RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Suppl. 82 (2010), pp. 1–16

Coconvex pointwise approximation

G.A. Dzyubenko, D. Leviatan and I.A. Shevchuk

Dedicated to Professor Giuseppe Mastroianni on occasion of his 70th birthday

Abstract. There are two kinds of estimates of the degree of approximation of continuous functions on $[-1, 1]$ by algebraic polynomials, Nikolskii-type pointwise estimates and Jackson-type uniform estimates, involving either ordinary moduli of smoothness, or the Ditzian-Totik (DT) ones, or the recent estimates involving the weighted DT-moduli of smoothness. The purpose of this paper is to complete the table of the validity or invalidity of the pointwise estimates in the case of coconvex polynomial approximation. This will enable us to compare this table with a corresponding one for coconvex Jackson-type (uniform) estimates.

Mathematical Subject Classification (2010). Primary 41A10, 41A17, 41A25, 41A29

Keywords and phrases. Coconvex polynomial approximation, pointwise estimates.

1 Introduction

There are two kinds of estimates of the degree of approximation of continuous functions on $[-1, 1]$ by algebraic polynomials, Nikolskii-type pointwise estimates and Jackson-type uniform estimates, involving either ordinary moduli of smoothness, or the Ditzian-Totik (DT) ones, or the recent estimates involving the weighted DT-moduli of smoothness. Specifically, if

This paper is in final form and no version of it will be submitted for publication elsewhere.

$\omega_k(f,1/n)$ denotes the ordinary modulus of smoothness (of order k), then the uniform estimates took the form

$$(1)\quad E_n(f) := \inf_{p_n\in\mathcal{P}_n} \|f-p_n\|_{C[-1,1]} \le c(k,r)n^{-r}\omega_k(f^{(r)},1/n), \quad n\ge k+r,$$

where $f\in C^r[-1,1] =: C^r$, the space of r times continuously differentiable functions on $[-1,1]$ (when $r=0$ we suppress the superscript r, that is, we write $C := C[-1,1]$ instead of $C^0[-1,1]$), $E_n(f)$ is the degree of approximation of f by polynomials of degree $< n$, the class of which we denote by $\mathcal{P}_n$, and $c(k,r)$ is a constant which may depend on k and r but is independent of f and n. Similar estimates hold for the other moduli of smoothness mentioned above.

In the sequel, we denote by c constants, which may depend only on the parameters indicated in the parentheses.

While we are dealing with the notation of function spaces, we denote as usual by $L_\infty := L_\infty[-1,1]$ the space of functions essentially bounded in $[-1,1]$, equipped with the norm $\|\cdot\| := \|\cdot\|_{L_\infty[-1,1]} := \operatorname{esssup} f$ and, in particular, if $f\in C$, then $\|f\| = \|f\|_{C[-1,1]}$. For $r\in\mathbb{N}$ we let W^r be the subspace of all functions $f\in C$, possessing an absolutely continuous $(r-1)$-st derivative in $(-1,1)$ and such that $f^{(r)}\in L_\infty$.

Since the early papers by Lorentz, Zeller, DeVore, and Newman, on shape preserving approximation most papers (many of them by some or all the authors of this paper), were dedicated to the question of the validity of analogs of these estimates for comonotone and coconvex approximations. Namely, for each triple (k,r,s), where k and r are as above and s is the number of changes of monotonicity or convexity, respectively, of the function f, it is now known whether or not (1) is valid where the approximation is restricted to polynomials which are comonotone or coconvex, respectively, with f (see, e.g., [15] and [17], respectively). Estimates for the analogue of (1) for the weighted DT-moduli of smoothness are also known (see e.g. [15, 12, 13]). Finally, in [15] are given the complete tables of validity for comonotone pointwise estimates (see [3] for details).

Recently, Dzyubenko and collaborators have dealt with coconvex Nikolskii-type (pointwise) estimates (see references) and have covered most but not all cases. The purpose of this paper is to complete the table of the validity or invalidity of the estimates in the case of coconvex polynomial approximation. This will enable us to compare this table with a corresponding one for coconvex Jackson-type (uniform) estimates.

2 Pointwise and uniform coconvex approximation: tables of validity

Let $f \in C[a,b]$, where $a < b$. For $k \in \mathbb{N}$, denote by

$$\omega_k(f,t,[a,b]) \tag{2}$$

$$:= \sup_{h\in[0,t]} \sup_{a\le x-kh/2<x+kh/2\le b} \left| \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x-kh/2+jh) \right|,$$

the ordinary k-th modulus of smoothness of f, in $[a,b]$. For the sake of uniformity in notation we also denote $\omega_0(f,t,[a,b]) := \|f\|_{[a,b]}$. When $f \in C$ we suppress mentioning the interval, namely, we write $\omega_k(f,t) := \omega_k(f,t,[-1,1])$.

