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A note on Hermite interpolation

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Abstract

Let $x_0, x_1, \ldots, x_n \in \mathbb{R}$, be pairwise disjoint, and let $\theta_0, \theta_1, \ldots, \theta_n \in \mathbb{N}$. Set $\theta := \sum_{\nu=0}^{n} \theta_\nu$. For each pair j, p such that $0 \le j \le n$ and $0 \le p \le \theta_j - 1$, let $y_{j,p}$ be a complex number. Then there is a unique polynomial, $H(x)$, of degree $\theta - 1$, such that

$$H^{(p)}(x_j) = y_{j,p}, \quad \text{for} \quad 0 \le p \le \theta_j - 1, \quad 0 \le j \le n.$$

In particular, there is a unique fundamental Hermite polynomial, $T_{j,p}(x)$, of degree $\theta - 1$, such that

$$T_{j,p}^{(r)}(x_s) = \delta_{j,s}\delta_{p,r}, \quad 0 \le r \le \theta_s - 1, \quad 0 \le s \le n,$$

δ being Kronecker's delta, and we have the representation

$$H(x) = \sum_{\substack{0 \le p \le \theta_j, \\ 0 \le j \le n}} H^{(p)}(x_j) T_{j,p}(x) = \sum_{\substack{0 \le p \le \theta_j, \\ 0 \le j \le n}} y_{j,p} T_{j,p}(x).$$

We give an explicit representation of the polynomials $T_{j,p}(x)$.

Keywords: Hermite interpolation, explicit representation of the fundamental polynomials.

MSC: 41A05, 41A10.

§1. Introduction

Fix $n \in \mathbb{N}$. Let $x_0, x_1, \ldots, x_n \in \mathbb{R}$, be pairwise disjoint, and let $\theta_0, \theta_1, \ldots, \theta_n \in \mathbb{N}$. Set $\theta := \sum_{\nu=0}^{n} \theta_\nu$, and let

$$\omega(x) := \prod_{\nu=0}^{n} (x - x_\nu)^{\theta_\nu} \quad \text{and} \quad \omega_j(x) := \omega(x)/(x - x_j)^{\theta_j}, \quad 0 \le j \le n.$$

For each pair j, p such that $0 \le j \le n$ and $0 \le p \le \theta_j - 1$, let $y_{j,p}$ be a complex number. Then there is a unique polynomial, $H(x)$, of degree $\theta - 1$, such that

$$H^{(p)}(x_j) = y_{j,p}, \quad \text{for} \quad 0 \le p \le \theta_j - 1, \quad 0 \le j \le n.$$

If $T_{j,p}(x)$ is the Hermite interpolation polynomial, of degree $\theta - 1$, such that,

$$T_{j,p}^{(r)}(x_s) = \delta_{j,s}\delta_{p,r}, \quad 0 \le r \le \theta_s - 1, \quad 0 \le s \le n,$$

δ being Kronecker's delta, then it readily follows that the above polynomial $H(x)$ has the representation

$$H(x) = \sum_{\substack{0 \le p \le \theta_j, \\ 0 \le j \le n}} H^{(p)}(x_j) T_{j,p}(x) = \sum_{\substack{0 \le p \le \theta_j, \\ 0 \le j \le n}} y_{j,p} T_{j,p}(x).$$

Thus, we are interested in representing the fundamental Hermite polynomials $T_{j,p}(x)$, $0 \le p \le \theta_j$, $0 \le j \le n$.

In the literature (see, e.g., [3, 2.5 Ex. 4, (2.5.25)-(2.5.27)], [6, pp. 844-849], [2, Chapter 1], [1, p. 147 (33) and (34)]) one may find a representation that involves derivatives of $\omega_j^{-1}(x)$ evaluated at $x = x_j$. Such a representation was useful, for instance, to Szabados [5], and could be used to obtain an explicit representation of $T_{j,p}(x)$. In fact, in [4, pp. 201–204] the authors work out the differentiation and give an explicit representation, and we would like to thank Giuseppe Mastroianni for bringing this book, in Rumanian, to our attention, but this book is hardly accessible.

We obtain here the explicit representation in a very simple way, with no differentiation and, in addition, we provide a recursive formula for computing the terms.

