields

Lemma 7. *The polynomials* τ_j *and* $\bar\tau_j$ *satisfy*

$$|\tau_j''(x)| \ge \frac{ch_j}{\rho^2}\pi(x)\left(\frac{\rho}{|x-x_j|+\rho}\right)^{25(s+1)}, \quad x \in I, \tag{4.23}$$

$$|\bar\tau_j''(x)| \le \frac{ch_j}{\rho^2}\pi(x), \quad x \in I_j,$$

and

$$\begin{aligned} |(x-x_j)_+ - \tau_j(x)| &\le c\rho\left(\frac{h_j}{|x-x_j|+\rho}\right)^2, \\ |(x-x_j)_+ - \bar\tau_j(x)| &\le c\rho\left(\frac{h_j}{|x-x_j|+\rho}\right)^2, \quad x \in I. \end{aligned} \tag{4.24}$$

In order to prove Lemma 10 below, we need two more auxiliary results.

Lemma 8. *Let* $l_0, l_1 \in \mathbb{N}$*, and assume that* $0 \le j_0 \le j_1 < \cdots < j_{2l_1} \le j_0 + l_0 \le n$*. Then*

$$\frac{1}{l_1}\sum_{\nu=1}^{l_1}(x_{j_\nu} - x_{j_{\nu+l_1}}) \ge \left(\frac{l_1}{l_0}\right)^2 (x_{j_0} - x_{j_0+l_0}). \tag{4.25}$$

Proof. With no loss of generality we may assume that $j_0 \le n - j_0 - l_0$. Then for each $1 \le \nu \le l_1$,

$$x_{j_\nu} - x_{j_{\nu+l_1}} \ge x_{j_\nu} - x_{j_\nu + l_1} \ge x_{j_0} - x_{j_0+l_1}.$$

Thus, in order to prove (4.25), it suffices to estimate

$$\begin{aligned} \frac{x_{j_0} - x_{j_0+l_1}}{x_{j_0} - x_{j_0+l_0}} &= \frac{\sin \pi l_1/2n}{\sin \pi l_0/2n}\,\frac{\sin \pi(2j_0+l_1)/2n}{\sin \pi(2j_0+l_0)/2n} \\ &\ge \frac{\sin^2 \pi l_1/2n}{\sin^2 \pi l_0/2n} \ge \left(\frac{l_1}{l_0}\right)^2, \end{aligned}$$

where in both inequalities we use the fact that $l_1 < l_0$ and in the last inequality also that $\sin x/x$ is decreasing in $(0,\pi)$. This completes the proof. $\square$

Lemma 9. *Let* $A := \{j_0, \dots, j_0 + l_0\}$*, and let* $A_1, A_2 \subseteq A$*, be such that* $\#A_1 = 2l_1$ *and* $\#A_2 = l_2$*. If* $\delta_j \in \{-1, 1\}$*,* $j \in A_2$*, then there exist* $2l_1$ *constants* a_i*,* $i \in A_1$*, such that*

$$|a_i| \le \left(\frac{l_0}{l_1}\right)^2, \quad i \in A_1, \tag{4.26}$$

and

$$\frac{1}{l_2}\sum_{j\in A_2}\delta_j(x-x_j)+\frac{1}{l_1}\sum_{i\in A_1}a_i(x-x_i)\equiv 0. \tag{4.27}$$

Proof. Without loss of generality we may take $l_2=1$, that is, $A_2=\{j_*\}$, and we may assume $\delta_{j_*}=-1$. We may write A_1 as $A_1=A_1^+\cup A_1^-$, where each set contains l_1 elements, and each index in A_1^+ is less than all indices in A_1^-. Denote

$$\frac{1}{l_1}\sum_{i\in A_1^+}(x-x_i)=:x-\alpha^+\quad\text{and}\quad\frac{1}{l_1}\sum_{i\in A_1^-}(x-x_i)=:x-\alpha^-,$$

and put

$$a_i:=\begin{cases}\frac{x_{j_*}-\alpha^-}{\alpha^+-\alpha^-}, & i\in A_1^+\\ \frac{x_{j_*}-\alpha^+}{\alpha^--\alpha^+}, & i\in A_1^-.\end{cases}$$

Then (4.27) for $l_2=1$, follows. By virtue of Lemma 8 we have

$$\alpha^+-\alpha^-\ge\left(\frac{l_1}{l_0}\right)^2(x_{j_0}-x_{j_0+l_0}),$$

whence (4.26) follows by the straightforward inequality $|x_{j_*}-\alpha^\pm|\le x_{j_0}-x_{j_0+l_0}$. This completes the proof of Lemma 9. □

We are ready to state and prove Lemma 10.

Lemma 10. *Let E be an interval which is the union of $l\ge 12s$ of the intervals I_j, and let a set $J\subseteq E$ be the union of $1\le\mu\le l/4$ of these intervals. Then there exists a polynomial $Q_n(x)=Q_n(x,E,J)$ of degree $\le cn$, satisfying*

$$Q_n''(x)\delta(x)\ge c_1\frac{l}{\mu}\left(\frac{\rho}{\max\{\rho,\operatorname{dist}(x,E)\}}\right)^{25(s+1)}\frac{\pi(x)}{\rho^2},\quad x\in J\cup(I\setminus E), \tag{4.28}$$

(we may take $c_1\le 1$)

$$Q_n''(x)\delta(x)\ge-\frac{\pi(x)}{\rho^2},\quad x\in E\setminus J \tag{4.29}$$

and

$$|Q_n(x)|\le cl^6\rho\sum_{I_j\subseteq E}\frac{h_j}{(|x-x_j|+\rho)^2},\quad x\in I. \tag{4.30}$$

Proof. Let $H(E) := \{j \in H \mid I_j \subseteq E\}$, $H(J) := \{j \in H \mid I_j \subseteq J\}$, $E(O) := \{j \mid I_j \subseteq E \cap \overline{O}\}$, and $H_*(E) := \{j \in H(E) \mid I_j \cap \overline{O} \neq \emptyset\}$, where $\overline{O}$ denotes the closure of O. Finally, let $j_* := \min\{j \in H(E)\}$ and $j^* := \max\{j \in H(E)\}$. Set

$$A_2 := H(J) \cup H_*(E) \cup \{j_*, j^*\}, \quad \text{and} \quad A_1 := H(E) \setminus (A_2 \cup E(O)).$$

Denote by l_1^* and l_2 the number of elements in A_1 and A_2, respectively, and set $l_1 := [\frac{l_1^*}{2}]$. Then it readily follows that

$$l_2 \leq \mu + 2s + 2 \leq c\mu, \tag{4.31}$$

(recall that we allow c to depend on s), and

$$l > l_1^* \geq l - (l_2 + 3s) \geq \frac{1}{6} l. \tag{4.32}$$

Denote by j_0 and $j^0 = j_0 + l - 1$ the smallest and the largest integers j, such that $I_j \subseteq E$. We consider three cases.

Case I. Let $l \geq j_0$. Set

$$Q_n(x) := \frac{l}{\mu} \sum_{j \in A_2} \frac{\delta_j \tau_j(x)}{h_j},$$

where $\delta_j := \operatorname{sgn} \Pi(x_j)$. Then $Q_n''(x)\delta(x) \geq 0$, $x \in I$, which implies (4.29), and (4.28) readily follows from (4.23). Thus we only have to prove (4.30). To this end, by (4.24) we obtain for any $j \in A_2$,

$$\frac{|\tau_j(x)|}{h_j} \leq \frac{1}{h_j} |\tau_j(x) - (x - x_j)_+| + \frac{(x - x_j)_+}{h_j} \leq c \frac{\rho h_j}{(|x - x_j| + \rho)^2} + \frac{(x - x_j)_+}{h_j}.$$

Now, if $x \leq x_j$, then $(x - x_j)_+ = 0$. Otherwise, observe that $x \in I_i$ for some $1 \leq i \leq j \leq 2l$. Thus,

$$\begin{aligned} \frac{(x - x_j)}{h_j} \frac{(x - x_j + \rho)^2}{\rho h_j} &\leq 10 \frac{x - x_j}{h_j} \frac{x - x_j + h_j}{h_j} \frac{x - x_j + h_i}{h_i} \\ &\leq 10 \Big(\frac{x_0 - x_{2l}}{h_1} + 1 \Big)^3 \leq c l^6. \end{aligned}$$

which implies (4.30).

Case II. Let $j_0 \geq n - 2l$. Set

$$Q_n(x) := \frac{l}{\mu} \sum_{j \in A_2} \frac{\delta_j}{h_j} (\tau_j(x) - (x - x_j)),$$

and proceed in the same manner as in the Case I.

