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Approximation by polynomials and ridge functions of classes of s-monotone radial functions

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Abstract

We obtain estimates on the order of best approximation by polynomials and ridge functions in the spaces L_q of classes of s-monotone radial functions which belong to the space L_p, $1 \leqslant q \leqslant p \leqslant \infty$.

MSC: 41A46

Keywords: s-Monotone functions; Radial functions; Polynomials; Ridge functions

1. Introduction and the main results

Let $I \subset \mathbb{R}$, be a finite interval (open, half-open, or closed). Given $s \geqslant 1$, a function $x : I \mapsto \mathbb{R}$ is called s-monotone on I if for every collection of $(s+1)$ distinct points $t_0, \ldots, t_s \in I$ the corresponding divided difference $[x; t_0, \ldots, t_s]$ is nonnegative. For $s = 1, 2$, s-monotone functions are nondecreasing or convex on I, respectively. Thus, the parameter s characterizes the shape of functions. Note that if a function x is s-times differentiable on I, then x is s-monotone if and only if $x^{(s)}(t) \geqslant 0$ for all $t \in I$.

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It is well known (see [2,18,21]) that for $s \geqslant 2$, if x is s-monotone, then $x^{(s-2)}$ is convex and locally absolutely continuous in I. Hence $x^{(s-1)}$ exists a.e. and is monotone nondecreasing, which in turn implies that $x^{(s)} \geqslant 0$ a.e. in I. Actually, the usual derivative $x^{(s-1)}$ exists except perhaps at a denumerable set of points of I, and the one-sided limits $x^{(s-1)}(t\pm)$ exist everywhere. The left and right derivatives $x_-^{(s-1)}(t)$ and $x_+^{(s-1)}(t)$ exist at any interior point $t \in I$, and at the end-points of I the respective one-sided derivatives of order $(s-1)$ exist but may be infinite. The one-sided derivatives $x_-^{(s-1)}$ and $x_+^{(s-1)}$ are nondecreasing on I, and at all interior points t, where the $s-1$th derivative does not exist, we will denote $x^{(s-1)}(t) := (x_-^{(s-1)}(t) + x_+^{(s-1)}(t))/2$. Finally, for every $k \leqslant s-2$ the derivative $x^{(k)}$ which exists in any open subinterval of I is $(s-k)$-monotone.

We denote by $\Delta_+^s(I)$, the set of all s-monotone functions on I. If W is a class of functions defined on I, then we set $\Delta_+^s W(I) := \Delta_+^s(I) \cap W$. By $L_p(I)$, $1 \leqslant p \leqslant \infty$, we denote the usual space of all Lebesgue measurable functions $x : I \to \mathbb{R}$ with finite norm $\|x\|_{L_p(I)}$, and its unit ball $B_p(I)$.

Let $d \geqslant 1$ and let $\mathbb{B}^d$ be the open d-dimensional unit ball in the space $\mathbb{R}^d$. A function $x : \mathbb{B}^d \mapsto \mathbb{R}$ is called radial on the ball $\mathbb{B}^d$ if $x(t) = y(|t|)$, $t = (t_1, \ldots, t_d) \in \mathbb{B}^d$, where $|t| := (t_1^2 + \cdots + t_d^2)^{1/2}$. By $\Delta_+^{\circ,s}(\mathbb{B}^d)$ we denote the set of all radial functions $x : \mathbb{B}^d \mapsto \mathbb{R}$ such that the univariate functions $y(\tau)$, $\tau \in [0,1)$, belong to the class $\Delta_+^s[0,1)$ and satisfy the conditions $y_+^{(k)}(0) = 0$, $k = 0, \ldots, s-1$. We call these functions s-monotone radial functions. If W is a class of functions defined on $\mathbb{B}^d$, then we denote $\Delta_+^{\circ,s} W(\mathbb{B}^d) := \Delta_+^{\circ,s}(\mathbb{B}^d) \cap W$. Again, $L_p(\mathbb{B}^d)$, $1 \leqslant p \leqslant \infty$, denotes the space of all Lebesgue measurable functions $x : \mathbb{B}^d \to \mathbb{R}$ with finite norm $\|x(\cdot)\|_{L_p(\mathbb{B}^d)}$, and $B_p(\mathbb{B}^d)$ is its unit ball.

The main goal of our paper is to estimate the orders of best approximation by polynomials and ridge functions of the classes $\Delta_+^{\circ,s} B_p(\mathbb{B}^d)$ in the spaces $L_q(\mathbb{B}^d)$, $1 \leqslant q \leqslant p \leqslant \infty$.

By $\mathcal{P}_n(\mathbb{R}^d)$ we denote the space of polynomials

$$P_n(t) := \sum_{|k| \leqslant n} a_k t^k, \quad t \in \mathbb{R}^d,$$

where $k = (k_1, \ldots, k_d) \in \mathbb{Z}_+^d$, $|k| := k_1 + \cdots + k_d$, $a_k \in \mathbb{R}$ and $t^k := t_1^{k_1} \ldots t_d^{k_d}$.

Denote by $\mathcal{P}_n(I)$ and $\mathcal{P}_n(\mathbb{B}^d)$ the restrictions of $\mathcal{P}_n(\mathbb{R})$ and $\mathcal{P}_n(\mathbb{R}^d)$ on I and $\mathbb{B}^d$, respectively, and let

$$E\left(\Delta_+^s B_p(I), \mathcal{P}_n(I)\right)_{L_q(I)} := \sup_{x \in \Delta_+^s B_p(I)} \inf_{P_n \in \mathcal{P}_n(I)} \|x - P_n\|_{L_q(I)},$$

$$E\left(\Delta_+^{\circ,s} B_p\left(\mathbb{B}^d\right), \mathcal{P}_n\left(\mathbb{B}^d\right)\right)_{L_q(\mathbb{B}^d)} := \sup_{x \in \Delta_+^{\circ,s} B_p(\mathbb{B}^d)} \inf_{P_n \in \mathcal{P}_n(\mathbb{B}^d)} \|x - P_n\|_{L_q(\mathbb{B}^d)}.$$

Let $\mathbb{S}^{d-1} := \partial \mathbb{B}^d$ be the unit sphere in $\mathbb{R}^d$. For $d > 1$, we denote by $\mathcal{R}_n(\mathbb{B}^d)$ the nonlinear manifold of ridge functions

$$R_n(t) := \sum_{k=1}^n r_k(a_k \cdot t), \quad t \in \mathbb{B}^d,$$

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where $r_k : I \mapsto \mathbb{R}$ is any univariate function, $a_k \in \mathbb{S}^{d-1}$, and $a_k \cdot t$ is the usual scalar product in $\mathbb{R}^d$. We also let $\mathcal{R}_{n,q}(\mathbb{B}^d)$ denote the collection of all elements of $\mathcal{R}_n(\mathbb{B}^d)$ such that $r_k \in L_q(I)$, $k = 1, \ldots, n$, and let

$$E\left(\Delta_+^{\circ,s} B_p\left(\mathbb{B}^d\right), \mathcal{R}_{n,q}\left(\mathbb{B}^d\right)\right)_{L_q(\mathbb{B}^d)} := \sup_{x\in\Delta_+^{\circ,s} B_p(\mathbb{B}^d)} \inf_{R_n\in\mathcal{R}_{n,q}(\mathbb{B}^d)} \|x - R_n\|_{L_q(\mathbb{B}^d)}.$$

For any radial function $x \in L_q(\mathbb{B}^d)$ we denote by

$$E\left(x, \mathcal{P}_n\left(\mathbb{B}^d\right)\right)_{L_q(\mathbb{B}^d)} := \inf_{P\in\mathcal{P}_n(\mathbb{B}^d)} \|x - P\|_{L_q(\mathbb{B}^d)}$$

and

$$E\left(x, \mathcal{R}_{n,q}\left(\mathbb{B}^d\right)\right)_{L_q(\mathbb{B}^d)} := \inf_{R\in\mathcal{R}_{n,q}(\mathbb{B}^d)} \|x - R\|_{L_q(\mathbb{B}^d)}$$

the deviations of x in the space $L_q(\mathbb{B}^d)$ from the space $\mathcal{P}_n(\mathbb{B}^d)$, and the manifold $\mathcal{R}_{n,q}(\mathbb{B}^d)$, respectively.