For $n \in \mathbb{N}$, let

$$\rho_n(x) := \frac{1}{n^2} + \frac{\sqrt{1-x^2}}{n}, \qquad x \in [-1,1].$$

If $f \in C^r$, then by the classical Nikolskii-type direct pointwise estimates a sequence $\{P_n\}_{n=k+r}^{\infty}$ of polynomials $P_n \in \mathcal{P}_n$, exists, such that

$$|f(x) - P_n(x)| \le c(k,r)\rho_n^r(x)\omega_k(f^{(r)},\rho_n(x)), \qquad x \in [-1,1]. \tag{3}$$

This direct theorem was proved by Timan for $k = 1$, by Dzyadyk and, independently, by Freud, for $k = 2$, and by Brudnyi for $k \ge 3$ (for details, see [1], p. 381). Note that since $\rho_n(x) \le \frac{2}{n}$, (3) implies (1).

Everywhere below we assume that $f \notin \mathcal{P}_{k+r}$, and for such an f, we find it convenient to rewrite (3) in the form

$$E_{n,k,r}(f) := \inf_{P_n\in\mathcal{P}_n} \left\| \frac{f-P_n}{\rho_n^r \omega_k(f^{(r)},\rho_n)} \right\| \le c(k,r), \qquad n \ge k+r.$$

It follows immediately by (3) that if $f \in W^r$, then a sequence $\{P_n\}_{n=r}^{\infty}$ of polynomials $P_n \in \mathcal{P}_n$, exists, such that

$$|f(x) - P_n(x)| \le c(r)\rho_n^r(x)\|f^{(r)}\|, \qquad x \in [-1,1].$$

In this paper we compare the validity or invalidity of the uniform and pointwise estimates for coconvex approximation, for a given triplet (k,r,s), where $k \ge 0$ is the order of the modulus of smoothness of $f^{(r)}$, $r \ge 0$, and s

is the number of changes of convexity. More precisely, denote by $\mathbb{Y}_s$, $s\in\mathbb{N}$, the set of all collections $Y_s:=\{y_i\}_{i=1}^s$ of points y_i, such that

$$-1<y_s<\cdots<y_1<1.$$

For $Y_s\in\mathbb{Y}_s$ denote by $\Delta^{(2)}(Y_s)$ the collection of functions $f\in C$, that change convexity at the points y_i, and are convex in $[y_1,1]$. That is $f\in\Delta^{(2)}(Y_s)$ if and only if f is convex in $[y_1,1]$, concave in $[y_2,y_1]$, convex in $[y_3,y_2]$, and so on. Note that if f is twice differentiable in $[-1,1]$, then $f\in\Delta^{(2)}(Y_s)$ if and only if

$$f''(x)\Pi(x)\ge 0,\quad x\in[-1,1],$$

where

$$\Pi(x):=\prod_{i=1}^s(x-y_i).$$

To unify the notation we set $Y_0:=\emptyset$, $\mathbb{Y}_0=\{\emptyset\}$, and denote by $\Delta^{(2)}(Y_0)$ the set of convex functions $f\in C$.

For $f\in\Delta^{(2)}(Y_s)$ let

$$E_n^{(2)}(f,Y_s):=\inf_{P_n\in\mathcal{P}_n\cap\Delta^{(2)}(Y_s)}\|f-P_n\|,$$

denote the degree of best uniform coconvex polynomial approximation, and denote for $f\in C^r\cap\Delta^{(2)}(Y_s)$ ($f\in W^r\cap\Delta^{(2)}(Y_s)$, if $k=0$),

$$E_{n,k,r}^{(2)}(f,Y_s):=\inf_{P_n\in\mathcal{P}_n\cap\Delta^{(2)}(Y_s)}\left\|\frac{f-P_n}{\rho_n^r\omega_k(f^{(r)},\rho_n)}\right\|,$$

so that we always have

$$E_n^{(2)}(f,Y_s)\le 2^{k+r}n^{-r}\omega_k(f^{(r)},1/n)E_{n,k,r}^{(2)}(f,Y_s).$$

Thus, with this notation the purpose of the paper is for a given triplet (k,r,s), to establish whether or not the inequality

$$E_{n,k,r}^{(2)}(f,Y_s)\le c,\qquad n\ge N,\tag{4}$$

is valid for a function $f\in C^r\cap\Delta^{(2)}(Y_s)$ ($f\in W^r\cap\Delta^{(2)}(Y_s)$, if $k=0$), and $f\notin\mathcal{P}_{k+r}$.