Case III. Let $l < j_0 < n-2l$. Denote by $h = |E| = x_{j_0-1} - x_{j_0+l-1}$, the length of the interval E. Then (4.6) implies

$$\frac{1}{2}h \le lh_j \le 2h, \quad I_j \subset E. \tag{4.33}$$

Lemma 9, (4.31) and (4.32), guarantee the existence of a_i, $i \in A_1$, such that

$$\frac{l}{\mu}\sum_{j\in A_2}\delta_j(x-x_j) + \sum_{i\in A_1} a_i(x-x_i) \equiv 0, \tag{4.34}$$

and

$$|a_i| \le \frac{l}{\mu}\left(\frac{l}{l_1}\right)^2 \frac{l_2}{l_1} \le c, \quad i \in A_1. \tag{4.35}$$

(Note that if l_1^* is odd, then we apply Lemma 9 to $A_1 \setminus \{i^*\}$, for some arbitrary i^*, and put $a_{i^*} = 0$ in (4.34).)
For each $i \in A_1$ set

$$\tau_i^* := \begin{cases} \tau_i, & \text{if} \quad \delta_i a_i \ge 0, \\ \bar{\tau}_i, & \text{otherwise,} \end{cases}$$

and let

$$Q_n(x) := c\frac{l}{h}\left(\frac{l}{\mu}\sum_{j\in A_2}\delta_j\tau_j(x) + \sum_{i\in A_1} a_i\tau_i^*(x)\right),$$

for some c to be prescribed. Then by virtue of (4.33) and (4.35), we see that (4.28) readily follows by (4.19) and (4.23), and that (4.29) is valid for a proper choice of the constant c. We conclude with the proof of (4.30). Take

$$L(x) := \frac{l}{\mu}\sum_{j\in A_2}\delta_j(x-x_j)_+ + \sum_{i\in A_1} a_i(x-x_i)_+.$$

Then by (4.24) we have

$$|Q_n(x)| \le cl\rho \sum_{j\in H(E)} \frac{h_j}{(|x-x_j|+\rho)^2} + c\frac{l}{h}|L(x)|, \quad x \in I.$$

So we only need to estimate $\frac{l}{h}|L(x)|$. To this end, note that if $x \notin E$, then (4.34) implies that $L(x) \equiv 0$. On the other hand, if $x \in E$, then

$$\frac{l}{h}|L(x)| \le \frac{cl}{h}\left(\frac{ll_2}{\mu}h + 2l_1 h\right) \le cl^2 \le cl^3\rho \sum_{I_j\subseteq E}\frac{h_j}{(|x-x_j|+\rho)^2},$$

where for the last inequality we have applied (4.3), (4.33) and the estimate

$$1 = h\sum_{I_j\subseteq E}\frac{h_j}{h^2} \le 16h\sum_{I_j\subseteq E}\frac{h_j}{(|x-x_j|+\rho)^2} \le 160l\rho\sum_{I_j\subseteq E}\frac{h_j}{(|x-x_j|+\rho)^2}.$$

This completes the proof of (4.30), and in turn of Lemma 10. $\square$

§5. Proof of Theorems 2 and 4

We begin with the

Proof of Theorem 4. Since Theorem 4 for $k=1$ is trivial, we have to prove Theorem 4 only for $k\ge 2$. Given $n\ge 1$, denote by $G_\nu=(x_{J_\nu},x_{j_\nu})$ the connected components of $O=O(n,Y_s)$. For $j=1,\dots,n-1$, let $\tilde\tau_j$ be polynomials of degree $\le cn$ defined as follows.
a. if $j\in H$, then

$$\tilde\tau_j(x):=\tau_j(x),$$

where τ_j are from Lemma 6;
b. if $j_\nu=0$ and $0<j<J_\nu$, then $\tilde\tau_j(x):=0$;
c. $J_\nu=n$ and $j_\nu<j<n$, then $\tilde\tau_j(x):=x-x_j$.

Finally, we have the j's for which $0<j_\nu<j<J_\nu<n$. We divide the ν's into two groups. Let $n_1:=22s(k-1)^3n$. We say that $\nu\in Od$ if there exists an $l_\nu\in H(n_1,Y_s)$ such that $I_{l_\nu,n_1}\cap G_\nu\ne\emptyset$, and the interval $(x_{l_\nu,n_1},x_{j_\nu,n})$ contains an odd number of points y_i. Note that if $\nu\notin Od$, then the set G_ν contains an even number, say $2m$, of points y_i, the points $y_{i_0+2m-1}<\dots<y_{i_0}$, say. In this case each two consecutive points y_{i_0+2v} and y_{i_0+2v+1}, $v=0,\dots,m-1$, must belong to the union of four consecutive intervals, say $[x_{l_v+2,n_1},x_{l_v-2,n_1})$, whence

$$\{x\in G_\nu:\Pi(x_{j_\nu})S''(x)<0\}\subseteq\cup_{v=0}^{m-1}[x_{l_v+2,n_1}x_{l_v-2,n_1}].$$

It follows by the left-hand side of (4.5) that,

$$\begin{aligned}\text{meas}\{x\in G_\nu:\Pi(x_{j_\nu})S''(x)<0\}&\le\frac{s}{2}4\max_{I_{l,n_1}\subseteq(x_{J_\nu},x_{j_\nu})}|I_{l,n_1}|\\&\le\frac{4s}{2}2\frac{|G_\nu|}{(J_\nu-j_\nu)\frac{n_1}{n}}\\&\le 4s\frac{|G_\nu|}{3\frac{n_1}{n}}\\&=\frac{1}{16(k-1)^3}|G_\nu|.\end{aligned}\tag{5.1}$$

We need the polynomials τ_{j_ν} and τ_{J_ν}, however, we note that j_ν might not be in H. Since $2j_\nu$ is always in $H(2n,Y_s)$, in the case $j_\nu\notin H$, we define $\tilde\tau_{j_\nu}:=\tau_{j_\nu}:=\tau_{2j_\nu,2n}$. Similarly, we always have $J_\nu\notin H$ and $2J_\nu\in H(2n,Y_s)$, so we define $\tilde\tau_{J_\nu}:=\tau_{J_\nu}:=\tau_{2J_\nu,2n}$. Now,
d. if $0<j_\nu<j<J_\nu<n$ and $\nu\notin Od$, then we let

$$\tilde\tau_j(x):=\tau_{j_\nu}(x),$$

if on the other hand,
e. $0<j_\nu<j<J_\nu<n$ and $\nu\in Od$, then we let

$$\tilde\tau_j(x):=\delta_j\tau_{j_\nu}(x)+(1-\delta_j)\tau_{l_\nu,n_1}(x),$$

where $\delta_j = 0$ or $= 1$, is to be prescribed.

We are in a position to define P_n. Recall that the piecewise linear function L that interpolates S, at the x_j's satisfies

$$\|S - L\| \le c\omega_2^\varphi\left(S, \frac{1}{n}\right), \tag{5.2}$$

and may be written in the form

$$L(x) = l(x) + \sum_{j=1}^{n-1}[S; x_{j+1}, x_j, x_{j-1}](x_{j-1} - x_{j+1})(x - x_j)_+,$$

where $l(x)$ is a linear function. Thus, denote

$$P_n(x) := l(x) + \sum_{j=1}^{n-1}[S; x_{j+1}, x_j, x_{j-1}](x_{j-1} - x_{j+1})\tilde{\tau}_j(x).$$

We begin with the proof of (2.6). To this end, we show that for each $j = 1, \dots, n-1$, we have

$$|(x - x_j)_+ - \tilde{\tau}_j(x)| \le ch_j\Gamma_j^2(x), \quad x \in I. \tag{5.3}$$

Indeed, going through the various cases we see that
a. (5.3) readily follows from (4.21);
b., *c*. (5.3) readily follows from the inequalities

$$h_j \le |G_\nu| < ch_j, \quad j_\nu < j < J_\nu; \tag{5.4}$$

d. by (4.21) and (5.4),

$$\begin{aligned}|(x - x_j)_+ - \tilde{\tau}_j(x)| &\le |(x - x_j)_+ - (x - x_{j_\nu})_+| + |(x - x_{j_\nu})_+ - \tilde{\tau}_{j_\nu}(x)| \\ &\le ch_j\Gamma_j^2(x) + ch_{j_\nu}\Gamma_{j_\nu}^2(x) \le ch_j\Gamma_j^2(x);\end{aligned}$$

and finally,
e. if $\delta_j = 1$, then we are back in Case *d*., and if $\delta_j = 0$, then similarly we have,

$$\begin{aligned}|(x - x_j)_+ - \tilde{\tau}_j(x)| &\le ch_j\Gamma_j^2(x) + |(x - x_{l_\nu,n_1})_+ - \tilde{\tau}_{l_\nu,n_1}(x)| \\ &\le ch_j\Gamma_j^2(x) + \frac{h_{l_\nu,n_1}^3}{(|x - x_{l_\nu,n_1}| + h_{l_\nu,n_1})^2} \\ &\le ch_j\Gamma_j^2(x),\end{aligned}$$

and (5.3) is proved. Since it is well-known that

$$|[S;x_{j+1},x_j,x_{j-1}]| \le ch_j^{-2}\omega_2^\varphi\left(S,\frac{1}{n}\right), \quad j=1,\dots,n-1,$$

and

$$\|\sum_{j=1}^n \Gamma_j^2\| \le c,$$

we obtain

$$\|L-P_n\| \le c\|\sum_{j=1}^{n-1}\Gamma_j^2\|\omega_2^\varphi\left(S,\frac{1}{n}\right).$$

This together with (5.2) concludes the proof of (2.6).