Ridge functions have many applications in various areas of mathematics and its applications. The question of approximation by ridge functions has been intensively investigated in recent years. However, very little is known about the exact orders of best approximation by ridge functions of any nontrivial function classes. The first such result was obtained in [14] for Sobolev classes $W_2^r(\mathbb{B}^d)$, namely,

$$E\left(W_2^r\left(\mathbb{B}^d\right), \mathcal{R}_{n,2}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)} \asymp n^{-r/(d-1)},$$

where for sequences a_n and b_n, $n \geqslant 1$, of positive numbers a_n and b_n we write $a_n \asymp b_n$, $n \geqslant 1$, if there exist constants $0 < c_1 \leqslant c_2$ such that $c_1 \leqslant a_n/b_n \leqslant c_2$, for all $n \geqslant 1$. The interested reader should see related results in [6,15,17,23].

Throughout the paper p' denotes the conjugate of $1 \leqslant p \leqslant \infty$, that is, $1/p + 1/p' = 1$. By $c := c(\alpha, \beta, \ldots, \gamma)$ we denote various constants which depend on the given parameters, but may differ from one another even if they appear in the same line. Finally, let $I := (-1, 1)$ and let $|J|$ denote the length of the interval $J \subset \mathbb{R}$.

We are ready to state the main results.

Theorem 1. *For $s = 1$, if $1 \leqslant q \leqslant p \leqslant \infty$ and $q \neq p/2$, and if $q = p = \infty$, then for $n > 1$,*

$$E\left(\Delta_+^1 B_p(I), \mathcal{P}_n(I)\right)_{L_q(I)} \asymp n^{-\min\{1/q, 2/q-2/p\}},$$

if $s = 1$ and $1 \leqslant q = p/2 < \infty$, then there exist constants $c_* = c_*(p) > 0$ and $c^* = c^*(p)$ such that for $n > 1$,

$$c_* n^{-2/p} \leqslant E\left(\Delta_+^1 B_p(I), \mathcal{P}_n(I)\right)_{L_{p/2}(I)} \leqslant c^* n^{-2/p} (\ln n)^{1/p'},$$

and if $s > 1$ and $1 \leqslant q \leqslant p \leqslant \infty$, then for $n > 1$,

$$E\left(\Delta_+^s B_p(I), \mathcal{P}_n(I)\right)_{L_q(I)} \asymp n^{-2/q+2/p}.$$

The reader may find it interesting to compare the results of Theorem 1 with earlier estimates of the widths of classes of s-monotone functions (see [5,9–11]).

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For the classes of s-monotone radial functions we have,

Theorem 2. *Let $d > 1$ and $n > 1$. For $s = 1$, $1 \leqslant q \leqslant p \leqslant \infty$ and $q \neq p/2$, and if $q = p = \infty$, then*

$$E\left(\Delta_+^{\circ,1} B_p\left(\mathbb{B}^d\right), \mathcal{P}_n\left(\mathbb{B}^d\right)\right)_{L_q(\mathbb{B}^d)} \asymp n^{-\min\{1/q, 2/q-2/p\}},$$

if $s = 1$ and $1 \leqslant q = p/2 < \infty$, then there exist constants $c_* = c_*(d, p) > 0$ and $c^* = c^*(d, p)$ such that

$$c_* n^{-2/p} \leqslant E\left(\Delta_+^{\circ,1} B_p\left(\mathbb{B}^d\right), \mathcal{P}_n\left(\mathbb{B}^d\right)\right)_{L_{p/2}(\mathbb{B}^d)} \leqslant c^* n^{-2/p} (\ln n)^{1/p'},$$

and if $s > 1$ and $1 \leqslant q \leqslant p \leqslant \infty$, then

$$E\left(\Delta_+^{\circ,s} B_p\left(\mathbb{B}^d\right), \mathcal{P}_n\left(\mathbb{B}^d\right)\right)_{L_q(\mathbb{B}^d)} \asymp n^{-2/q+2/p}.$$

It is well known (see, e.g., [20, p. 169]), that for $d > 1$, the space $\mathcal{P}_n\left(\mathbb{B}^d\right)$ can be embedded in the manifold $\mathcal{R}_{cn^{d-1}}\left(\mathbb{B}^d\right)$ where $c = c(d)$. Thus, an immediate consequence of Theorem 2 is,

Corollary 3. *Let $d > 1$ and $n > 1$. For $s = 1$, if $1 \leqslant q \leqslant p \leqslant \infty$ and $q \neq p/2$, and if $q = p = \infty$, then*

$$E\left(\Delta_+^{\circ,1} B_p\left(\mathbb{B}^d\right), \mathcal{R}_n\left(\mathbb{B}^d\right)\right)_{L_q(\mathbb{B}^d)} \leqslant c n^{-\min\{1/(q(d-1)), (2/q-2/p)/(d-1)\}},$$

if $s = 1$, $2 \leqslant p < \infty$, and $q = p/2$, then

$$E\left(\Delta_+^{\circ,1} B_p\left(\mathbb{B}^d\right), \mathcal{R}_n\left(\mathbb{B}^d\right)\right)_{L_{p/2}(\mathbb{B}^d)} \leqslant c n^{-2/(p(d-1))} (\ln n)^{1/(p'(d-1))},$$

and if $s > 1$ and $1 \leqslant q \leqslant p \leqslant \infty$, then

$$E\left(\Delta_+^{\circ,s} B_p\left(\mathbb{B}^d\right), \mathcal{R}_n\left(\mathbb{B}^d\right)\right)_{L_q(\mathbb{B}^d)} \leqslant c n^{-(2/q-2/p)/(d-1)},$$

where $c = c(d, s, p, q) > 0$.

Our next result generalizes to $d > 2$ the corresponding result by Oskolkov [17, Theorem 1], which was obtained for $d = 2$. Its proof is closely related to that of error estimates of optimal cubature formulas, in the sense of Kolmogorov–Nikolskii, for spherical harmonics (see details in [3]). Our proof closely follows Oskolkov's.

Theorem 4. *Let $n, d \in \mathbb{N}$ and $d > 1$. There exist $\bar{c} = \bar{c}(d) > 0$, and integers $\hat{c} = \hat{c}(d)$ and $\check{c} = \check{c}(d)$, such that for any radial function $x \in L_2(\mathbb{B}^d)$,*

$$\bar{c} E\left(x, \mathcal{P}_{\hat{c}n}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)} \leqslant E\left(x, \mathcal{R}_{n^{d-1},2}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)} \leqslant E\left(x, \mathcal{P}_{\check{c}n}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)}.$$

Finally, we show that for $q = 2$, in most cases the estimates of Corollary 3 are exact in order. Since the space $\mathcal{P}_n\left(\mathbb{B}^d\right)$ may be embedded in $\mathcal{R}_{cn^{d-1}}\left(\mathbb{B}^d\right)$, where $c = c(d)$, the following is an immediate consequence of Corollary 3 and Theorem 4. The problem is open for other values of q.