§2. Main result

Given a pair j, p, $0 \le p \le \theta_j - 1$, $0 \le j \le n$, straightforward calculations show that the function

$$f_{j,p}(x) := \frac{(x - x_j)^p}{p!} \frac{\omega_j(x)}{\omega_j(x_j)} g_{j,p}(x), \tag{2.1}$$

where $g_{j,p}(x)$ is an infinitely differentiable function with the only requirement that $g_{j,p}(x_j) = 1$, satisfies

$$f_{j,p}^{(r)}(x_s) = \delta_{j,s}\delta_{p,r}, \text{ for } 0 \le r \le \theta_s - 1,\ s \ne j, \text{ and } 0 \le r \le p,\ s = j. \tag{2.2}$$

Hence, we look for $g_{j,p}(x)$ so that $f_{j,p}(x)$ also satisfies

$$f_{j,p}^{(r)}(x_j) = 0, \quad p < r \le \theta_j - 1. \tag{2.3}$$

We will construct a polynomial $g_{j,p}(x)$ of degree $\theta_j - 1 - p$ that guarantees (2.3) and such that $g_{j,p}(x_j) = 1$. Thus, (2.1) would yield a representation for $T_{j,p}(x)$.

For $q \in \mathbb{N}_0$, denote

$$H_{j,q} := \{(\tau_\mu)_{\mu \ne j} : \tau_\mu \in \mathbb{N}_0 \quad \text{and} \quad \sum_{\mu \ne j} \tau_\mu = q\}.$$

We have the following result.

Theorem 2.1. *Given a pair j, p, $0 \le p \le \theta_j - 1$, $0 \le j \le n$, the function*

$$g_{j,p}(x) = \sum_{q=0}^{\theta_j - 1 - p} B_{j,q}(x - x_j)^q,$$

where

$$B_{j,q} := \sum_{(\tau_\nu)_{\nu \ne j} \in H_{j,q}} \prod_{\nu \ne j} \binom{\tau_\nu + \theta_\nu - 1}{\tau_\nu} (x_\nu - x_j)^{-\tau_\nu}, \quad 0 \le q \le \theta_j - 1 - p. \tag{2.4}$$

Corollary 2.2. *Given a pair j, p, $0 \le p \le \theta_j - 1$, $0 \le j \le n$, we have*

$$T_{j,p}(x) = \frac{(x - x_j)^p}{p!} \frac{\omega_j(x)}{\omega_j(x_j)} \sum_{q=0}^{\theta_j - 1 - p} B_{j,q}(x - x_j)^q,$$

where $B_{j,q}$ is given in (2.4).

In order to prove the theorem we need the following simple lemma.

Lemma 2.3. *Given j, denote*

$$D_j := \{x : |x - x_j| < \min_{s \ne j} |x_s - x_j|\}.$$

The function $\omega_j^{-1}(x)$ is analytic in D_j and possesses the Taylor expansion about x_j,

$$\frac{1}{\omega_j(x)} = \frac{1}{\omega_j(x_j)} \sum_{q=0}^{\infty} B_{j,q}(x - x_j)^q, \quad x \in D_j, \tag{2.5}$$

where

$$B_{j,q} := \sum_{(\tau_\nu)_{\nu \neq j} \in H_{j,q}} \prod_{\nu \neq j} \binom{\tau_\nu + \theta_\nu - 1}{\tau_\nu} (x_\nu - x_j)^{-\tau_\nu}, \quad q \geq 0.$$

Proof. Recall that $\omega_j(x) = \prod_{\nu \neq j}(x - x_\nu)^{\theta_\nu}$. Hence,

$$\omega_j^{-1}(x) = \prod_{\nu \neq j} \left((x_j - x_\nu) \left(1 - \frac{x - x_j}{x_\nu - x_j} \right) \right)^{-\theta_\nu} = \omega_j^{-1}(x_j) \prod_{\nu \neq j} \left(1 - \frac{x - x_j}{x_\nu - x_j} \right)^{-\theta_\nu}. \tag{2.6}$$

Now, for each $\nu \neq j$,

$$\left(1 - \frac{x - x_j}{x_\nu - x_j} \right)^{-\theta_\nu} = \sum_{q=0}^{\infty} \binom{q + \theta_\nu - 1}{q} (x_\nu - x_j)^{-q} (x - x_j)^q, \quad x \in D_j,$$

and the Cauchy product of the $n - 1$ absolutely convergent series in D_j, combined with (2.6) yield,

$$\omega_j^{-1}(x) = \omega_j^{-1}(x_j) \sum_{q=0}^{\infty} B_{j,q}(x - x_j)^q, \quad x \in D_j.$$

This completes the proof. ∎

Proof of Theorem 2.1. Given $0 \leq p \leq \theta_j - 1$, observe that the function $h_j(x) := (x - x_j)^p / p!$ satisfies

$$h_j^{(r)}(x_j) = \delta_{r,p}, \quad r \geq 0. \tag{2.7}$$

For $x \in D_j$, by Lemma 2.3, write

$$h_j(x) = \frac{(x - x_j)^p}{p!} \frac{\omega_j(x)}{\omega_j(x)} = \frac{(x - x_j)^p}{p!} \frac{\omega_j(x)}{\omega_j(x_j)} \sum_{q=0}^{\infty} B_{j,q}(x - x_j)^q.$$