In order to prove that $P_n \in \Delta^2(Y_s)$ we denote

$$L_j(x) := [S;x_{j+1},x_j,x_{j-1}](x_{j-1}-x_{j+1})\tilde{\tau}_j(x), \quad j=1,\dots,n-1,$$

and

$$P_n(x) =: l(x)+A(x)+B(x)+C(x)+D(x)+E(x),$$

where

$$A(x)=\sum_{j\in H}L_j(x)+\sum_{J_\nu<n}L_{J_\nu}(x),$$

$$B(x)=\sum_{j=1}^{J_\nu-1}L_j(x), \quad \text{if} \quad j_\nu=0,$$

$$C(x)=\sum_{j=j_\nu+1}^{n-1}L_j(x), \quad \text{if} \quad J_\nu=n,$$

$$D(x)=\sum_{\nu\in Od}\sum_{j=j_\nu+1}^{J_\nu-1}L_j(x),$$

and

$$E(x)=\sum_{\nu\notin Od}\sum_{j=j_\nu+1}^{J_\nu-1}L_j(x) =: \sum_{\nu\notin Od}E_\nu(x).$$

It is important to emphasize that we either have $j_\nu \in H$ or $j_\nu = J_{\nu+1}$, so that indeed all $1 \le j \le n-1$ are taken care of.

Again we have to investigate each case separately.

a. If $j \in H$, then by definition of $\Delta^2(Y_s)$ we have, $\Pi(x_j)[S;x_{j+1},x_j,x_{j-1}] \ge 0$. Hence by (4.19),

$$\Pi(x)L_j''(x) = \Pi(x)[S;x_{j+1},x_j,x_{j-1}](x_{j-1}-x_{j+1})\tau_j''(x) \ge 0,$$

and similarly $\Pi(x)L''_{J_\nu}(x) \geq 0$, $J_\nu < n$, so that $\Pi(x)A''(x) \geq 0, \quad x \in I$.
b., *c.* Since B and C are linear functions, we have $B''(x) \equiv 0$ and $C''(x) \equiv 0$.
e. If $\nu \in Od$, then by definition, we have an odd number of points $y_i \in (x_{l_\nu,n_1}, x_{j_\nu})$, which in turn implies that

$$\Pi(x_{l_\nu,n_1})\Pi(x_{j_\nu}) < 0.$$

Hence, (4.19) implies

$$\tau''_{l_\nu,n_1}(x)\tau''_{j_\nu}(x) \leq 0, \quad x \in I.$$

Hence for each $j = j_\nu + 1, \dots, J_\nu - 1$, we may prescribe δ_j so that

$$\Pi(x)L''_j(x) \geq 0, \quad x \in I.$$

With this choice

$$\Pi(x)D''(x) \geq 0, \quad x \in I.$$

Finally we conclude with the proof of Case
d. If $\nu \notin Od$, then

$$\begin{aligned}
E_\nu(x) &= \sum_{j=j_\nu+1}^{J_\nu-1} L_j(x) \\
&= \tau_{j_\nu}(x) \sum_{j=j_\nu+1}^{J_\nu-1} [S; x_{j+1}, x_j, x_{j-1}](x_{j-1} - x_{j+1}) \\
&= \tau_{j_\nu}(x)\big([S; x_{J_\nu}, x_{j_\nu+1}, x_{j_\nu}](x_{j_\nu+1} - x_{J_\nu}) + [S; x_{J_\nu}, x_{J_\nu-1}, x_{j_\nu}](x_{j_\nu} - x_{J_\nu-1})\big) \\
&=: \tau_{j_\nu}(x)e_\nu.
\end{aligned}$$

By virtue of Lemma 4 and (5.1), it now follows that

$$\Pi(x_{j_\nu})e_\nu \geq 0.$$

Therefore, (4.19) implies

$$\Pi(x)E''_\nu(x) = \tau''_{j_\nu}(x)\Pi(x)\Pi(x_{j_\nu})\frac{e_\nu}{\Pi(x_{j_\nu})} \geq 0, \quad x \in I.$$

Since $l''(x) \equiv 0$, we have shown that $P_n \in \Delta^2(Y_s)$, and concluded the proof of Theorem 4. $\square$

Proof of Theorem 2. Analyzing the above proof, one notes that the only place one needs the assumption that our function is a piecewise polynomial, is in order to apply Lemma 4. Thus for a general $f \in \Delta^2(Y_s)$, if one is guaranteed that n is sufficiently big so that each component G_ν contains an odd number of points of Y_s, in particular one point, then one may conclude the same. If f changes convexity just once, then obviously the requirement that each component G_ν contains an odd number of points of Y_s, specifically one point, holds for all $n \geq 1$. This proves Theorem 2. $\square$

Remark. In view of the above discussion we see that we always have the estimate (2.4) for $n \geq N = N(Y_s)$. This is of course weaker than (2.1) and we only mention it since we have got it for free.

§6. Smoothing lemmas

Let $I_{i,j}$ be the smallest closed interval, containing I_i and I_j, and denote $h_{i,j} := |I_{i,j}|$. For $S \in \Sigma_{k,n}$ set

$$a_{i,j}(S) := \|p_i - p_j\|_{I_i} \left(\frac{h_j}{h_{i,j}}\right)^k, \quad i,j = 1,\dots,n, \tag{6.1}$$

where $\|p\|_{I_i} = \max\{|p(x)| : x \in I_i\}$.
We are going to call an interval A a *proper* interval, if its endpoints belong to the Chebyshev partition, that is, are among the x_j's. For any proper interval A, let

$$a_k(S, A) := \max a_{i,j}(S),$$

where the maximum is taken over all i, j, such that $I_j \subseteq A$ and $I_i \subseteq A$. Finally, write

$$a_k(S) := a_k(S, I).$$

Then, by virtue of [10, Lemma 9] we have

$$a_k(S) \le c\omega_k^\varphi\left(S, \frac{1}{n}\right) \le c a_k(S). \tag{6.2}$$

Given $x \in I$, if $\theta \in [x - h\varphi(x), x + h\varphi(x)] \subseteq I$, then we have $\varphi(x) \le 2(h + \varphi(\theta))$. If $S \in \Sigma^1_{k,n}$, S' is absolutely continuous in I, whence for $0 < h \le 1/n$,

$$\begin{aligned}
|\Delta^2_{h\varphi(x)} S(x)| &= \left| \int_x^{x+h\varphi(x)} \big(S'(t) - S'(t - h\varphi(x))\big) dt \right| \\
&= \left| \int_x^{x+h\varphi(x)} \int_{t-h\varphi(x)}^t S''(u) du\, dt \right| \\
&\le \frac{1}{\min(h^2 + h\varphi(\theta))^2} \left| \int_x^{x+h\varphi(x)} \int_{t-h\varphi(x)}^t \rho_n^2(u) S''(u) du\, dt \right| \\
&\le \frac{(h\varphi(x))^2}{\min(h^2 + h\varphi(\theta))^2} \|\rho^2 S''\| \\
&\le 4\|\rho^2 S''\|,
\end{aligned}$$

where the minimum is taken above on $\theta \in [x - h\varphi(x), x + h\varphi(x)]$. Hence, if $S \in \Sigma^1_{k,n}$, then

$$\omega_2^\varphi\left(S, \frac{1}{n}\right) \le c\|\rho^2 S''\|, \tag{6.3}$$

which in turn by (6.2), and the inequality $\omega_k^\varphi(S,t) \le c\omega_2^\varphi(S,t)$, $k \ge 3$, readily implies

Lemma 11. *If $S \in \Sigma^1_{k,n}$, then*

$$a_k(S) \le \ c\|\rho^2 S''\|.$$

Finally we have

Lemma 12. *Suppose $k \ge 3$ and $S \in \Sigma^1_{k,n}$ is such that*

$$a_k(S) \le 1. \tag{6.4}$$

If an interval $I_{\mu,\nu}$ contains at least $2k-5$ intervals I_i, and points $x_i^* \in \overset{\circ}{I}_i$, such that

$$\rho_n^2(x_i^*)|S''(x_i^*)| \le 1, \tag{6.5}$$

then for every $0 \le j \le n$, we have

$$\|\rho^2 S''\|_{I_j} \le c\big((j-\mu)^{4k} + (j-\nu)^{4k}\big). \tag{6.6}$$

Proof. Fix j and $x \in \overset{\circ}{I}_j$. It follows by (6.1) and (6.4) that for every i,

$$\|p_i - p_j\|_{I_i} \le \left(\frac{h_{i,j}}{h_j}\right)^k.$$

Since p_i and p_j are polynomials of degree $k-1$, we get

$$\|p_i'' - p_j''\|_{I_i} \le \frac{c}{h_i^2}\left(\frac{h_{i,j}}{h_j}\right)^k.$$

In view of (4.3) and (4.5), we see that (6.5) implies

$$\begin{aligned} |p_j''(x_i^*)| &\le \frac{c}{h_i^2}\left(\frac{h_{i,j}}{h_j}\right)^k + \frac{c}{h_i^2} \\ &\le \frac{c}{h_i^2}\left(\frac{h_{i,j}}{h_j}\right)^k \\ &\le \frac{c}{h_j^2}(|i-j|+1)^{2k} \end{aligned} \tag{6.7}$$