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Theorem 5. *Let $d>1$, $n>1$, and $2\leqslant p\leqslant\infty$. If $s=1$ and $p\neq 4$, then*

$$E\left(\Delta_+^{\circ,1}B_p\left(\mathbb{B}^d\right),\mathcal{R}_{n,2}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)}\asymp n^{-\min\{1/(2(d-1)),(1-2/p)/(d-1)\}}.$$

If $s=1$ and $p=4$ then there exist constants $c_*=c_*(d)$ and $c^*=c^*(d)$ such that

$$c_*n^{-1/(2(d-1))}\leqslant E\left(\Delta_+^{\circ,1}B_4\left(\mathbb{B}^d\right),\mathcal{R}_{n,2}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)}\leqslant c^*n^{-1/(2(d-1))}(\ln n)^{3/(4(d-1))}.$$

If $s>1$, then

$$E\left(\Delta_+^{\circ,s}B_p\left(\mathbb{B}^d\right),\mathcal{R}_{n,2}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)}\asymp n^{-(1-2/p)/(d-1)}.$$

2. Auxiliary lemmas

Let

$$t_{n,i}:=\cos(n+1-i)\pi/(2(n+1)),\quad i=0,\pm 1\ldots,\pm(n+1) \tag{2.1}$$

be a partition of $I:=(-1,1)$, and denote $I_{n,0}:=(t_{n,-1},t_{n,1})$, $I_{n,i}:=\left[t_{n,i},t_{n,i+1}\right)$, $i=1,\ldots,n$, and $I_{n,i}:=(t_{n,i-1},t_{n,i}]$, $i=-1,\ldots,-n$. It is readily seen that

$$c_1(n-|i|+1)/n^2\leqslant|I_{n,i}|\leqslant c_2(n-|i|+1)/n^2,\quad i=0,\pm 1,\ldots,\pm n, \tag{2.2}$$

where $0<c_1<c_2$ are absolute constants, and that

$$|t_{n,i}-t_{n,j}|\leqslant\pi^2(|i-j|)(2(n+1)-i-j)/(8n^2),\quad i,j=0,\pm 1,\ldots,\pm(n+1). \tag{2.3}$$

For the proof of our first lemma see [4, Chapter VII, \$ 4, p. 274].

Lemma 6. *For each $v\geqslant 1$, there exist polynomials $P_{v,n,i}(\cdot)$, $i=0,\pm 1,\ldots,\pm n$, of degree $\leqslant 2v(2n-1)+1$ and a constant $c=c(v)>0$ such that*

$$P_{v,n,-i}(-t)=P_{v,n,i}(t),\quad t\in I,\ \ i=0,1,\ldots,n,$$

$$\sum_{i=-n}^{n}P_{v,n,i}(t)\equiv 1,\quad t\in I, \tag{2.4}$$

and

$$|P_{v,n,i}(t)|\leqslant c\left(|i-j|+1\right)^{-2v+1},\quad t\in I_{n,j},\ \ i,j=0,\pm 1,\ldots,\pm n. \tag{2.5}$$

For $s\geqslant 1$ and $m\in\mathbb{Z}$ let

$$(m)_s:=\begin{cases}\dbinom{m+s-2}{s-1}, & m\geqslant 1,\\ 0, & m<1.\end{cases}$$

The next result is proved in [5, Lemma 2].

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Lemma 7. *Given $a, b \in \mathbb{R}^n$ such that b has nonzero entries. Let $1 \leqslant p \leqslant \infty$ and $M \geqslant 0$, and let*

$$\Omega_{s,n,p}(b) := \left\{ \omega \in \mathbb{R}^n \mid \|\tilde{\omega}\|_{\ell_p} \leqslant M, \quad \tilde{\omega}_i := b_i \sum_{j=1}^{i} (i-j+1)_s \omega_j, \ \ i = 1, \dots, n \right\}.$$

Then,

$$\max_{\omega \in \Omega_{s,n,p}(b)} |\langle a, \omega \rangle| = M \|u\|_{\ell_{p'}}, \quad u_i := b_i^{-1} \sum_{k=0}^{n-i} (-1)^k \binom{s}{k} a_{i+k}, \ \ i = 1, \dots, n,$$

where $1/p + 1/p' = 1$, and where $\langle a, \omega \rangle := \sum_{i=1}^{n} a_i \omega_i$.

Next is a lemma which was proved in [12, Lemma 3].

Lemma 8. *Let $I = (-1, 1)$, $s \geqslant 1$, and $1 \leqslant p \leqslant \infty$. For $x \in \Delta_+^s L_p(I)$, let*

$$\tilde{x}(t) := x(t) - \pi_s(t; x; 0), \quad t \in I,$$

where

$$\pi_s(t; x; 0) := \sum_{k=0}^{s-1} x^{(k)}(0) \frac{t^k}{k!}, \quad t \in I$$

is the Taylor polynomial of x about $t = 0$, and we recall that $x^{(s-1)}(0) := (x_-^{(s-1)}(0) + x_+^{(s-1)}(0))/2$. Then there exists a constant $c = c(s, p)$ such that

$$\|\tilde{x}(\cdot)\|_{L_p(I)} \leqslant c \|x(\cdot)\|_{L_p(I)}.$$

We need some Remez-type inequalities, the first of which is well known (see, e.g., [16, p. 113, Theorem 14].

Lemma 9. *Let $n \geqslant 1$, $1 \leqslant q \leqslant \infty$, $I = (-1, 1)$, and*

$$I_n := \left(-1 + 1/n^2, 1 - 1/n^2\right).$$

Then there exists a constant $\bar{c} = c(q) \geqslant 1$ such that for any polynomial $P_n \in \mathcal{P}_n(I)$ the inequality holds

$$\|P_n\|_{L_q(I)} \leqslant \bar{c} \|P_n\|_{L_q(I_n)}.$$

The next Remez-type inequality is known for $q = \infty$ (see, e.g., [1, p. 414, E21]. We have not found reference for the case $1 \leqslant q < \infty$ so we prove it below.

Lemma 10. *Let $n \geqslant 1$, $1 \leqslant q \leqslant \infty$, $I := (-1, 1)$, and*

$$I_n := (-1/(4n), 1/(4n)).$$

Then there exists $\hat{c} = c(q) > 0$ such that for any polynomial $P_n \in \mathcal{P}_n(I)$ the inequality holds

$$\|P_n\|_{L_q(I)} \leqslant \hat{c} \|P_n\|_{L_q(I \setminus I_n)}.$$

Proof. Lemma 10 is trivial for $n = 1$, so let $n > 1$. Set

$$\tau_{n,i} := \begin{cases} (i-1/2)/n, & i = 1, \dots, n, \\ (i+1/2)/n, & i = -1, \dots, -n, \end{cases}$$

and for $\tau \in \mathbb{R}$, denote by $l_{n,i}(t;\tau)$, $i = \pm 1, \dots, \pm n$, the Lagrange fundamental polynomials of degree $(2n-1)$ the points $\{\tau_{n,i} + \tau\}$, namely,

$$l_{n,i}(t;\tau) = \prod_{j=\pm 1, j\neq i}^{\pm n} \frac{t - \tau_{n,j} - \tau}{\tau_{n,i} - \tau_{n,j}}, \quad i = \pm 1, \dots, \pm n.$$