With $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, let $(k,r,s)\in\mathbb{N}_0\times\mathbb{N}_0\times\mathbb{N}_0$ and $k+r\ne 0$. We say that *a*) (4) is not valid with both c and N independent on Y_s, if for each $A > 0$ there is a positive integer N, such that for every $n \ge N$ there are, a collection $Y_s \in \mathbb{Y}_s$ and a function $f \in C^r \cap \Delta^{(2)}(Y_s)$ ($f \in W^r \cap \Delta^{(2)}(Y_s)$, if $k = 0$), satisfying

$$E^{(2)}_{n,k,r}(f, Y_s) \ge A; \tag{5}$$

b) (4) is not valid with $c = c(k, r, s)$ and N independent of f, if for each $A > 0$ there are, a positive integer N and a collection $Y_s \in \mathbb{Y}_s$, such that for every $n \ge N$, a function $f \in C^r \cap \Delta^{(2)}(Y_s)$ ($f \in W^r \cap \Delta^{(2)}(Y_s)$, if $k = 0$) exists, satisfying (5);
c) (4) is not valid for any Y_s with both c and N independent of f, if for each $A > 0$ and $Y_s \in \mathbb{Y}_s$, there is a positive integer N, such that for every $n \ge N$, a function $f \in C^r \cap \Delta^{(2)}(Y_s)$ ($f \in W^r \cap \Delta^{(2)}(Y_s)$, if $k = 0$) exists, satisfying (5);
d) (4) cannot be had even if we allow both constants c and N to depend on all parameters k, r, Y_s and f, if for each $Y_s \in \mathbb{Y}_s$ there is a function $f \in C^r \cap \Delta^{(2)}(Y_s)$, such that

$$\limsup_{n\to\infty} E^{(2)}_{n,k,r}(f, Y_s) = \infty.$$

It turns out we have to distinguish between five different cases for a triplet (k, r, s). Namely,

Definition 1. *Let $(k, r, s) \in \mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$ and $k + r \ne 0$. We write*
1. $(k, r, s) \in$ "$+$" *if* (4) *holds with $c = c(k, r, s)$ and $N = k + r$;*
2. $(k, r, s) \in$ "$\oplus$" *if* (4) *holds with $c = c(k, r, Y_s)$ and $N = k + r$, as well as, with $c = c(k, r, s)$ and $N = (k, r, Y_s)$, but* (4) *is not valid with both c and N independent on Y_s;*
3. $(k, r, s) \in$ "$\oslash$" *if* (4) *holds with $c = c(k, r, Y_s)$ and $N = k + r$, as well as, with $c = c(k, r, s)$ and $N = (k, r, Y_s, f)$, but* (4) *is not valid with $c = c(k, r, s)$ and N independent of f;*
4. $(k, r, s) \in$ "$\ominus$" *if* (4) *holds with $c = c(k, r, s)$ and $N = (k, r, Y_s, f)$ but* (4) *is not valid for any Y_s, with both c and N independent of f;*
5. $(k, r, s) \in$ "$-$"*:* (4) *cannot be had even if we allow both constants c and N to depend on all parameters k, r, Y_s and f.*

We summarize the results in the following theorem.

Theorem 1.
1. $(k, r, s) \in$ "$+$", *if $s = 0$ and either $k \le 3$ and $r \le 3 - k$, or $k \ge 0$ and $r \ge 2$; or $s = 1$ and $k \le 2$ and $r \le 2 - k$;*

2. $(k, r, s) \in$ "$\oplus$", *if* $s \geq 2$ *and* $k \leq 2$ *and* $r \leq 2 - k$;
3. $(k, r, s) \in$ "$\oslash$", *if* $s \geq 2$ *and either* $k = 3$ *and* $r = 0$, *or* $k = 2$ *and* $r = 1$, *or* $1 \leq k \leq 3$ *and* $r = 2$, *or* $k \geq 0$ *and* $r \geq 3$;
4. $(k, r, s) \in$ "$\ominus$", *if* $s = 1$ *and either* $k = 3$ *and* $r = 0$, *or* $k = 2$ *and* $r = 1$, *or* $1 \leq k \leq 3$ *and* $r = 2$, *or* $k \geq 0$ *and* $r \geq 3$.
5. $(k, r, s) \in$ "$-$", *otherwise.*

We find it easier and more convenient for the reader to comprehend Theorem 1 by describing the results in the following truth tables for the pairs (k, r), for "pointwise estimates, $s = 0$", "pointwise estimates, $s = 1$" and "pointwise estimates, $s \geq 2$". We also take the opportunity to compare these tables with tables for "uniform" coconvex approximation (see [17], p. 110 and 114), namely, when the estimates

$$E_n^{(2)}(f, Y_s) \leq c n^{-r} \omega_k(f^{(r)}, 1/n), \quad n \geq N, \tag{6}$$

are valid.