By assumption there are $k-2$ points $x_{i_m}^* \in I_{\mu,\nu}$, $\quad m = 1,\dots,k-2$, each two being separated by an interval $I_i \subseteq I_{\mu,\nu}$. Recalling that $x \in I_j$, we have for each $1 \le l \le k-2$ and $1 \le m \le k-2$, with $l \ne m$,

$$\frac{|x - x_{i_m}^*|}{|x_{i_l}^* - x_{i_m}^*|} \le c\frac{h_{j,i_m}}{h_{i_m}} \le c(|j-i_m|+1)^2 \le c\big((j-\mu)^2 + (j-\nu)^2\big). \tag{6.8}$$

Now, by virtue of the representation

$$p_j''(x) \equiv \sum_{l=1}^{k-2} p_j''(x_{i_l}^*) \prod_{m=1, m\ne l}^{k-2} \frac{x - x_{i_m}^*}{x_{i_l}^* - x_{i_m}^*},$$

we obtain from (6.7) and (6.8),

$$\rho^2|S''(x)| = \rho^2|p_j''(x)| \le h_j^2|p_j''(x)| \le c\big((j-\mu)^{4k-6} + (j-\nu)^{4k-6}\big), \quad x \in \overset{\circ}{I}_j,$$

and the proof is complete. $\square$

§7. Zero-preserving approximation

We begin with a technical result. Namely,

Lemma 13. *For $s \in \mathbb{N}$, let 2^s vectors $\bar{a}_l = (a_{0,l}, a_{1,l}, ..., a_{s-1,l})$, $l = 0, \dots, 2^s - 1$, be given so that $sgn\, a_{\nu,l} = (-1)^{\delta_{\nu,l}}$, $0 \le \nu \le s-1$, where $\delta_{\nu,l} \in \{0,1\}$ is from the representation $l = \sum_{\nu=0}^{s-1} \delta_{\nu,l} 2^\nu$. Then there are 2^s positive numbers α_l such that*

$$\sum_{l=0}^{2^s-1} \alpha_l \bar{a}_l = (0, 0, ..., 0). \tag{7.1}$$

Proof. The proof by induction is straightforward. $\square$

Next we need

Lemma 14. *Let $K(x)$ be a continuous strictly positive function on I, and let $0 \le i^* \le s$ be fixed. Then there exist s interlacing points $y_{i+1} < t_i < y_i$, $i = 0, \dots, s$, $i \ne i^*$, such that the function*

$$\Phi(x) = \Phi(x, K, i^*, Y_s) := \int_{-1}^{x} K(u) \Pi^2(u) \prod_{i=0, i\ne i^*}^{s} (u - t_i) du, \tag{7.2}$$

(if $s = 0$, then the empty product $= 1$), satisfies

$$\Phi'(y_i) = \Phi''(y_i) = 0, \quad 1 \le i \le s, \tag{7.3}$$

and

$$\Phi(y_i) = \begin{cases} \Phi(1), & 0 \le i \le i^*, \\ \Phi(-1), & i^* < i \le s+1. \end{cases} \tag{7.4}$$

Proof. Since (7.3) is self-evident for any choice of $\{t_i\}$, we prove that we may select them so as to yield (7.4). For each $0 \le l \le 2^s - 1$ and every $0 \le i \le s$, $i \ne i^*$, we take $y_{i,l} \in \{y_i, y_{i+1}\}$, such that for $u \in (y_{i+1}, y_i)$,

$$\operatorname{sgn}\big(\Pi(u)(u - y_{i^*})(u - y_{i,l})\big) = \begin{cases} (-1)^{\delta_{i,l}}, & i < i^*, \\ (-1)^{\delta_{i-1,l}}, & i > i^*, \end{cases}$$

and denote

$$\Phi_l(x) := \int_{-1}^{x} K(u) \Pi^2(u) \prod_{i=0, i\ne i^*}^{s} (u - y_{i,l}) du.$$

Now, for

$$a_{i,l} := \begin{cases} \Phi_l(y_i) - \Phi_l(y_{i+1}), & i < i^*, \\ \Phi_l(y_{i+1}) - \Phi_l(y_{i+2}), & i \geq i^*, \end{cases}, \tag{7.5}$$

it follows that $\operatorname{sgn} a_{i,l} = (-1)^{\delta_{i,l}}$, therefore by Lemma 13 there are 2^s positive numbers α_l such that

$$\sum_{l=0}^{2^s-1} \alpha_l (a_{0,l}, a_{1,l}, , ..., a_{s-1,l}) = (0, 0, ..., 0).$$

Set

$$\Phi(x) := \left(\sum_{l=0}^{2^s-1} \alpha_l \right)^{-1} \sum_{l=0}^{2^s-1} \alpha_l \Phi_l(x). \tag{7.6}$$

Then Φ is the required function. Indeed, for each $0 \leq i < i^*$, we have

$$\sum_{l=0}^{2^s-1} \alpha_l \big(\Phi_l(y_{i+1}) - \Phi_l(y_i) \big) = \sum_{l=0}^{2^s-1} \alpha_l a_{i,l} = 0,$$

which implies (7.4) for $0 \leq i \leq i^*$. Similarly we have (7.4) for $i^* < i \leq s+1$. By its definition,

$$\Phi(x) := \int_{-1}^{x} K(u) \Pi^2(u) P_s(u) du, \tag{7.7}$$

where $P_s(x)$ is a monic polynomial of degree s. By Rolle's theorem (7.4) implies that $\Phi'(x)$ has a zero in (y_{i+1}, y_i), $0 \leq i \leq s$, $i \neq i^*$. Hence by (7.7), $\Phi(x)$ possesses the representation (7.2). $\square$

Let $j \in H$ and let $0 \leq i_j \leq s$ be such that $y_{i_j+1} < x_j < y_{i_j}$. For a fixed integer $b \geq 6(3s+1)$, denote

$$\check{T}_j(x) := \check{T}_{j,n}(x, b, Y_s) := d^{-1}_{j,b,Y_s,n} \Phi(x, t_j^b, i_j, Y_s), \tag{7.8}$$

where t_j was defined in (4.12) and where $d_{j,b,Y_s,n}$ is chosen so that $\check{T}_j(1) = 1$. Evidently, it is a polynomial of degree $\leq Cn$. A proof similar to that of (4.18) yields

$$|\chi_j(x) - \check{T}_j(x)| \leq C \left(\frac{h_j}{|x - x_j| + h_j} \right)^{b_1}, \quad x \in I, \tag{7.9}$$

where $b_1 = 2b - 3s - 1$.

For the rest of this section we assume that $s > 0$ otherwise many of the statements are vacuous and there is nothing to prove. For $j \notin H$, let j^* be the closest element to it from H (if there are two such elements, then we take the bigger one), and denote by I_j^* the connected component of $\overline{O}$ (the closure of O), that contains x_j. Since the interval I_j^* contains at most $3s$ intervals I_ν, we conclude from (4.5) that

$$h_j \le |I_j^*| \le (3s)^2 h_j. \tag{7.10}$$

In order to use a unified notation we denote for $j \in H$, $j^* := j$, and $I_j^* := I_j$. It follows by (7.10) that (7.9) is valid also for the polynomial

$$\check{T}_j(x) := \check{T}_{j,n}(x, b, Y_s) := \check{T}_{j^*,n}(x, b, Y_s), \quad j \notin H. \tag{7.11}$$

We summarize the above in the following

Lemma 15. *For every* $1 \le j \le n$*,*

$$\check{T}_j'(y_i) = \check{T}_j''(y_i) = 0, \quad 1 \le i \le s, \tag{7.12}$$

$$\chi_j(y_i) - \check{T}_j(y_i) = 0, \quad 1 \le i \le s, \quad y_i \notin I_j^*, \tag{7.13}$$

and

$$|\chi_j(x) - \check{T}_j(x)| < C \left(\frac{h_j}{|x - x_j| + h_j} \right)^{b_1}, \quad x \in I. \tag{7.14}$$

Set

$$\begin{aligned}
\hat{T}_1(x) &= \hat{T}_{1,n}(x, b, Y_s) := \check{T}_{1,n}(x, b, Y_s), \\
\hat{T}_n(x) &= \hat{T}_{n,n}(x, b, Y_s) := 1 - \check{T}_{n-1,n}(x, b, Y_s), \\
\hat{T}_j(x) &= \hat{T}_{j,n}(x, b, Y_s) := \check{T}_{j,n}(x, b, Y_s) - \check{T}_{j-1,n}(x, b, Y_s), \quad 2 \le j \le n-1.
\end{aligned} \tag{7.15}$$

Then we prove

Lemma 16. *The following relations hold*

$$\sum_{j=1}^n \hat{T}_j(x) \equiv 1, \tag{7.16}$$

$$\begin{aligned}
\hat{T}_j'(y_i) =& \hat{T}_j''(y_i) = 0, \quad 1 \le i \le s, \quad 1 \le j \le n, \\
&\hat{T}_j(y_i) = 0, \quad 1 \le i \le s, \quad 1 \le j \le n, \quad y_i \notin I_j^*,
\end{aligned} \tag{7.17}$$

and

$$(7.18) \qquad |\hat{T}_j^{(q)}(x)| < \frac{C}{\rho^q}\left(\frac{h_j}{|x-x_j|+h_j}\right)^{b_1}, \quad x \in I, \quad 1 \le j \le n, \quad 0 \le q \le s+2.$$