Evidently, every polynomial P_n of degree $\leqslant n$ may be represented as

$$P_n(t) = \sum_{i=\pm 1}^{\pm n} P_n(\tau_{n,i} + \tau) l_{n,i}(t;\tau), \quad t \in \mathbb{R}.$$

Now

$$\begin{aligned} |l_{n,i}(\tau;\tau)| &= \prod_{j=\pm 1, j\neq i}^{\pm n} \frac{|\tau_{n,j}|}{|\tau_{n,i} - \tau_{n,j}|} \\ &= \frac{1}{i-1/2} \cdot \frac{(1-1/2)\cdots(n-1/2)}{(n+i-1)!} \cdot \frac{(1-1/2)\cdots(n-1/2)}{(n-i)!} \\ &= \frac{1}{2i-1} \cdot \frac{1\cdot 3\cdots(2n-1)}{2^{n-1}(n+i-1)!} \cdot \frac{1\cdot 3\cdots(2n-1)}{2^n(n-i)!}. \end{aligned}$$

Hence, for $1 \leqslant i < n$,

$$\begin{aligned} |l_{n,i}(\tau;\tau)| &\leqslant \frac{1}{2i-1} \cdot \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n\cdots 2(n+i-1)} \cdot \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2(n-i)} \\ &= \frac{1}{2i-1} \cdot \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n} \cdot \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2(n-i)\cdot 2(n+1)\cdots 2(n+i-1)} \\ &\leqslant \frac{1}{2i-1} \cdot \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n} \cdot \frac{3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots 2(n-i)\cdot 2(n-i+1)\cdots 2(n-1)} \\ &= \frac{1}{2i-1} \cdot \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n} \cdot \frac{3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots 2(n-1)} \\ &= \frac{1}{2i-1} \cdot \frac{2n-1}{2n} \prod_{j=1}^{n-1} \left(\frac{2j-1}{2j} \cdot \frac{2j+1}{2j} \right) \\ &= \frac{1}{2i-1} \cdot \frac{2n-1}{2n} \prod_{j=1}^{n-1} \frac{4j^2-1}{4j^2} \leqslant \frac{1}{2i-1}. \end{aligned}$$

Similarly for $i = n$,

$$\begin{aligned} |l_{n,n}(\tau;\tau)| &= \frac{1}{2n-1} \cdot \frac{1\cdot 3\cdots(2n-1)\cdot 1\cdot 3\cdots(2n-1)}{2^{2n-1}(2n-1)!} \\ &\leqslant \frac{1}{2n-1} \cdot \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n} \cdot \frac{3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots 2(n-1)} \\ &= \frac{1}{2n-1} \cdot \frac{2n-1}{2n} \prod_{j=1}^{n-1} \frac{4j^2-1}{4j^2} \leqslant \frac{1}{2n-1}. \end{aligned}$$

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Due to symmetry similar estimates are valid for $i = -1, \dots, -n$. Thus, we conclude that

$$|l_{n,i}(\tau;\tau)| \leqslant 1/|i|, \quad i = \pm 1, \dots, \pm n, \quad \tau \in \mathbb{R}.$$

Therefore,

$$\begin{aligned} |P_n(\tau)| &\leqslant \sum_{i=\pm 1}^{\pm n} |P_n(\tau_{n,i} + \tau)||l_{n,i}(\tau;\tau)| \\ &\leqslant \sum_{i=\pm 1}^{\pm n} |P_n(\tau_{n,i} + \tau)||i|^{-1} \\ &\leqslant \left(\sum_{i=\pm 1}^{\pm n} |P_n(\tau_{n,i} + \tau)|^q\right)^{1/q} \left(\sum_{i=\pm 1}^{\pm n} |i|^{-q'}\right)^{1/q'}, \quad \tau \in \mathbb{R}, \end{aligned}$$

where for the last inequality we have applied Hölder's inequality.

Since $q' > 1$, we have

$$\left(\sum_{i=\pm 1}^{\pm n} |i|^{-q'}\right)^{1/q'} \leqslant \left(2\int_1^\infty \tau^{-q'}\, d\tau\right)^{1/q'} = \left(2(q'-1)\right)^{1/q'} =: \check{c}.$$

Hence,

$$|P_n(\tau)|^q \leqslant \check{c}^q \sum_{i=\pm 1}^{\pm n} |P_n(\tau_{n,i} + \tau)|^q, \quad \tau \in I_n,$$

and integrating both sides of the inequality over $\tau \in I_n$ yields

$$\int_{I_n} |P_n(\tau)|^q\, d\tau \leqslant \check{c}^q \sum_{i=\pm 1}^{\pm n} \int_{\tau_{n,i}-1/(4n)}^{\tau_{n,i}+1/(4n)} |P_n(\tau)|^q\, d\tau.$$

Note that the intervals $(\tau_{n,i} - 1/(4n), \tau_{n,i} + 1/(4n))$, $i = \pm 1, \dots, \pm n$, are piecewise disjoint and are contained in $I \setminus I_n$. Hence,

$$\int_{I_n} |P_n(\tau)|^q\, d\tau \leqslant \check{c}^q \int_{I\setminus I_n} |P_n(\tau)|^q\, d\tau,$$

so that,

$$\|P_n\|_{L_q(I)} \leqslant (\check{c}^q + 1)^{1/q} \|P_n\|_{L_q(I\setminus I_n)} =: \hat{c}\|P_n\|_{L_q(I\setminus I_n)}.$$

This completes the proof. □

We need a simple result for the multivariate case.

Lemma 11. *Let $d \geqslant 1$, $s \geqslant 1$, and $1 \leqslant p \leqslant \infty$. There exist constants $c_* = c_*(d,p) > 0$ and $c^* = c^*(d,p) > 0$ such that for any $x \in \Delta_+^{\circ,s} L_p(\mathbb{B}^d)$ the inequalities hold*

$$c_*(d,p)\|x\|_{L_p(\mathbb{B}^d)} \leqslant \|y\|_{L_p[0,1)} \leqslant c^*(d,p)\|x\|_{L_p(\mathbb{B}^d)}, \tag{2.6}$$

where $x(t) = y(|t|)$, $t \in \mathbb{B}^d$.

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Proof. Using spherical coordinates we have

$$\begin{aligned}\|x\|_{L_p(\mathbb{B}^d)} &= c(d,p)\left(\int_0^1 \left(\tau^{(d-1)/p}|y(\tau)|\right)^p \, d\tau\right)^{1/p}\\ &\leqslant c(d,p)\left(\int_0^1 |y(\tau)|^p \, d\tau\right)^{1/p}\\ &= c(d,p)\|y\|_{L_p[0,1)}.\end{aligned} \tag{2.7}$$

Thus,

$$(c(d,p))^{-1}\,\|x\|_{L_p(\mathbb{B}^d)} \leqslant \|y\|_{L_p[0,1)}.$$

On the other hand,

$$\begin{aligned}\left(\int_0^1 \left(\tau^{(d-1)/p}y(\tau)\right)^p \, d\tau\right)^{1/p} &\geqslant \left(\int_{1/2}^1 \left(\tau^{(d-1)/p}y(\tau)\right)^p \, d\tau\right)^{1/p}\\ &\geqslant 2^{-(d-1)/p}\left(\int_{1/2}^1 (y(\tau))^p \, d\tau\right)^{1/p}.\end{aligned}$$