Then for $p < r \le \theta_j - 1$, it follows by (2.7) that,

$$\begin{aligned} 0 = h_j^{(r)}(x_j) &= \sum_{k=0}^{r} \binom{r}{k} \left(\frac{(x-x_j)^p}{p!} \right)^{(k)} \Bigg|_{x=x_j} \left(\frac{\omega_j(x)}{\omega_j(x_j)} \sum_{q=0}^{\infty} B_{j,q}(x-x_j)^q \right)^{(r-k)} \Bigg|_{x=x_j} \qquad (2.8) \\ &= \binom{r}{p} \left(\frac{\omega_j(x)}{\omega_j(x_j)} \sum_{q=0}^{\infty} B_{j,q}(x-x_j)^q \right)^{(r-p)} \Bigg|_{x=x_j} \\ &= \binom{r}{p} \left(\frac{\omega_j(x)}{\omega_j(x_j)} \sum_{q=0}^{\theta_j-1-p} B_{j,q}(x-x_j)^q \right)^{(r-p)} \Bigg|_{x=x_j} \\ &= p_j^{(r)}(x_j), \end{aligned}$$

where

$$p_j(x) := \frac{(x-x_j)^p}{p!} \frac{\omega_j(x)}{\omega_j(x_j)} \sum_{q=0}^{\theta_j-1-p} B_{j,q}(x-x_j)^q. \qquad (2.9)$$

Comparing (2.1) with (2.9), we conclude that we may take

$$g_{j,p}(x) := \sum_{q=0}^{\theta_j-1-p} B_{j,q}(x-x_j)^q,$$

a polynomial of degree $\theta_j - 1 - p$, such that $g_{j,p}(x_j) = 1$, and (2.3) is satisfied. This completes the proof. ∎

Proof of Corollary 2.2. The polynomial $g_{j,p}(x)$ is the truncated Taylor expansion of $\omega_j(x_j)/\omega_j(x)$, and once we have the concrete polynomial $g_{j,p}(x)$, we do not limit x to be in D_j. ∎

§3. Recursive formula

Following an idea of Szabados [5], we provide an inductive way to compute the coefficients $B_{j,q}$.

Theorem 3.1. *For $\theta_j \ge 2$, $0 \le j \le n$, let*

$$a_{i,j} := \sum_{\substack{\nu=0 \\ \nu \ne j}}^{n} \frac{\theta_\nu}{(x_\nu - x_j)^i}, \quad 1 \le i \le \theta_j - 1.$$

Then

$$B_{j,q} = \frac{1}{q}\sum_{i=1}^{q} a_{i,j} B_{j,q-i}, \quad 1 \le q \le \theta_j - 1.$$

Proof. First, we observe that

$$a_{i,j} = \frac{1}{(i-1)!}\left(\frac{\theta_j}{x-x_j} - \frac{\omega'(x)}{\omega(x)}\right)^{(i-1)}\Big|_{x=x_j}, \quad 1 \le i \le \theta_j - 1,$$

and that,

$$\left(\frac{1}{\omega_j(x)}\right)' = \frac{1}{\omega_j(x)}\left(\frac{\theta_j}{x-x_j} - \frac{\omega'(x)}{\omega(x)}\right). \tag{3.1}$$

It follows by (2.5) that,

$$B_{j,q} = \frac{\omega_j(x_j)}{q!}\left(\frac{1}{\omega_j(x)}\right)^{(q)}\Big|_{x=x_j}.$$

Hence, for $1 \le q \le \theta_j - 1$,

$$\begin{aligned}
\frac{1}{q}\sum_{i=1}^{q} a_{i,j} B_{j,q-i} &= \frac{\omega_j(x_j)}{q}\sum_{i=1}^{q}\frac{1}{(i-1)!}\left(\frac{\theta_j}{x-x_j} - \frac{\omega'(x)}{\omega(x)}\right)^{(i-1)}\Big|_{x=x_j}\frac{1}{(q-i)!}\left(\frac{1}{\omega_j(x)}\right)^{(q-i)}\Big|_{x=x_j} \\
&= \frac{\omega_j(x_j)}{q!}\left[\left(\frac{\theta_j}{x-x_j} - \frac{\omega'(x)}{\omega(x)}\right)\frac{1}{\omega_j(x)}\right]^{(q-1)}\Big|_{x=x_j} \\
&= \frac{\omega_j(x_j)}{q!}\left[\left(\frac{1}{\omega_j(x)}\right)'\right]^{(q-1)}\Big|_{x=x_j} \\
&= \frac{\omega_j(x_j)}{q!}\left(\frac{1}{\omega_j(x)}\right)^{(q)}\Big|_{x=x_j} = B_{j,q},
\end{aligned}$$

where for the third equality we have applied (3.1). ∎

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