We do this before proceeding to give references to the already known results and to prove the new ones.

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
3	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
2	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
1	$+$	$+$	$+$	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	$+$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Pointwise, $s = 0$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
3	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
2	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
1	$+$	$+$	$+$	$\ominus$	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	$+$	$\ominus$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	6	k

Uniform, $s = 0$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
3	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
2	$+$	$\ominus$	$\ominus$	$\ominus$	$-$	$-$	$\cdots$
1	$+$	$+$	$\ominus$	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	$\ominus$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Pointwise, $s = 1$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
3	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
2	$+$	$+$	$+$	$\oplus$	$-$	$-$	$\cdots$
1	$+$	$+$	$\oplus$	$-$	$-$	$-$	$\cdots$
0		$+$	$+$	$\oplus$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Uniform, $s = 1$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\cdots$
3	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\oslash$	$\cdots$
2	$\oplus$	$\oslash$	$\oslash$	$\oslash$	$-$	$-$	$\cdots$
1	$\oplus$	$\oplus$	$\oslash$	$-$	$-$	$-$	$\cdots$
0		$\oplus$	$\oplus$	$\oslash$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Pointwise, $s \geq 2$

r	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
4	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$\cdots$
1	$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$-$	$\cdots$
0		$\oplus$	$\oplus$	$\oplus$	$-$	$-$	$\cdots$
	0	1	2	3	4	5	k

Uniform, $s \geq 2$

Note that the "Uniform" tables do not contain the case " $\oslash$ ".

Credits. Theorem 1.1 (that is, the " $+$ " case) was proved for $s = 0$, first by Leviatan [14] for $k \leq 2$ and $r \leq 2 - k$, and later by Kopotun [10] for $k \leq 3$ and $r \leq 3 - k$, and by Mania and Shevchuk (see [1], Theorem 7.6.5) for $k = 0$ and $r \geq 2$. For $s = 1$, $k \leq 2$ and $r \leq 2 - k$, it was proved by Dzyubenko, Gilewicz and Shevchuk [4].

The positive result in Theorem 1.2 (that is, the " $\oplus$ " cases) was proved in [4], and the negative result in [16].

The first variant of the positive result in Theorem 1.3 (that is, the " $\oslash$ " cases) was proved in [7, 8] and the second one for $r \geq 2$ follows from Remark 1.3 in [6] ("... the arguments of this paper can be easily extended to the case $s > 1$, so that the exact analogs of Theorems 1.1 through 2.2 hold also for $s > 1$."). We prove here the second variant for $r < 2$, that is, for $s > 1$ and either $k = 3$ and $r = 0$, or $k = 2$ and $r = 1$. The negative result was proved in [5].

The positive result in Theorem 1.4 (that is, the " $\ominus$ " cases) for $r \geq 2$ was proved in [6]. We are going to prove it here for $r < 2$, that is, for $s = 1$ and either $k = 3$ and $r = 0$, or $k = 2$ and $r = 1$. The negative result was proved in [5].

Finally, Theorem 1.5 (that is, the " $-$ " case) was proved by Wu and Zhou [18, 20], for $s = 0$ and either $k \geq 5$ and $r = 0$, or $k \geq 4$ and $r = 1$; and for $s \geq 1$ and either $k \geq 4$ and $r = 0$, or $k \geq 3$ and $r = 1$. It was proved by Gilewicz and Yushchenko [9] for $s \geq 1$ and $k \geq 4$ and $r = 2$, and by Yushchenko [19] for $s = 0$ and either $k = 4$ and $r = 0$, or $k = 3$ and $r = 1$.

3 The case $(k = 3, r = 0, s \geq 1)$

Our aim is to prove,

Theorem 2. *For each $Y_s \in \mathbb{Y}_s$ and every function $f \in \Delta^{(2)}(Y_s)$ there are a number $N = N(Y_s, f)$ and a sequence $\{P_n\}_{n=N}^{\infty}$ of polynomials $P_n \in \mathcal{P}_n$, such that*

$$|f(x) - P_n(x)| \le c\omega_3(f, \rho_n(x)), \quad x \in [-1, 1],$$

where c is an absolute constant.