Keeping in mind that y is nondecreasing on $[0,1)$, we conclude that

$$\left(\int_0^{1/2} (y(\tau))^p \, d\tau\right)^{1/p} \leqslant \left(\int_{1/2}^1 (y(\tau))^p \, d\tau\right)^{1/p},$$

so that

$$\left(\int_0^1 (y(\tau))^p \, d\tau\right)^{1/p} \leqslant 2^{1/p}\left(\int_{1/2}^1 (y(\tau))^p \, d\tau\right)^{1/p}.$$

Hence,

$$\begin{aligned}\left(\int_0^1 (y(\tau))^p \, d\tau\right)^{1/p} &\leqslant 2^{d/p}\left(\int_0^1 \left(\tau^{(d-1)/p}y(\tau)\right)^p \, d\tau\right)^{1/p}\\ &\leqslant 2^{d/p}\,(c(d,p))^{-1}\,\|x\|_{L_p(\mathbb{B}^d)}.\end{aligned}$$

This completes the proof. □

For $d \in \mathbb{N}$, $d > 1$, we denote by $G_{d,n}(t)$, $-1 \leqslant t \leqslant 1$, the Gegenbauer polynomials defined by the generating function (see, e.g., [19, p. 158])

$$(1-2tz+z^2)^{-d/2} =: \sum_{n=0}^{\infty} G_{d,n}(t)z^n, \quad |z| < 1.$$

The Gegenbauer polynomials satisfy the following Rodriguez' formula (see, e.g., [19, p. 158]):

$$G_{d,n}(t) = (-1)^n \alpha_{d,n}(1-t^2)^{-(d-1)/2}\left(\frac{d}{dt}\right)^n (1-t^2)^{n+(d-1)/2},$$

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where

$$\alpha_{d,n} := \frac{(d)_n}{n!2^n(d/2+1/2)_n},$$

$$(d)_0 := 0, \quad (d)_n := d(d+1)\cdots(d+n-1) = \Gamma(d+n)/\Gamma(d).$$

It is well known that deg $G_{d,n} = n$, and the family $\{G_{d,n}\}_{n=0}^{\infty}$ is a complete orthogonal system for the weighted space $L_2(I; w_d)$, where $w_d(t) := (1-t^2)^{(d-1)/2}$, $t \in I := (-1, 1)$. Also,

$$\int_I G_{d,n}^2(t) w_d(t)\,dt = \frac{\pi^{1/2}(d)_n \Gamma(d/2+1/2)}{(n+d/2)n!\Gamma(d/2)} =: v_{d,n}.$$

Thus, if we denote

$$U_{d,n}(t) := v_{d,n}^{-1/2} G_{d,n}(t), \quad t \in [-1, 1],$$

then $\|U_{d,n}\|_{L_2(I,w_d)} = 1$, hence the family $\{U_{d,n}\}_{n=0}^{\infty}$ is a complete orthonormal system for the weighted space $L_2(I; w_d)$. Also $U_{d,n}(-t) = (-1)^n U_{d,n}(t)$, $t \in [-1, 1]$.

The following result is due to Petrushev [19, p. 163], where one may find comments explaining the nature of the decomposition below (see also [3]). Also note that some ideas of the proof of Lemma 12 are based on the paper by Logan and Schepp [13] about reconstruction of a function from its projections. In the paper Logan and Schepp considered the special case of $d = 2$, and the Chebyshev ridge polynomials of the second kind as an orthonormal set in the space $L_2(\mathbb{B}^2)$.

Lemma 12. *If $d \in \mathbb{N}$, $d > 1$, then each function $x \in L_2(\mathbb{B}^d)$ has the unique representation*

$$x = \sum_{n=0}^{\infty} Q_{d,n}(\cdot; x), \tag{2.8}$$

where the convergence is in $L_2(\mathbb{B}^d)$, and

$$Q_{d,n}(t; x) := v_{d,n} \int_{\mathbb{S}^{d-1}} A_{d,n}(\xi; x) U_{d,n}(\xi \cdot t)\,d\xi, \quad t \in \mathbb{B}^d \tag{2.9}$$

with

$$A_{d,n}(\xi; x) := \int_{\mathbb{B}^d} x(\tau) U_{d,n}(\xi \cdot \tau)\,d\tau, \quad \xi \in \mathbb{S}^{d-1} \tag{2.10}$$

and

$$v_{d,n} := \frac{(n+1)_{d-1}}{2(2\pi)^{d-1}}. \tag{2.11}$$

Moreover, the operators $Q_{d,n}(\cdot; x)$, $n \in \mathbb{N}_0$, are orthogonal projectors from $L_2(\mathbb{B}^d)$ onto $\mathcal{P}_n(\mathbb{B}^d) \ominus \mathcal{P}_{n-1}(\mathbb{B}^d)$, and the following Parseval identity holds:

$$\|x\|_{L_2(\mathbb{B}^d)}^2 = \sum_{n=0}^{\infty} \|Q_{d,n}(\cdot; x)\|_{L_2(\mathbb{B}^d)}^2 = \sum_{n=0}^{\infty} v_{d,n} \|A_{d,n}(\cdot; x)\|_{L_2(\mathbb{S}^{d-1})}^2. \tag{2.12}$$

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Let $d \in \mathbb{N}$ and $\mathbb{T}^d := [0, 2\pi)^d$ be the d-dimensional torus. By $\mathcal{T}_n(\mathbb{T}^d)$ we denote the spaces of all (real-valued) trigonometric polynomials

$$T_n(t) = \sum_{|k| \leqslant n} a_k e^{i(k\cdot t)}, \quad t \in \mathbb{T}^d,$$

where $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, $|k| := |k_1| + \cdots + |k_d|$, $a_k \in \mathbb{C}$, and $a_{-k} = \bar{a}_k$.

The following lemma plays an important role in the proof of Theorem 4.

Lemma 13. *Let* $d, n \in \mathbb{N}$. *For each constant* $0 < c_* < 1$ *and every subspace* $\mathcal{T}_* \subseteq \mathcal{T}_n(\mathbb{T}^d)$ *such that* $\dim \mathcal{T}_* \geqslant c_* \dim \mathcal{T}_n(\mathbb{T}^d)$, *there exists a trigonometric polynomial* $T_* \in \mathcal{T}_*$ *such that*

$$\|T_*\|_{L_\infty(\mathbb{T}^d)} = 1 \quad \textit{and} \quad \|T_*\|_{L_2(\mathbb{T}^d)} \geqslant c^*,$$

where $0 < c^* = c^*(d, c_*) < 1$.

Proof. The proof is based on estimating of volumes of sets for Fourier coefficients of bounded trigonometric polynomials, and can be found in [24] (see also [25, Chapter 2, Section 1]). Note that the first result for $d = 1$ about estimating of volumes of sets for Fourier coefficients is due to Kashin [8,7]. □

3. Proof of Theorem 1—the upper bounds

Proof. For $1 \leqslant q = p \leqslant \infty$ the upper bounds are trivial, because any $x \in \Delta_+^s B_p(I)$ is approximated at this order by the polynomial $P_n(t) \equiv 0$. Thus we assume that $1 \leqslant q < p \leqslant \infty$.