Proof. Obviously (7.16) is self-evident, and (7.17) and (7.18) with $q=0$, readily follow by (7.12) through (7.14). One can deduce (7.18) for $q>0$ from the case $q=0$ in the standard way, using Dzyadyk's inequality (see, e.g., [4, p. 262], see also [12, p. 118])

$$\|\rho^{\alpha+1}P_n'\| \le d\|\rho^\alpha P_n\|,$$

where $d=d(\alpha)$ is independent of n. $\square$

Now let n_1 be divisible by n and for every $1 \le j \le n$, denote

$$\tilde{T}_{j,n_1}(x) = \tilde{T}_{j,n_1}(x,b,Y_s) := \sum_{I_{\nu,n_1}\subseteq I_j} \hat{T}_{\nu,n_1}(x,b,Y_s).$$

Clearly it is a polynomial of degree $\le Cn_1$. We have

Lemma 17. *The following relations hold.*

$$(7.19) \qquad \sum_{j=1}^n \tilde{T}_{j,n_1}(x) \equiv 1,$$

$$(7.20) \qquad \begin{aligned} &\tilde{T}'_{j,n_1}(y_i) = \tilde{T}''_{j,n_1}(y_i) = 0, \quad 1 \le i \le s, \quad 1 \le j \le n,\\ &\tilde{T}_{j,n_1}(y_i) = 0, \quad 1 \le i \le s, \quad 1 \le j \le n, \quad y_i \notin I_j^*, \end{aligned}$$

and

$$(7.21) \qquad \begin{aligned} |\tilde{T}^{(q)}_{j,n_1}(x)| \le \frac{C}{\rho^q_{n_1}(x)}&\left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x)+\operatorname{dist}(x,I_j)}\right)^{b_2},\\ x\in I, \quad 1 \le j &\le n, \quad 0 \le q \le s+2, \end{aligned}$$

where $b_2 = \frac{1}{2}(b_1-1)$.

Proof. Relations (7.19) and (7.20) follow immediately from (7.16) and (7.17), when we observe that if $I_{\nu,n_1} \subseteq I_j$, then $I^*_{\nu,n_1} \subseteq I_j^*$. Thus we just have to prove (7.21). Note that (4.3) and (4.4) yield

$$\left(\frac{h_{\nu,n_1}}{|x-x_{\nu,n_1}|+h_{\nu,n_1}}\right)^2 \le c\frac{\rho_{n_1}(x)}{|x-x_{\nu,n_1}|+\rho_{n_1}(x)}.$$

Now if $x < x_j$, then it follows by (7.18) that

$$\begin{aligned}
\rho_1^q|\tilde{T}_{j,n_1}^{(q)}(x)| &\le C \sum_{I_{\nu,n_1}\subseteq I_j} \left(\frac{h_{\nu,n_1}}{|x-x_{\nu,n_1}|+h_{\nu,n_1}}\right)^{b_1} \\
&\le C\rho_{n_1}^{b_2}(x) \sum_{I_{\nu,n_1}\subseteq I_j} \frac{h_{\nu,n_1}}{(|x-x_{\nu,n_1}|+\rho_{n_1}(x))^{b_2+1}} \\
&\le C\rho_{n_1}(x)^{b_2} \int_{x_j-x}^{\infty} \frac{du}{(u+\rho_{n_1}(x))^{b_2+1}} \\
&= C\left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x)+x_j-x}\right)^{b_2} = C\left(\frac{\rho_{n_1}(x)}{\rho_{n_1}(x)+\operatorname{dist}(x,I_j)}\right)^{b_2}.
\end{aligned}$$

Similar proofs yield (7.21) if $x_{j-1} < x$, and if $x \in I_j$. $\square$

Let $S \in \Sigma_{k,n}$, take n_1 divisible by n and set

$$D_{n_1}(x) := D_{n_1}(x,S) := \sum_{j=1}^{n_1} p_j(x)\tilde{T}_{j,n_1}(x,b,Y_s), \tag{7.22}$$

evidently a polynomial of degree $\le Cn_1$. Finally, denote

$$O_e := \{u \in \overline{O} : [u-\frac{1}{2}\rho_n(u), u+\frac{1}{2}\rho_n(u)] \subseteq \overline{O}\} \cup (\overline{O}\cap(I_1\cup I_n)).$$

Recall that A is a proper interval if its endpoints belong to the Chebyshev partition. We have

Lemma 18. *Let $b_3 = b_2-s-2k-6 > 0$, and let A be a proper interval. For $S \in \Sigma_{k,n}(Y_s)$,*

$$\begin{gathered}
|S^{(q)}(x) - D_{n_1}^{(q)}(x)| \le \frac{C}{\rho^q}\left(a_k(S,A) + a_k(S)\frac{n}{n_1}\left(\frac{\rho}{\rho+\operatorname{dist}(x,I\setminus A)}\right)^{b_3}\right), \\
x \in A\cap\overline{O}_e, \quad q = 0,\dots,s+2.
\end{gathered} \tag{7.23}$$

Furthermore, if $S \in \Sigma_{k,n}^1$, then for $x \ne x_j$, $0 \le j \le n$,

$$|S''(x) - D''_{n_1}(x)| \le \frac{C}{\rho^2}\left(a_k(S,A) + a_k(S)\frac{n}{n_1}\left(\frac{\rho}{\operatorname{dist}(x,I\setminus A)}\right)^{b_3}\right), \quad x \in A. \tag{7.24}$$

Proof. The proof of the two statements is similar and we will proceed simultaneously in both. Fix $I_\nu \subseteq A\cap\overline{O}$ (or simply $I_\nu \subseteq A$, if we prove (7.24)), and let $x \in I_\nu\cap\overline{O}_e$ (or simply $x \in I_\nu$) be such that, say,

$$x - x_\nu \le x_{\nu-1} - x. \tag{7.25}$$

For the sake of brevity, we will write in this proof ρ_1 for $\rho_{n_1}(x)$, $\tilde{T}_j$ for $\tilde{T}_{j,n_1}$, and $a_{\nu,j}$ for $a_{\nu,j}(S)$. By (6.1),

$$\|p_\nu - p_j\|_{I_\nu} = a_{\nu,j}\left(\frac{h_{\nu,j}}{h_j}\right)^k,$$

whence, for each $r \in \mathbb{N}$,

$$\|p_\nu^{(r)} - p_j^{(r)}\|_{I_\nu} \le \frac{c a_{\nu,j}}{h_\nu^r}\left(\frac{h_{\nu,j}}{h_j}\right)^k.$$

First let $j \ne \nu, \nu+1$. Then (4.3) and (7.25) imply $\operatorname{dist}(x, I_j) > \frac{1}{2}\rho$. Hence (7.21) combined with (4.3) and (4.4) yields

$$\begin{aligned}
&\|p_\nu^{(r)} - p_j^{(r)}\|_{I_\nu}|\tilde{T}_j^{(q-r)}(x)| \\
&\le \frac{C a_{\nu,j}}{h_\nu^r}\left(\frac{h_{\nu,j}}{h_j}\right)^k \frac{1}{\rho_1^{q-r}}\left(\frac{\rho_1}{\rho_1 + \operatorname{dist}(x, I_j)}\right)^{b_2} \\
&\le \frac{C a_{\nu,j}}{h_\nu^r}\left(\frac{h_{\nu,j}}{h_j}\right)^{k+1} \frac{h_j}{h_{\nu,j}}\frac{1}{\rho_1^{q-r}}\left(\frac{\rho_1}{\rho_1 + \operatorname{dist}(x, I_j)}\right)^{b_2} \\
&\le \frac{C a_{\nu,j}}{h_\nu^r}\left(\frac{\rho + \operatorname{dist}(x, I_j)}{\rho}\right)^{2(k+1)} \frac{h_j}{h_\nu}\frac{1}{\rho_1^{q-r}}\left(\frac{\rho_1}{\rho_1 + \operatorname{dist}(x, I_j)}\right)^{q-r+1} \\
&\quad\times\left(\frac{\rho}{\rho + \operatorname{dist}(x, I_j)}\right)^{b_2-q+r-1} \\
&\le \frac{C a_{\nu,j}}{h_\nu^{r+1}} h_j \frac{\rho_1}{\rho}\frac{1}{\rho^{q-r}}\left(\frac{\rho}{\rho + \operatorname{dist}(x, I_j)}\right)^{b_3+1} \\
&\le \frac{C a_{\nu,j}}{\rho^q}\frac{n}{n_1}\rho^{b_3} h_j\left(\frac{1}{\rho + \operatorname{dist}(x, I_j)}\right)^{b_3+1}, \quad 0 \le r \le q,
\end{aligned} \tag{7.26}$$

where in the third inequality we applied the third inequality in (4.3)and (4.4), in the next one we used the fact that $\operatorname{dist}(x, I_j) > \frac{1}{2}\rho$, and in the last we have applied the straightforward inequality

$$\frac{\rho_1}{\rho} \le \frac{n}{n_1}.$$

Now, by virtue of (7.19) we may represent $S^{(q)}(x) - D_{n_1}^{(q)}(x)$ as

$$\begin{aligned}
S^{(q)}(x) - D_{n_1}^{(q)}(x) &= \Big((p_\nu(x) - p_{\nu+1}(x))\tilde{T}_{\nu+1}(x)\Big)^{(q)} \\
&\quad + \left(\sum_{I_j \subseteq A, j \ne \nu,\nu+1} + \sum_{I_j \not\subseteq A, j \ne \nu,\nu+1}\right)\Big((p_\nu(x) - p_j(x))\tilde{T}_j(x)\Big)^{(q)} \\
&=: \sigma_1(x) + \sigma_2(x) + \sigma_3(x),
\end{aligned}$$

where we write $p_{n+1} := p_n$, if $\nu = n$.