Theorem 2 readily follows from Theorems 3 and 4 below, but in order to formulate them we need some notations.

Let $x_{j,n} := \cos(j\pi/n)$, $j = 0, \dots, n$, be the Chebyshev partition of $[-1, 1]$. For $n \in \mathbb{N}$ and $Y_s = \{y_i\}_{i=1}^{s} \in \mathbb{Y}_s$, denote by

$$O_{i,n} := (x_{j+1,n}, x_{j-2,n}), \quad \text{if} \quad y_i \in [x_{j,n}, x_{j-1,n}),$$

where $x_{n+1,n} := -1$ and $x_{-1,n} := 1$. Let

$$O = O(n, Y_s) := \bigcup_{i=1}^{s} O_{i,n}.$$

Set $I_{j,n} := [x_{j,n}, x_{j-1,n}]$, $j = 1, \dots, n$, and let $|I_{j,n}| = x_{j-1,n} - x_{j,n}$, be its length. It is well known that

$$|I_{j\pm1,n}| < 3|I_{j,n}|, \tag{7}$$

and

$$\rho_n(x) < |I_{j,n}| < 5\rho_n(x), \quad x \in I_{j,n}. \tag{8}$$

In particular

$$|O_{i,n}| < c_1\rho_n(x), \quad x \in O_{i,n}, \tag{9}$$

where $|O_{i,n}|$ denotes the length of $O_{i,n}$ and $c_1 < 65$ is an absolute constant.

Denote by $\Sigma_{k,n}$ the collection of all continuous piecewise polynomials of degree $< k$, on the Chebyshev partition $\{x_{j,n}\}_{j=0}^{n}$, and by $\Sigma_{k,n}(Y_s)$, the subset of $\Sigma_{k,n}$ consisting of those continuous piecewise polynomials $S \in \Sigma_{k,n}$ such that for every $i = 1, \dots, s$, $S|_{O_{i,n}}$ is a polynomial.

The following Theorem allows us to reduce the proofs of the direct estimates for (co)convex polynomial approximation to those for (co)convex piecewise polynomial approximation, proofs which are much easier.

Theorem 3[2]. *Let $k \in \mathbb{N}$ and $s \in \mathbb{N}_0$. For every $Y_s \in \mathbb{Y}_s$, an $f \in \Delta^{(2)}(Y_s)$, an $n \ge N(k, Y_s)$, and a piecewise polynomial $S \in \Sigma_{k,n}(Y_s) \cap \Delta^{(2)}(Y_s)$, there is a polynomial $P_n \in \Delta^{(2)}(Y_s)$ of degree $< c^*(k, s)n$, such that*

$$|f(x) - P_n(x)| \le c(k, s)\left(1 + \left\|\frac{f - S}{\omega_k(f, \rho_n)}\right\|\right)\omega_k(f, \rho_n(x)), \quad x \in [-1, 1].$$

Thus, our main result is,

Theorem 4. *If $f \in \Delta^{(2)}(Y_s)$, then there exists an $N = N(f, Y_s)$, such that for each $n \geq N$, a piecewise polynomial $S \in \Sigma_{3,n}(Y_s) \cap \Delta^{(2)}(Y_s)$ exists, such that*

$$|f(x) - S(x)| \leq c\omega_3(f, \rho_n(x)), \quad x \in [-1, 1],$$

where c is an absolute constant.

4 Proof of Theorem 4

We begin with three lemmas.

Everywhere below c_i are absolute constants.

Lemma 1. *For each convex (concave) function $f \in C[a, b]$ there exists a number $d = d([a, b], f) > 0$, such that for each h,*

$$|h| \leq E_2(f)_{[a,b]} := \inf_{l \in \mathcal{P}_2} \|f - l\|_{[a,b]},$$

there is convex (concave) function $g \in C[a, b]$, $g(x) = g(x, h, f, [a, b])$, satisfying

$$g(x) = f(x) + h, \quad x \in [a, a + d]; \qquad g(x) = f(x), \quad x \in [b - d, b],$$

and

$$\|f - g\|_{[a,b]} = |h|.$$

Proof. We prove Lemma 1 for a convex function f. Let l be a linear function of best approximation of the function f, that is $\|f - l\|_{[a,b]} = E_2(f)_{[a,b]} =: E$. Set $f^* := f - l$, and note that $f^*(a) = f^*(b) = E$. If $h = 0$, then there is nothing to prove. Hence, we may assume that $0 < |h| \leq E \neq 0$. Denote by x_1 and x_5 the (exactly two) roots of the equation $f^*(x) = 0$, $x_1 < x_5$, and let x_2 and x_4 be the (exactly two) roots of the equation $f^*(x) = |h| - E$, $x_2 < x_4$. Finally, let x_3 be a root of the equation $f^*(x) = -E$. Clearly we have