We fix $s \geqslant 1$, $n \geqslant 1$, and for the sake of simplicity, we omit them in our notations whenever it is obvious which s and n apply. Let $I := (-1, 1)$, and let $t_i := t_{n,i}$, $i = 0, \pm 1, \ldots, \pm n$, be defined by (2.1). Given $x \in \Delta_+^s B_p(I)$, denote

$$\pi_i(t; x) := \pi_{s,i}(t; x) := \sum_{k=0}^{s-1} x^{(k)}(t_i) \frac{(t - t_i)^k}{k!}, \quad t \in I, \ \ i = 0, \pm 1, \ldots, \pm n.$$

We fix $v > (3s - 1/q')/2$ and use the polynomials obtained in Lemma 6, to set

$$P_n(t; x) := P_{s,n}(t; x) := \sum_{i=-n}^{n} \pi_{s,i}(t; x) P_{v,n,i}(t), \quad t \in I.$$

By virtue of (2.4),

$$x(t) - P_n(t; x) = \sum_{i=-n}^{n} (x(t) - \pi_i(t; x))\, P_{v,n,i}(t), \quad t \in I.$$

Hence,

$$|x(t) - P_n(t; x)| \leqslant \sum_{i=-n}^{n} |x(t) - \pi_i(t; x)||P_{v,n,i}(t)|, \quad t \in I. \tag{3.1}$$

We first assume that

$$x^{(k)}(0) = 0, \quad k = 0, \ldots, s - 1, \tag{3.2}$$

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which in turn implies $x^{(k)}(t) \geqslant 0$, $k = 0, \ldots, s-1$, $t \in [0,1)$, and $(-1)^{s-k} x^{(k)}(t) \geqslant 0$, $k = 0, \ldots, s-1$, $t \in (-1, 0]$.

Fix $0 \leqslant j < n$, and let $t \in I_j \cap [0,1)$, where $I_j := I_{n,j}$ is defined after (2.1). For $s = 1$, and for $i = 0, \pm 1, \ldots, \pm n$, if $i \leqslant j$, then

$$|x(t) - \pi_{1,i}(t;x)| \leqslant \left| x(t_{j+1}) - x(t_i) \right| = \sum_{k=i}^{j} |x(t_{k+1}) - x(t_k)|, \tag{3.3}$$

since x is nondecreasing on $(-1,1)$. If $i > j$, then for the same reason,

$$|x(t) - \pi_{1,i}(t;x)| \leqslant \left| x(t_j) - x(t_i) \right| = \sum_{k=j}^{i-1} |x(t_{k+1}) - x(t_k)|. \tag{3.4}$$

For $s > 1$, by the Taylor remainder formula,

$$x(t) - \pi_{s,i}(t;x) = \frac{1}{(s-2)!} \int_{t_i}^{t} \left(x^{(s-1)}(\tau) - x^{(s-1)}(t_i) \right) (t-\tau)^{s-2}\, d\tau.$$

Again for $i = 0, \pm 1, \ldots, \pm n$, if $i \leqslant j$,

$$\begin{aligned} |x(t) - \pi_{s,i}(t;x)| &\leqslant \frac{1}{(s-1)!} \left| x^{(s-1)}(t_{j+1}) - x^{(s-1)}(t_i) \right| |t_{j+1} - t_i|^{s-1} \\ &= \frac{1}{(s-1)!} |t_{j+1} - t_i|^{s-1} \sum_{k=i}^{j} \left| x^{(s-1)}(t_{k+1}) - x^{(s-1)}(t_k) \right|, \end{aligned} \tag{3.5}$$

since $x^{(s-1)}$ is nondecreasing on $(-1,1)$. And if $i > j$, then for the same reason

$$\begin{aligned} |x(t) - \pi_{s,i}(t;x)| &\leqslant \frac{1}{(s-1)!} \left| x^{(s-1)}(t_j) - x^{(s-1)}(t_i) \right| |t_j - t_i|^{s-1} \\ &= \frac{1}{(s-1)!} |t_j - t_i|^{s-1} \sum_{k=j}^{i-1} \left| x^{(s-1)}(t_{k+1}) - x^{(s-1)}(t_k) \right|. \end{aligned} \tag{3.6}$$

Put

$$\omega_{s,k} := \begin{cases} \left| x^{(s-1)}(t_{k+1}) - x^{(s-1)}(t_k) \right|, & k = 1, \ldots, n-1, \\ \left| x^{(s-1)}(t_{-1}) \right| + \left| x^{(s-1)}(t_1) \right|, & k = 0, \\ \left| x^{(s-1)}(t_{k-1}) - x^{(s-1)}(t_k) \right|, & k = -1, \ldots, -n+1. \end{cases}$$

Since $j \geqslant 0$, we have for every $i = 0, \pm 1, \ldots, \pm n$,

$$\begin{aligned} 0 \leqslant 2n + 2 - i - j &= 2(n - j + 1) - (i - j) \\ &\leqslant 2(n - j + 1) + 2|i - j| \\ &\leqslant 2(n - j + 1)(|i - j| + 1). \end{aligned}$$

Hence, combining (3.3)–(3.6) with (2.3) and (2.5), for every $s \geqslant 1$, we obtain for $0 \leqslant i \leqslant j$

$$\begin{aligned} &|x(t) - \pi_i(t;x)||P_{v,n,i}| \\ &\quad \leqslant c n^{-2s+2} (|i-j|+1)^{s-2v} (2n + 2 - i - j)^{s-1} \sum_{k=i}^{j} \omega_{s,k} \\ &\quad \leqslant c n^{-2s+2} (|i-j|+1)^{2s-2v-1} (n - j + 1)^{s-1} \sum_{k=i}^{j} \omega_{s,k}, \end{aligned} \tag{3.7}$$

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for $-n \leqslant i < 0$,

$$\begin{aligned} &|x(t) - \pi_i(t; x)||P_{v,n,i}| \\ &\quad \leqslant cn^{-2s+2}(|i-j|+1)^{2s-2v-1}(n-j+1)^{s-1}\sum_{k=i+1}^{j}\omega_{s,k}, \end{aligned} \tag{3.8}$$

and finally for $j < i$,

$$\begin{aligned} &|x(t) - \pi_i(t; x)||P_{v,n,i}| \\ &\quad \leqslant cn^{-2s+2}(|i-j|+1)^{s-2v+1}(2n+2-i-j)^{s-1}\sum_{k=j}^{i-1}\omega_{s,k} \\ &\quad \leqslant cn^{-2s+2}(|i-j|+1)^{2s-2v-1}(n-j+1)^{s-1}\sum_{k=j}^{i-1}\omega_{s,k}. \end{aligned} \tag{3.9}$$

Clearly, similar estimates hold for $-n < j < 0$, and $t \in I_j \cap (-1, 0]$. Therefore by (2.4), we summarize that for each $-n < j < n$ and $t \in I_j$,

$$\begin{aligned} &|x(t) - P_n(t; x)| \\ &\quad \leqslant cn^{-2s+2}\sum_{i=-n}^{n}(|i-j|+1)^{2s-2v-1}(n-|j|+1)^{s-1}\sum_{k=k_0(i,j)}^{k_1(i,j)}\omega_{s,k}, \end{aligned}$$

where we denote

$$k_0(i,j) := \begin{cases} i, & 0 \leqslant i < j \leqslant n, 0 \leqslant i = j < n, \\ i+1, & -n \leqslant i < 0, i < j \leqslant n, \\ j, & 0 \leqslant j < i \leqslant n, \\ j+1, & -n < j < 0, j \leqslant i \leqslant n, \end{cases} \tag{3.10}$$

and

$$k_1(i,j) := \begin{cases} j, & -n \leqslant i < j, 0 \leqslant j < n, \\ j+1, & -n \leqslant i < j < 0, \\ i-1, & -n \leqslant j \leqslant i, 0 \leqslant i \leqslant n, \\ i, & -n \leqslant j \leqslant i < 0. \end{cases} \tag{3.11}$$