We begin with the estimate of σ_1. Note that if $\nu = n$, then $\sigma_1 \equiv 0$, so that we may assume that $\nu < n$. We need separate arguments for (7.23) and (7.24).

First we deal with (7.24). Since $S \in \Sigma^1_{k,n}$, $q = 2$, and $I_\nu \subseteq A$, it readily follows that

$$\|p''_\nu - p''_{\nu+1}\|_{I_\nu} \leq \frac{c}{\rho^2} a_{\nu,\nu+1},$$

which in turn implies

$$|p'_\nu(x) - p'_{\nu+1}(x)| = \left| \int_{x_\nu}^{x} (p''_\nu - p''_{\nu+1}) du \right| \leq \frac{c}{\rho^2} a_{\nu,\nu+1}(x - x_\nu),$$

and

$$|p_\nu(x) - p_{\nu+1}(x)| \leq \frac{c}{\rho^2} a_{\nu,\nu+1}(x - x_\nu)^2.$$

Therefore, by (7.21)

$$\begin{aligned} |\sigma_1(x)| &\leq \frac{c}{\rho^2} a_{\nu,\nu+1} \left(1 + \frac{x - x_\nu}{\rho_1} + \frac{(x - x_\nu)^2}{\rho_1^2}\right) \left(\frac{\rho_1}{\rho_1 + |x - x_\nu|}\right)^{b_2} \\ &\leq \frac{c}{\rho^2} a_{\nu,\nu+1} \left(\frac{\rho_1}{\rho_1 + |x - x_\nu|}\right)^{b_2 - 2}. \end{aligned} \tag{7.27}$$

Now, if $I_{\nu+1} \subseteq A$, then (7.27) implies

$$|\sigma_1(x)| \leq \frac{C}{\rho^2} a_k(S, A), \tag{7.28}$$

and if $I_{\nu+1} \nsubseteq A$, then (7.27) yields

$$\begin{aligned} |\sigma_1(x)| &\leq \frac{C}{\rho^2} a_k(S) \frac{\rho_1}{\rho} \frac{\rho}{\rho_1 + |x - x_\nu|} \left(\frac{\rho_1}{\rho_1 + |x - x_\nu|}\right)^{b_2 - 3} \\ &\leq \frac{C}{\rho^2} a_k(S) \frac{n}{n_1} \frac{\rho}{|x - x_\nu|} \left(\frac{\rho}{\rho + |x - x_\nu|}\right)^{b_2 - 3} \\ &\leq \frac{C}{\rho^2} a_k(S) \frac{n}{n_1} \left(\frac{\rho}{\operatorname{dist}(x, I \setminus A)}\right)^{b_3}. \end{aligned} \tag{7.29}$$

As for (7.23). Since $x \in \overline{O}_e$, $\nu \notin H$. If also $(\nu + 1) \notin H$, then $S \in \Sigma_{k,n}(Y_s)$ implies $p_\nu \equiv p_{\nu+1}$. Hence $\sigma_1 = 0$. Otherwise, $(\nu + 1) \in H$, so that $x \in \overline{O}_e$ implies $x - x_\nu \geq \rho$. Therefore (7.26) holds for $j = \nu + 1$, and we may absorb σ_1 either in σ_2 or in σ_3, as the case may be, and which we estimate below.

What is left is to estimate σ_2 and σ_3. It follows from (7.26) that

$$
\begin{aligned}
|\sigma_3(x)| &\le \frac{Ca_k(S)}{\rho^q}\frac{n}{n_1}\rho^{b_3}\sum_{I_j\not\subseteq A, j\ne\nu,\nu+1}\frac{h_j}{(\rho+\operatorname{dist}(x,I_j))^{b_3+1}},\\
&\le \frac{Ca_k(S)}{\rho^q}\frac{n}{n_1}\left(\frac{\rho}{\rho+\operatorname{dist}(x,I\setminus A)}\right)^{b_3}.
\end{aligned}
\tag{7.30}
$$

Similarly, if $\operatorname{dist}(x,I_{v^*}):=\min\{\operatorname{dist}(x,I_{\nu-1}),\operatorname{dist}(x,I_{\nu+2})\}$, then we obtain

$$
|\sigma_2(x)| \le \frac{Ca_k(S,A)}{\rho^q}\frac{n}{n_1}\left(\frac{\rho}{\rho+\operatorname{dist}(x,I_{\nu^*})}\right)^{b_3} \le \frac{Ca_k(S,A)}{\rho^q}.
\tag{7.31}
$$

Thus (7.23) follows by combining (7.30) and (7.31) with the above discussion of σ_1, and (7.24) is obtained by combining (7.28) through (7.31). This completes the proof. $\square$

The following result is almost trivial.

Lemma 19. *If $S\in\Sigma_{k,n}$, then*

$$
\|S-D_{n_1}\| \le Ca_k(S).
\tag{7.32}
$$

Moreover, if $S\in\Sigma_{k,n}(Y_s)$ and

$$
S''(y_i)=0,\quad i=1,\dots,s,
\tag{7.33}
$$

then

$$
D''_{n_1}(y_i)=0,\quad i=1,\dots,s.
\tag{7.34}
$$

Proof. The proof of (7.32) is similar to that of (7.24), in fact easier, so we only prove (7.34).

To this end fix $1\le i\le s$, and let ν be such that $y_i\in I_\nu$. Since $p_j\equiv p_\nu$, for all $I_j\subseteq I^*_\nu$, then

$$
D''_{n_1}(y_i)=\sum_{j=1}^{n}(p_j(y_i)\tilde T''_j(y_i)+p'_j(y_i)\tilde T'_j(y_i))+\sum_{I_j\not\subseteq I^*_\nu}p''_j(y_i)\tilde T_j(y_i)+p''_\nu(y_i)\sum_{I_j\subseteq I^*_\nu}\tilde T_j(y_i).
$$

Now, by virtue of (7.20), the first and the second sums are zero, and since $p''_\nu(y_i)=S''(y_i)=0$, it follows that the third term vanishes. $\square$.

Finally we have,

Lemma 20. *If A is a proper interval, $S \in \Sigma^1_{k,n}(Y_s)$, and (7.33) holds, then*

$$|S''(x) - D''_{n_1}(x)| \le \frac{C_0\pi(x)}{\rho^2}\left(a_k(S,A) + a_k(S)\frac{n}{n_1}\left(\frac{\rho}{\operatorname{dist}(x, I\setminus A)}\right)^{b_3}\right), \quad x \in A, \tag{7.35}$$

where $C_0 = C_0(k,s,b)$, and recall that $\pi(x)$ is from (4.8).

Proof. Let $x \in A$. First observe that if $x \notin \overline{O}_e$, then $\pi(x) > c$. Indeed, if $x \notin \bar{O}$, then it follows from (4.9), and we only have to check the case where x is in a connected component, say $[x_\mu, x_\nu]$, of $\bar{O}$ and either $x + \rho/2 > x_\nu$ and $\nu > 0$, or $x - \rho/2 < x_\mu$ and $\mu < n$. Clearly, we have to worry only about y_i's in this component, so let $y_i \in [x_\mu, x_\nu]$. It is easily seen that $x + \rho/2$ is increasing in $[-1, x_1]$ and that $x - \rho/2$ is increasing in $[x_{n-1}, 1]$. We will show that $x_\nu < x + \rho/2$ and $x < \frac{x_\nu + x_{\nu+1}}{2}$, cannot hold simultaneously. Indeed if $x_\nu < x + \rho/2$ and $x_{\nu+1} \le x \le x_\nu$, then $x_\nu < x + \rho/2 \le x + |I_{\nu+1}|/2$, which yields that $x - x_{\nu+1} > |I_{\nu+1}|/2$. Since $x + \rho/2$ is increasing, this in turn implies that if $x < x_{\nu+1}$, then $x + \rho/2 < x_\nu$. Hence if $x_\nu < x + \rho/2$, then $x - y_i \ge x - x_{\nu+1} > |I_{\nu+1}|/2$, so that

$$\frac{x - y_i}{x - y_i + \rho} \ge \frac{|I_{\nu+1}|/2}{|I_{\nu+1}|/2 + |I_{\nu+1}|} \ge \frac{1}{3}.$$

The case $x - \rho/2 < x_\mu$ is symmetric. Thus (7.35) follows by (7.24).