$$a < x_1 < x_2 < x_3 < x_4 < x_5 < b.$$

Then Lemma 1 is valid with $d = \min\{x_1 - a, b - x_5\}$, and the evidently convex function,

$$g(x) = l(x) + \begin{cases} f^*(x) + h, & \text{if} \quad x \in [a, x_3], \\ h - E & \text{if} \quad x \in [x_3, x_4], \\ f^*(x) & \text{if} \quad x \in [x_4, b], \end{cases}$$

if $h > 0$, and

$$g(x) = l(x) + \begin{cases} f^*(x) + h, & \text{if} \quad x \in [a, x_2], \\ -E & \text{if} \quad x \in [x_2, x_3], \\ f^*(x) & \text{if} \quad x \in [x_3, b], \end{cases}$$

if $h < 0$. □

Denote by

$$[t_1, t_2; f] := \frac{f(t_2) - f(t_1)}{t_2 - t_1},$$

the first divided difference of the function f at the points t_1 and t_2.

Lemma 2. *Given the interval $[a, b]$, let $y \in (a, b)$ be such that $(b - y)/3 < y - a < 3(b - y)$. If a function $f \in C[a, b]$ is concave on $[a, y]$ and convex on $[y, b]$, then there is a linear function $l(x) = l(x, f, [a, b], y)$, such that*

$$l' \le [a, y; f] \quad \text{and} \quad l' \le [y, b; f], \tag{10}$$

we may choose $l(a) = f(a)$ or $l(b) = f(b)$, and

$$\|f - l\|_{[a.b]} \le c_2 \omega_3(f, b - a, [a, b]), \tag{11}$$

where the modulus of smoothness for the interval $[a, b]$ was defined in (2).

Proof. Since f is concave on $[a, y]$ and convex on $[y, b]$, then (see [11]) $E_2(f)_{[a,b]} \le c_3 E_3(f)_{[a,b]} := c_3 \inf_{p \in \mathcal{P}_3} \|f - p\|_{[a,b]}$, whence $E_2(f)_{[a,b]} \le c_4 \omega_3(f, b - a, [a, b])$. Therefore, it readily follows that the choice of either $l_1(x) = f(a) + (x - a)[a, y; f]$, or $l_2(x) = f(b) + (x - b)[y, b; f]$, according to which of the two divided differences is smaller, clearly fulfils (10), while (11) follows immediately by Whitney's Theorem and the fact that l_j $j = 1$ or $j = 2$, interpolates f at two points, the distance between which is proportional to $b - a$. In order to guarantee that $l(a) = f(a)$, if the choice is l_1, then we take $l = l_1$. Otherwise, if $h := l_2(a) - f(a)$, then by (11) we may take $l = l_2 - h$, which satisfies (10) and (11), and we obtain $l(a) = f(a)$. Guaranteeing, instead, that $l(b) = f(b)$ is done in a similar way. □

Lemma 3. *For $f \in \Delta^{(2)}(Y_s)$, there exists an $N = N(f, Y_s)$, such that for each $n \geq N$, a function $f_n \in \Delta^{(2)}(Y_s)$ exists, such that $f_n|_{0_{i,n}}$ is linear for all $i = 1, \dots, s$, and*

$$|f(x) - f_n(x)| \leq c_5\, \omega_3\left(f, \rho_n(x)\right), \quad x \in [-1, 1].$$

Proof. Set

$$y_i^+ := \min\{y_i + \frac{1}{2}(1 - |y_i|), \frac{1}{2}(y_{i-1} + y_i)\},$$

and

$$y_i^- := \max\{y_i - \frac{1}{2}(1 - |y_i|), \frac{1}{2}(y_i + y_{i+1})\},$$

where $y_0 := 1$ and $y_{s+1} := -1$. Now write

$$J_i^- := [y_i^-, y_i] \quad \text{and} \quad J_i^+ := [y_i, y_i^+].$$

We divide $1 \leq i \leq s$ into two sets. We write $i \in A$, if there are two linear functions l_{i-} and l_{i+}, such that,

$$f|_{J_i^-} = l_{i-} \qquad \text{and} \qquad f|_{J_i^+} = l_{i+}.$$

Otherwise $i \notin A$.