Hence, integrating over I_j, $-n < j < n$, yields

$$\begin{aligned} &\|x - P_n(\cdot; x)\|_{L_q(I_j)} \\ &\quad \leqslant cn^{-2s+2}|I_j|^{1/q}\sum_{i=-n}^{n}(|i-j|+1)^{2s-2v-1}(n-|j|+1)^{s-1}\sum_{k=k_0(i,j)}^{k_1(i,j)}\omega_{s,k} \\ &\quad \leqslant cn^{-2s+2/q'}\sum_{i=-n}^{n}(|i-j|+1)^{2s-2v-1}(n-|j|+1)^{s-1/q'}\sum_{k=k_0(i,j)}^{k_1(i,j)}\omega_{s,k}. \end{aligned} \tag{3.12}$$

Finally, we have to consider the case $j = \pm n$. To this end, let $t \in I_n$ and take $i = n$. Then $x(t) \geqslant 0$, and further, if $s = 1$, then $x(t) - \pi_{1,n}(t; x) = x(t) - x(t_n) \geqslant 0$ and if $s > 1$, then

$$x(t) - \pi_{s,n}(t; x) = \frac{1}{(s-2)!}\int_{t_n}^{t}\left(x^{(s-1)}(\tau) - x^{(s-1)}(t_n)\right)(t-\tau)^{s-2}\,d\tau \geqslant 0.$$

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Hence,

$$0 \leqslant x(t) - \pi_{s,n}(t;x) \leqslant x(t),$$

and in turn, for $i < n$,

$$\begin{aligned}|x(t) - \pi_{s,i}(t;x)| &\leqslant |x(t) - \pi_{s,n}(t;x)| + |\pi_{s,n}(t;x) - \pi_{s,i}(t;x)| \\ &\leqslant x(t) + |\pi_{s,n}(t;x) - \pi_{s,i}(t;x)|. \end{aligned} \tag{3.13}$$

So we wish to estimate

$$\pi_{s,n}(t;x) - \pi_{s,i}(t;x) = \sum_{k=0}^{s-1} \left(x^{(k)}(t_n) - \pi_{s-k,i}\left(t_n; x^{(k)}\right)\right) \frac{(t-t_n)^k}{k!}.$$

For $s > 1$ and $k < s-1$, we have

$$x^{(k)}(t_n) - \pi_{s-k,i}\left(t_n; x^{(k)}\right) = \frac{1}{(s-k-2)!} \int_{t_i}^{t_n} \left(x^{(s-1)}(\tau) - x^{(s-1)}(t_i)\right)(t_n - \tau)^{s-k-2}\, d\tau.$$

Therefore, for $i \geqslant 0$,

$$\begin{aligned}
&|\pi_{s,n}(t;x) - \pi_{s,i}(t;x)| \\
&\quad\leqslant \left| \sum_{k=0}^{s-2} \frac{(t-t_n)^k}{k!(s-k-2)!} \int_{t_i}^{t_n} \left(x^{(s-1)}(\tau) - x^{(s-1)}(t_i)\right)(t_n-\tau)^{s-k-2}\, d\tau \right| \\
&\qquad + |x^{(s-1)}(t_n) - x^{(s-1)}(t_i)| \frac{(t-t_n)^{s-1}}{(s-1)!} \\
&\quad\leqslant \left(\sum_{k=0}^{s-1} |t_n - t_i|^{s-k-1} |1-t_n|^k \right) \sum_{k=i}^{n-1} \omega_{s,k} \\
&\quad\leqslant c(n-i+2)^{2s-2} n^{-2s+2} \sum_{k=i}^{n-1} \omega_{s,k},
\end{aligned} \tag{3.14}$$

where we estimated the sum applying (2.3). Similarly for $-n \leqslant i < 0$,

$$\begin{aligned}|\pi_{s,n}(t;x) - \pi_{s,i}(t;x)| &\leqslant \left(\sum_{k=0}^{s-1} |t_n - t_i|^{s-k-1}|1-t_n|^k\right) \sum_{k=i+1}^{n-1} \omega_{s,k} \\ &\leqslant c(n-i+2)^{2s-2} n^{-2s+2} \sum_{k=i+1}^{n-1} \omega_{s,k}. \end{aligned} \tag{3.15}$$

Note that (3.14) and (3.15) trivially hold for $s = 1$.

Substituting (3.14), (3.15) into (3.13) and combining with (2.5) we get

$$|x(t) - \pi_{s,n}(t;x)||P_{v,n,n}(t)| \leqslant c|x(t)|,$$

and for $0 \leqslant i < n$,

$$\begin{aligned}|x(t) - \pi_{s,i}(t;x)||P_{v,n,i}(t)| &\leqslant c|x(t)|(n-i+1)^{-2v+1} \\ &\quad + cn^{-2s+2}(n-i+1)^{2s-2v-1} \sum_{k=i}^{n-1} \omega_{s,k}.\end{aligned}$$

Finally, for $-n \leqslant i < 0$,

$$|x(t) - \pi_{s,i}(t;x)||P_{v,n,i}(t)| \leqslant c|x(t)|(n-i+1)^{-2v+1} + cn^{-2s+2}(n-i+1)^{2s-2v-1} \sum_{k=i+1}^{n-1} \omega_{s,k}(x).$$

Similar inequalities are valid for $t \in I_{-n}$. Hence, for $t \in I_n$, we obtain by (3.1) that

$$|x(t) - P_n(t;x)| \leqslant c|x(t)| \sum_{i=-n}^{n} (n-i+1)^{-2v+1} + cn^{-2s+2} \sum_{i=-n}^{n-1} (n-i+1)^{2s-2v-1} \sum_{k=k_0(i,n)}^{n-1} \omega_{s,k}, \tag{3.16}$$

and similarly for $t \in I_{-n}$,

$$|x(t) - P_n(t;x)| \leqslant c|x(t)| \sum_{i=-n}^{n} (n+i+1)^{-2v+1} + cn^{-2s+2} \sum_{i=-n+1}^{n} (n+i+1)^{2s-2v-1} \sum_{k=-n+1}^{k_1(i,-n)} \omega_{s,k}. \tag{3.17}$$

Since $2v - 1 > 1$,

$$\sum_{i=-n}^{n} (|n-i|+1)^{-2v+1} \leqslant 2 \int_1^\infty \tau^{-2v+1}\, d\tau = (v-1)^{-1} = c.$$

We conclude from (3.16) that for $t \in I_n$,

$$|x(t) - P_n(t;x)| \leqslant \check{c} n^{-2s+2} \sum_{i=-n}^{n-1} (n-i+1)^{2s-2v-1} \sum_{k=k_0(i,n)}^{n-1} \omega_{s,k} + c|x(t)|, \tag{3.18}$$

and by (3.17), we have for $t \in I_{-n}$,

$$|x(t) - P_n(t;x)| \leqslant cn^{-2s+2} \sum_{i=-n+1}^{n} (n+i+1)^{2s-2v-1} \sum_{k=-n+1}^{k_1(i,-n)} \omega_{s,k} + c|x(t)|. \tag{3.19}$$

Now, integrating (3.18) over I_n yields

$$\|x - P_n(\cdot;x)\|_{L_q(I_n)} \leqslant \check{c}|I_n|^{1/q} n^{-2s+2} \sum_{i=-n}^{n-1} (n-i+1)^{2s-2v-1} \sum_{k=k_0(i,n)}^{n-1} \omega_{s,k} + c|I_n|^{1/q-1/p} \|x\|_{L_p(I_n)}, \tag{3.20}$$

where we applied the inequalities

$$(|a|^q + |b|^q)^{1/q} \leqslant |a| + |b| \leqslant 2^{1/q'}(|a|^q + |b|^q)^{1/q},$$

and for the last term we used Hölder's inequality.