If, on the other hand, $x \in \overline{O}_e \subseteq \overline{O}$, then $x \in I^*_j$, where I^*_j is a connected component of $\overline{O}$, such that

$$\rho_n(u) \le |I^*_j| \le c\rho_n(u), \quad u \in I^*_j, \tag{7.36}$$

and we have

$$S(u) = p_j(u), \quad u \in I^*_j. \tag{7.37}$$

This together with (7.36) implies that for $A_1 := A \cup I^*_j$, which is a proper interval, we have $a_k(S, A_1) \le c a_k(S, A)$. Set

$$I^*_e := I^*_j \cap \overline{O}_e$$

Since $x \in I^*_e$, $\operatorname{dist}(x, I \setminus I^*_j) \ge \rho/2$, and by (7.36), $\operatorname{dist}(x, I\setminus I^*_j) \le |I^*_j| \le c \operatorname{dist}(x, I \setminus I^*_j)$. Hence

$$\begin{aligned} \operatorname{dist}(x, I\setminus A_1) &\le |I^*_j| + \operatorname{dist}(I^*_j, I\setminus A_1) \\ &\le c\operatorname{dist}(x, I\setminus I^*_j) + \operatorname{dist}(x, I\setminus A_1) \\ &\le c\operatorname{dist}(x, I\setminus A_1), \quad x \in I^*_e. \end{aligned}$$

By virtue of (7.23) we thus obtain,

$$\|S^{(q)} - D^{(q)}_{n_1}\|_{I^*_e} \le \frac{C}{|I^*_j|^q}\Omega, \quad q = 0, \dots, s+2, \tag{7.38}$$

with

$$\Omega := a_k(S,A) + a_k(S)\frac{n}{n_1}\left(\frac{|I_j^*|}{|I_j^*| + \operatorname{dist}(I_j^*, I\setminus A)}\right)^{b_3},$$

where we used the fact that $\operatorname{dist}(I_j^*, I\setminus A_1) \geq \operatorname{dist}(I_j^*, I\setminus A)$. It remains to prove that

$$|S''(x) - D''_{n_1}(x)| \leq \frac{C\pi(x)}{|I_j^*|^2}\Omega. \tag{7.39}$$

To this end, let

$$\pi_1(x) := \prod_{y_i\in I_j^*}\frac{|x-y_i|}{|x-y_i|+\rho}, \quad \pi_2(x) := \prod_{y_i\notin I_j^*}\frac{|x-y_i|}{|x-y_i|+\rho},$$

so that $\pi(x) = \pi_1(x)\pi_2(x)$. If $y_i \notin I_j^*$, then $|x-y_i| > \rho/2$, whence $\pi_2(x) \geq 3^{-s}$. Therefore we have to prove (7.39) with $\pi_1(x)$ in place of $\pi(x)$. Now by (7.37) $S - D_{n_1}$ is a polynomial in I_j^*, and (7.33) and (7.34) imply

$$S''(y_i) - D''_{n_1}(y_i) = 0, \quad i = 1\dots, s.$$

Hence, if y_{i_μ}, $1 \leq \mu \leq l \leq s$, are the points of Y_s in I_j^*, then there is a $\theta \in I_e^*$, such that

$$\begin{aligned}
|S''(x) - D''_{n_1}(x)| &= |S^{(l+2)}(\theta) - D^{(l+2)}_{n_1}(\theta)|\prod_{\mu=1}^{l}|x-y_{i_\mu}| \\
&\leq \frac{C\Omega}{|I_j^*|^2}\prod_{\mu=1}^{l}\frac{|x-y_{i_\mu}|}{|I_j^*|} \\
&\leq \frac{C\pi_1(x)}{|I_j^*|^2}\Omega,
\end{aligned}$$

where in the first inequality we applied (7.38) and for the second we used the inequality $|x-y_{i_\mu}| + \rho \leq c|I_j^*|$. This completes the proof of (7.39), and of our lemma. □

We are in a position to prove Theorem 5.

§8 Proof of Theorem 5

Recall that we may assume that $k \geq 3$. We begin with notation. Given $A \subseteq I$ denote

$$A^e := \cup_{I_j\cap A\neq\emptyset} I_j, \quad A^{2e} := (A^e)^e \quad \text{and} \quad A^{3e} := (A^{2e})^e.$$

Without loss of generality we may assume that

$$a_k(S) \leq 1, \tag{8.1}$$

so that in view of (6.2), in order to prove our assertion, we have to find a polynomial P_n of degree $\le cn$, such that

$$\|S - P_n\| \le c, \tag{8.2}$$

and

$$P_n''(x)\delta(x) \ge 0, \quad x \in I, \tag{8.3}$$

where $\delta(x)$ was defined in (4.7). We fix b so big that $b_3 \ge 25(s+1)$, (b_3 was defined in (7.29)). This makes $C_0(k,s,b)$, the constant in (7.35), dependent only on k and s so we denote $c_2 := C_0$. Fix an integer c_3 such that

$$c_3 \ge \max\{8k/c_1, 12s\}, \tag{8.4}$$

where c_1 is the constant from (4.28), and without loss of generality we may assume, that n is divisible by c_3, i.e., $n = Nc_3$, where this defines N.

We divide I into N intervals

$$E_q := [x_{qc_3}, x_{(q-1)c_3}] = I_{qc_3} \cup \cdots \cup I_{(q-1)c_3+1}, \quad q = 1, \dots, N.$$

We will write $j \in UC$ (for "Under Control"), if there is an $x \in I_j$, such that

$$|S''(x)| \le \frac{5c_2}{\rho^2}, \tag{8.5}$$

and we will say that $q \in G_1$, if E_q contains at least $2k-5$ intervals I_j with $j \in UC$. We will say that $q \in G$, if either $q \in G_1$, or there is a $q^* \in G_1$, such that

$$E^e_{q+\nu} \cap O \ne \emptyset, \quad \begin{cases} \nu = 0, 1, \dots, q^* - q, & \text{if} \quad q^* \ge q \\ \nu = 0, -1, \dots, q^* - q, & \text{if} \quad q^* < q. \end{cases} \tag{8.6}$$

Note that if $q \in G \setminus G_1$, then $|q - q^*| \le 2s$, hence (8.1), (8.5) and Lemma 12 imply

$$\|\rho^2 S''\|_{E_q} \le c, \quad q \in G. \tag{8.7}$$

Now set

$$E := \cup_{q \notin G} E_q,$$

and decompose S into a "small" part and a "big" one by setting

$$s_1(x) := \begin{cases} S''(x), & \text{if} \quad x \notin E^e, \\ 0, & \text{if} \quad x \in E^e, \end{cases}$$

and $s_2 := S'' - s_1$, and finally putting

$$S_1(x) := S(-1) + (x+1)S'(-1) + \int_{-1}^{x} (x-u)s_1(u)du,$$
$$S_2(x) := \int_{-1}^{x} (x-u)s_2(u)du.$$

(Note that s_1 and s_2 are well defined for $x \neq x_j$, $0 \leq j \leq n$, so that S_1 and S_2 are well defined everywhere and possess a second derivative again for $x \neq x_j$, $0 \leq j \leq n$. Thus from now on whenever we write $S_l''(x)$ we will mean $x \neq x_j$, $0 \leq j \leq n$.) It follows from (5.6) that $S_1, S_2 \in \Sigma^1_{k,n}(Y)$. Evidently,

$$S_1''(x)\delta(x) \geq 0, \quad x \in I, \quad \text{and} \quad S_2''(x)\delta(x) \geq 0, \quad x \in I.$$

Lemma 10 and (8.7) imply

$$a_k(S_1) \leq c,$$

which by virtue of (8.1) yields

$$a_k(S_2) \leq c + 1 < [c+2] =: c_4. \tag{8.8}$$

The set E is a union of disjoint intervals $F_p = [a_p, b_p]$, between any two of which there is an interval E_q with $q \in G$. We may assume that $n > c_3c_4$, and write $p \in AG$ (for "Almost Good"), if F_p consists of no more than c_4 intervals E_q, in particular it consists of no more than c_3c_4 intervals I_j. Set

$$F := \cup_{p \notin AG} F_p,$$

and let

$$s_4 := \begin{cases} S''(x), & \text{if} \quad x \in F^e, \\ 0, & \text{otherwise}, \end{cases}$$

and $s_3 := S'' - s_4$. Now put

$$S_3(x) := S(-1) + (x+1)S'(-1) + \int_{-1}^{x} (x-u)s_3(u)du,$$
$$S_4(x) := \int_{-1}^{x} (x-u)s_4(u)du.$$

Then evidently

$$S_3, S_4 \in \Sigma^1_{k,n}(Y_s), \tag{8.9}$$

$$S_3''(x)\delta(x) \geq 0, \quad x \in I, \tag{8.10}$$

and

$$S_4''(x)\delta(x) \ge 0, \quad x \in I. \tag{8.11}$$

For $p \in AG$, Lemma 12 and (8.8) imply

$$|S_3''(x)| = |S_2''(x)| \le \frac{c}{\rho^2}, \quad x \in F_p.$$

Hence

$$|S_3''(x)| \le \frac{c}{\rho^2}, \quad x \in I, \tag{8.12}$$

which by virtue of Lemma 10 yields, $a_k(S_3) \le c$, whence by (8.1),

$$a_k(S_4) \le c + 1 < [c+2] =: c_5. \tag{8.13}$$

In view of (8.9), (8.10), combining Theorem 4 with (8.12) and (6.3), we obtain the existence of a polynomial r_n which is coconvex with S, and such that

$$\|S_3 - r_n\| \le c. \tag{8.14}$$

Since

$$s_4(x) = S''(x), \quad x \in F^e,$$

then by (8.1) we have for $p \notin AG$

$$a_k(S_4, F_p^e) = a_k(S, F_p^e) \le a_k(S) \le 1. \tag{8.15}$$

Also for such p,

$$s_4(x) = S_2''(x), \quad x \in F_p^{3e}.$$

Hence from (8.8)

$$a_k(S_4, F_p^{3e}) = a_k(S_2, F_p^{3e}) \le a_k(S_2) \le c_4. \tag{8.16}$$

We still have to approximate S_4. To this end, applying Lemma 9 we construct three polynomials Q_n and M_n of degree $< cn$ and we let $D_{n_1}(\cdot, S_4)$ of degree cn_1, be defined by (7.22).