Let N_0 be so big that for $n \geq N_0$,

$$O_{i,n} \subset J_i^- \cup J_i^+ = [y_i^-, y_i^+], \quad 1 \leq i \leq s,$$

and note that in particular $O_{i,n} \cap O_{i+1,n} = \emptyset$, $1 \leq i < s$. Denote by $y_{i,n}^+$ and $y_{i,n}^-$ the right and left ends of $O_{i,n}$, respectively. Let $n \geq N_0$ and $i \in A$. We denote by $L_{i,n}^+$, the polygonal line consisting of three segments, such that

$$L_{i,n}^+(-1) = L_{i,n}^+(y_i) = L_{i,n}^+(1) = 0 \quad \text{and} \quad L_{i,n}^+(y_{i,n}^+) = 1,$$

and similarly, we denote by $L_{i,n}^-$, the polygonal line consisting of three linear pieces, such that,

$$L_{i,n}^-(-1) = L_{i,n}^-(y_i) = L_{i,n}^-(1) = 0 \quad \text{and} \quad L_{i,n}^-(y_{i,n}^-) = 1,$$

and define

$$L_{i,n} := \begin{cases} (l_{i-}(y_{i,n}^+) - l_{i+}(y_{i,n}^+))L_{i,n}^+, & \text{if} \quad (-1)^i(l'_{i-} - l'_{i+}) \geq 0; \\ (l_{i+}(y_{i,n}^-) - l_{i-}(y_{i,n}^-))L_{i,n}^-, & \text{otherwise.} \end{cases}$$

Clearly,

$$L_{i,n} \in \Delta^{(2)}(Y_s), \tag{12}$$

and either $(f + L_{i,n})|_{O_{i,n}} = l_{i-}$, or $(f + L_{i,n})|_{O_{i,n}} = l_{i+}$, so that $f + L_{i,n}$ is linear on $O_{i,n}$. Hence, with

$$L_n := \sum_{i \in A} L_{i,n},$$

we conclude that $(f + L_n)|_{O_{i,n}}$ is linear for each $i \in A$. Also, clearly, $L_n|_{O_{i,n}}$ is linear for each $i \notin A$.

If $M := \max_{i \in A} |l'_{i+} - l'_{i-}|$, then by (9),

$$|l'_{i+} - l'_{i-}|(y^+_{i,n} - y_i) \le M(y^+_{i,n} - y_i) \le M|O_{i,n}| \le c_1 M \rho_n(y^+_{i,n}),$$

and

$$|l'_{i+} - l'_{i-}|(y_i - y^-_{i,n}) \le c_1 M \rho_n(y^-_{i,n}).$$

Hence, it follows that

$$|L_{i,n}(x)| \le c_1 M \rho_n(x), \quad x \in [-1, 1],$$

which in turn implies,

$$|L_n(x)| \le s c_1 M \rho_n(x), \quad x \in [-1, 1]. \tag{13}$$

On the other hand, for each $i \in A$ there exists $t^0_i > 0$, such that

$$\omega_3(f, t, [y^-_i, y^+_i]) = |l'_{i+} - l'_{i-}|t, \quad t \le t^0_i.$$

Therefore, if $A \neq \emptyset$, then we have

$$\omega_3(f, t) \ge Mt, \quad t \le t^0 := \min_{i \in A} t^0_i.$$

Combining with (13) this in turn yields, for all $n \ge N^0 := \max\{N_0, [1/t^0]\}$,

$$|L_n(x)| \le s c_1 \omega_3(f, \rho_n(x)), \quad x \in [-1, 1], \tag{14}$$

Next, for $i \notin A$, denote $J_i := J^-_i$, if $f|_{J^+_i}$ is linear. Otherwise, we put $J_i := J^+_i$.

Now we need an auxiliary function g_n, which coincides with f on a major part of the interval $[-1, 1]$. In fact, it differs from f only on the $O_{i,n}$'s and on the J_i's, where $i \notin A$.

We fix $i \notin A$ and assume that $J_i := J_i^+$ and that i is odd, that is, f is convex on J_i. The other cases are similar. By Lemma 2, there is a linear function $l_{i,n}$, such that

$$l'_{i,n} \le [y_{i,n}^-, y_i; f], \quad l'_{i,n} \le [y_i, y_{i,n}^+; f], \tag{15}$$

$$f(y_{i,n}^-) = l_{i,n}(y_{i,n}^-), \tag{16}$$

and

$$\|f - l_{i,n}\|_{\overline{O}_i} \le c_2\omega_3(f, |O_{i,n}|), \tag{17}$$

in particular,

$$h_{i,n} := l_{i,n}(y_{i,n}^+) - f(y_{i,n}^+) \le c_2\omega_3(f, |O_{i,n}|).$$

Since $i \notin A$, it follows that $E_2(f)_{J_i} > 0$, and we may apply Lemma 1 with $d_i := d(J_i, f)$ and $g(x, h_{i,n}, f, J_i)$, guaranteed by that lemma.