We begin with Q_n. For each q for which $E_q \subseteq F$, let J_q be the union of all intervals $I_j \subseteq E_q$ with $j \in UC$. Recall that $q \notin G$, therefore by (8.4), the number of such intervals is at most $2k - 6 < c_3/4$, and the total number of intervals in E_q is c_3. Thus Lemma 9 is applicable for each E_q and if we set

$$Q_n := \sum_{E_q \subseteq F} Q_n(\cdot, E_q, J_q),$$

where on the right-hand side are the polynomials guaranteed by Lemma 9 ($Q_n(\cdot, E_q, J_q) \equiv 0$, if $J_q = \emptyset$), and denote

$$J := \bigcup_{E_q \subseteq F} J_q,$$

then we conclude that Q_n satisfies

$$Q_n''(x)\delta(x) \geq 0, \quad x \in I \setminus F, \tag{8.17}$$

$$Q_n''(x)\delta(x) \geq -\frac{\pi(x)}{\rho^2}, \quad x \in F \setminus J, \tag{8.18}$$

$$Q_n''(x)\delta(x) \geq \frac{4\pi(x)}{\rho^2}, \quad x \in J. \tag{8.19}$$

Note that (8.17), (8.18) and (8.19) follow since for any given x all relevant $Q_n''(x, E_q, J_q)$, except perhaps one, have the same sign. Finally, it follows from (4.30) that

$$\|Q_n\| \leq c. \tag{8.20}$$

Next we define the polynomial M_n. For each F_p with $p \notin AG$, let J_{p^-} denote the union of two intervals in the left side of $F_p^{2e} \setminus \overset{\circ}{F}_p$, and let J_{p^+} denote the union of two intervals in the right side of $F_p^{2e} \setminus \overset{\circ}{F}_p$. Similarly, let F_{p^-} and F_{p^+} be closed intervals each consisting of $l := c_3 c_4$ intervals I_j and such that $J_{p^-} \subseteq F_{p^-} \subseteq F_p^{2e}$ and $J_{p^+} \subseteq F_{p^+} \subseteq F_p^{2e}$. Now we set

$$M_n := \sum_{p \notin AG} \big(Q_n(\cdot, F_{p^+}, J_{p^+}) + Q_n(\cdot, F_{p^-}, J_{p^-}) \big).$$

Since $l = c_3 c_4$ and $\mu = 2$, it follows from (8.4) that $c_1 \frac{l}{\mu} \geq 2c_4$. Again by Lemma 9

$$M_n''(x)\delta(x) \geq -2\frac{\pi(x)}{\rho^2}, \quad x \in F, \tag{8.21}$$

$$M_n''(x)\delta(x) \geq \frac{2c_4\pi(x)}{\rho^2}, \quad x \in F^{2e} \setminus F, \tag{8.22}$$

and

$$M_n''(x)\delta(x) \geq \frac{\pi(x)}{\rho^2} \left(\frac{\rho}{\operatorname{dist}(x, F^e)} \right)^{25(s+1)}, \quad x \in I \setminus F^{2e}, \tag{8.23}$$

where in (8.23) we used the inequality

$$\max\{\rho, \operatorname{dist}(x, F^{2e})\} \le \operatorname{dist}(x, F^e), \quad x \in I \setminus F^{2e}.$$

Finally, it readily follows from (4.30) that

$$\|M_n\| \le c. \tag{8.24}$$

The third auxiliary polynomial the properties of which we need to recall is $D_{n_1} := D_{n_1}(\cdot, S_4)$. By (8.13) and the choice of b, Lemma 19 yields

$$\|S_4 - D_{n_1}\| \le c, \tag{8.25}$$

and Lemma 20 combined with (8.9) and (8.11) implies that for any proper interval A

$$|S_4''(x) - D_{n_1}''(x)| \le \frac{c_2\pi(x)}{\rho^2} a_k(S_4, A) + \frac{c_2c_5\pi(x)}{\rho^2}\frac{n}{n_1}\left(\frac{\rho}{dist(x, I \setminus A)}\right)^{13(s+1)}, \quad x \in A. \tag{8.26}$$

Put $n_1 := c_5 n$, and write

$$R_n := D_{n_1} + c_2 Q_n + c_2 M_n. \tag{8.27}$$

By virtue of (8.20), (8.24), and (8.25), we obtain

$$\|S_4 - R_n\| \le c.$$

Combined with (8.14), this proves (8.2) for $P_n := R_n + r_n$. Thus in order to conclude the proof of Theorem 5, we should prove that (8.3) holds for our P_n. To this end, we recall that r_n is coconvex with S, so that we only have to deal with R_n. Since (8.26) holds for any proper interval A, we will prescribe different ones as needed. As long as $x \in F$, it suffices to take $A = F_p^e$, where p is such that $x \in F_p$. Then the quotient inside the big parentheses in (8.26) is bounded by 1, for all $x \in F$, and (8.15) and (8.26) yield

$$|S_4''(x) - D_{n_1}''(x)| \le \frac{c_2\pi(x)}{\rho^2} a_k(S_4, F_p^e) + \frac{c_2c_5\pi(x)}{\rho^2}\frac{n}{n_1} \le 2\frac{c_2\pi(x)}{\rho^2}, \quad x \in F. \tag{8.28}$$

If $x \in F^{2e} \setminus F$, then it suffices to take $A = F_p^{3e}$, where p is such that $x \in F_p^{2e}$, and similarly (8.16) and (8.26) imply

$$|S_4''(x) - D_{n_1}''(x)| \le \frac{c_2\pi(x)}{\rho^2} a_k(S_4, F_p^{3e}) + \frac{c_2c_5\pi(x)}{\rho^2}\frac{n}{n_1} \le 2\frac{c_2c_4\pi(x)}{\rho^2}, \quad x \in F^{2e}. \tag{8.29}$$

Finally, if $x \in I \setminus F^{2e}$, then we take A, to be the connected component of $I \setminus \overset{\circ}{F}{}^e$, that contains x. Then by (8.26),

$$
\begin{aligned}
&|S_4''(x) - D_{n_1}''(x)| \\
&\le \frac{c_2\pi(x)}{\rho^2} a_k(S_4, A) + \frac{c_2 c_5 \pi(x)}{\rho^2} \frac{n}{n_1} \left(\frac{\rho}{\operatorname{dist}(x, I \setminus A)} \right)^{25(s+1)} \\
&= \frac{c_2\pi(x)}{\rho^2} \left(\frac{\rho}{\operatorname{dist}(x, F^e)} \right)^{25(s+1)}, \quad x \in I \setminus F^{2e}.
\end{aligned} \tag{8.30}
$$

Since by (8.27)

$$R_n''(x)\delta(x) \ge c_2 Q_n''(x)\delta(x) + c_2 M_n''(x)\delta(x) + S_4''(x)\delta(x) - |S_4''(x) - D_{n_1}''(x)|, \quad x \in I,$$

it follows by (8.19), (8.21), (8.11) and (8.28), that

$$R_n''(x)\delta(x) \ge \frac{c_2\pi(x)}{\rho^2}(4 - 2 + 0 - 2) = 0, \quad x \in J.$$

If $x \in F \setminus J$, then (8.5) is violated so that

$$S_4''(x)\delta(x) > \frac{5c_2}{\rho^2} \ge \frac{5c_2}{\rho^2}\pi(x).$$

Hence by virtue of (8.18), (8.21), (8.28), we get

$$R_n''(x)\delta(x) \ge \frac{c_2\pi(x)}{\rho^2}(-1 - 2 + 5 - 2) = 0, \quad x \in F \setminus J.$$

Next, if $x \in F^{2e} \setminus F$, then by (8.17), (8.22), (8.11) and (8.29), we obtain

$$R_n''(x)\delta(x) \ge 0. \tag{8.31}$$

Finally, (8.11), (8.17), (8.23) and (8.30) imply (8.31) for $x \in I \setminus F^{2e}$.

Thus, (8.31) holds for all $x \in I$, and so we have constructed a polynomial P_n, satisfying (8.2) and (8.3), for each $n > c$, divisible by c_3. For all other n's Theorem 5 follows by the inclusion

$$\Sigma^1_{k,n}(Y_s) \subseteq \Sigma^1_{k,c_3 n}(Y_s).$$

This completes the proof. □

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School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel
leviatan@math.tau.ac.il

Faculty of Mechanics and Mathematics, National Taras Shevchenko University of Kyiv, 01017 Kyiv, Ukraine
shevchuk@univ.kiev.ua