We take $N_i \ge N^0$ so big that for all $n \ge N_i$ we have

$$y_{i,n}^+ - y_i < d_i \quad \text{and} \quad c_2\omega_3(f, |O_{i,n}|) < E_2(f)_{J_i},$$

and we define

$$g_{i,n}(x) := \begin{cases} f(x), & \text{if} \quad x \in J_i^- \setminus O_{i,n}; \\ l_{i,n}, & \text{if} \quad x \in O_{i,n}; \\ g(x, h_{i,n}, f, J_i), & \text{if} \quad x \in J_i \setminus O_{i,n}. \end{cases}$$

Then by (17),

$$|f(x) - g_{i,n}(x)| \le c_2\omega_3(f, |O_{i,n}|), \quad x \in J_i^- \cup J_i.$$

Hence, the inequality

$$\max_{x \in J_i^- \cup J_i^+} \rho_n(x) \le 2 \min_{x \in J_i^- \cup J_i^+} \rho_n(x),$$

combined with (7) through (9), yields

$$|f(x) - g_{i,n}(x)| \le c_4\omega_3(f, \rho_n(x)), \quad x \in J_i^- \cup J_i. \tag{18}$$

Also, $g_{i,n}$, is continuous on $J_i^- \cup J_i$, it is concave on J_i^- and convex on J_i, and $g_{i,n}(x) = f(x)$ near both ends of $J_i^- \cup J_i$.

Thus, for each $n \ge N := \max_{i \notin A} N_i$, the function

$$g_n(x) := \begin{cases} g_{i,n}(x), & \text{if} \quad x \in O_{i,n} \cup J_i \quad \text{with} \quad i \notin A; \\ f(x), & \text{otherwise} \end{cases}$$

satisfies

$$g_n \in \Delta^{(2)}(Y_s), \tag{19}$$

$$|f(x) - g_n(x)| \le c_9 \omega_3(f, \rho_n(x)), \quad x \in [-1, 1], \tag{20}$$

and

$$g_n \quad \text{is a linear function on each} \quad O_i \quad \text{with} \quad i \notin A. \tag{21}$$

Combining (19) through (21) with (12) and (14), we conclude that

$$f_n := g_n + L_n,$$

is the required function. This completes the proof of Lemma 3. □

Denote by

$$[t_1, t_2, t_3; f] := \frac{[t_1, t_2; f] - [t_2, t_3; f]}{t_1 - t_3},$$

the second divided difference of the function f at the points t_1, t_2, t_3.

Proof of Theorem 4. Let N and f_n be defined by Lemma 3. For each $n \ge N$ and $j = 2, \dots, n-1$ let

$$\begin{aligned} p_{j,n} \quad &:= f_n(x_{j,n}) + (x - x_{j,n})[x_{j,n}, x_{j-1,n}; f_n] \\ &\quad +(\operatorname{sgn}\Pi(x_{j,n}))(x - x_{j,n})(x - x_{j-1,n}) \\ &\quad\quad \cdot \min\{|[x_{j,n}, x_{j-1,n}, x_{j-2,n}; f_n]|, |[x_{j,n}, x_{j-1,n}, x_{j+1,n}; f_n]|\}, \end{aligned}$$

be the quadratic polynomial that interpolates f_n at the points $x_{j,n}$, $x_{j-1,n}$ and either $x_{j-2,n}$ or $x_{j+1,n}$. It is readily seen (see e.g.,[10]), that the required piecewise polynomial S may be taken in the form

$$S|_{I_{j,n}} = p_{j,n}, \quad j = 2, \dots, n-1, \quad S|_{I_{1,n}} = p_{2,n} \quad \text{and} \quad S|_{I_{n,n}} = p_{n-1,n}.$$

This concludes the proof.

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G.A. Dzyubenko
International Mathematical Center NAS of Ukraine
Tereschenkivska str., 3, 01601 Kyiv, Ukraine
e-mail: `dzyuben@imath.kiev.ua`

D. Leviatan
Raymond and Beverly Sackler School of Mathematics Tel Aviv University
69978 Tel Aviv, Israel
e-mail: `leviatan@post.tau.ac.il`

I.A. Shevchuk
Faculty of Mechanics and Mathematics
National Taras Shevchenko University of Kyiv
01017 Kyiv, Ukraine
e-mail: `shevchuk@mail.univ.kiev.ua`