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Similarly, integrating (3.19) over I_{-n} yields

$$\|x - P_n(\cdot; x)\|_{L_q(I_{-n})} \leqslant \check{c}|I_{-n}|^{1/q} n^{-2s+2} \sum_{i=-n+1}^{n} (n+i+1)^{2s-2v-1} \sum_{k=-n+1}^{k_1(i,-n)} \omega_{s,k} + c|I_{-n}|^{1/q-1/p} \|x\|_{L_p(I_n)}. \tag{3.21}$$

We combine now (3.12) with (3.20) and (3.21), to obtain

$$\|x - P_n(\cdot; x)\|_{L_q(I)} \leqslant cn^{-2s+2/q'} \sum_{i=-n}^{n} \sum_{j=-n}^{n} (|i-j|+1)^{2s-2v-1} \times (n-|j|+1)^{s-1/q'} \sum_{k=k_0(i,j)}^{k_1(i,j)} \omega_{s,k} + cn^{-2/q+2/p} \|x\|_{L_p(I)}, \tag{3.22}$$

where $k_0(i, j)$ and $k_1(i, j)$ where defined in (3.10) and (3.11) for all pairs i, j except for $i = j = \pm n$, where we put $k_0(n, n) = k_0(-n, -n) = 1$ and $k_1(n, n) = k_1(-n, -n) = -1$, so that it is an empty set and thus $= 0$.

Note that for $k_0(i, j) \leqslant k \leqslant k_1(i, j)$,

$$\begin{aligned} n - |j| + 1 &= (n - |k| + 1) + |k| - |j| \leqslant (n - |k| + 1) + |k - j| \\ &\leqslant (n - |k| + 1) + (|i - j| + 1) \leqslant 2(|i - j| + 1)(n - |k| + 1). \end{aligned}$$

We first estimate the inner sum dealing with the summation on j from $i \geqslant 0$ to $n - 1$. By (3.10) and (3.11), we deal with

$$\begin{aligned} &\sum_{j=i}^{n-1} (|i-j|+1)^{2s-2v-1} (n-|j|+1)^{s-1/q'} \sum_{k=i}^{j} \omega_{s,k} \\ &\leqslant 2^{s-1/q'} \sum_{j=i}^{n-1} \sum_{k=i}^{j} (|i-j|+1)^{3s-1/q'-2v-1} (n-|k|+1)^{s-1/q'} \omega_{s,k} \\ &= c \sum_{k=i}^{n-1} \left(\sum_{j=k}^{n-1} (|i-j|+1)^{3s-1/q'-2v-1} \right) (n-|k|+1)^{s-1/q'} \omega_{s,k} \\ &\leqslant c \sum_{k=i}^{n-1} (|i-k|+1)^{3s-1/q'-2v} (n-|k|+1)^{s-1/q'} \omega_{s,k}, \end{aligned}$$

where we used the fact that $3s - 1/q' - 2v - 1 < -1$ to obtain

$$\begin{aligned} \sum_{j=k}^{n-1} (|i-j|+1)^{3s-1/q'-2v-1} &\leqslant \int_k^{\infty} (|i-\tau|+1)^{3s-1/q'-2v-1} \, d\tau \\ &= \int_{|i-k|+1}^{\infty} \tau^{3s-1/q'-2v-1} d\tau \\ &= (2v + 1/q' - 3s)^{-1} (|i-k|+1)^{3s-1/q'-2v} \\ &= c(|i-k|+1)^{3s-1/q'-2v}. \end{aligned}$$

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The other part of the inner sum is dealt similarly. Thus, substituting in (3.22), we conclude that

$$\begin{aligned}\|x - P_n(\cdot;x)\|_{L_q(I)} &\leqslant cn^{-2s+2/q'} \sum_{i=-n+1}^{n-1} \sum_{k=-n+1}^{n-1} (|k-i|+1)^{3s-1/q'-2v} \\ &\quad \times (n-|k|+1)^{s-1/q'} \omega_{s,k} + cn^{-2/q+2/p} \|x\|_{L_p(I)} \\ &= cn^{-2s+2/q'} \sum_{k=-n+1}^{n-1} \left(\sum_{i=-n+1}^{n-1} (|k-i|+1)^{3s-1/q'-2v} \right) \\ &\quad \times (n-|k|+1)^{s-1/q'} \omega_{s,k} + cn^{-2/q+2/p} \|x\|_{L_p(I)} \\ &\leqslant cn^{-2s+2/q'} \sum_{k=-n+1}^{n-1} (n-|k|+1)^{s-1/q'} \omega_{s,k} + cn^{-2/q+2/p} \|x\|_{L_p(I)},\end{aligned} \tag{3.23}$$

where again, we used the fact that $3s - 1/q' - 2v < -1$ to obtain

$$\begin{aligned}\sum_{i=-n+1}^{n-1} (|k-i|+1)^{3s-1/q'-2v} &\leqslant \sum_{i=k}^{n-1} (|k-i|+1)^{3s-1/q'-2v} \\ &\quad + \sum_{i=-n+1}^{k} (|k-i|+1)^{3s-1/q'-2v} \\ &\leqslant 2 \int_1^\infty \tau^{3s-1/q'-2v}\, d\tau \\ &= (3s + 1/q - 2v)^{-1} = c.\end{aligned}$$

Therefore we should estimate how big may the sum on the right-hand side of (3.23) be, when the only constraints on the collection $\{\omega_{s,k}\}$, $-n+1 \leqslant k \leqslant n-1$, is that $x \in \Delta_+^s B_p(I)$.

To this end we set

$$\omega_{s,0}^+ := |x^{(s-1)}(t_1)|, \quad \omega_{s,i}^+ := \omega_{s,i}, \quad i = 1, \ldots, n-1,$$

and

$$\omega_{s,0}^- := |x^{(s-1)}(t_{-1})|, \quad \omega_{s,i}^- := \omega_{s,i}, \quad i = -1, \ldots, -n+1,$$

and we will estimate the two sums separately. In order to estimate the first sum we write $I^+ := [0, 1)$, and we will estimate from below the values $\|x\|_{L_p(I^+)}$. Let $I_0^+ := [t_0, t_1]$ and put $I_i^+ := I_i$, $i = 1, \ldots, n$. We take $1 \leqslant p < \infty$, the case $p = \infty$ is analogous. Then

$$\|x(\cdot)\|_{L_p(I^+)}^p = \sum_{i=0}^{n} \|x(\cdot)\|_{L_p(I_i^+)}^p. \tag{3.24}$$

By virtue of (3.2), $x(t) \geqslant 0$ and $\pi_{s,0}(t;x) \equiv 0$, for $t \in I^+$, so that for $t \in I_i^+$, $i = 0, \ldots, n$,

$$x(t) = (x(t) - \pi_i(t;x)) + \sum_{j=1}^{i} \left(\pi_j(t;x) - \pi_{j-1}(t;x) \right),$$

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where for $i = 0$, the second sum is empty, thus $= 0$. For $s > 1$, applying the Taylor remainder formula, we get

$$x(t) - \pi_{s,i}(t; x) = \frac{1}{(s-2)!} \int_{t_i}^{t} \left(x^{(s-1)}(\tau) - x^{(s-1)}(t_i) \right) (t - \tau)^{s-2}\, d\tau \geqslant 0,$$

as $x^{(s-1)}$ is nondecreasing. Hence, we proceed with $i \geqslant 1$, and obtain

$$x(t) \geqslant \sum_{j=1}^{i} \left(\pi_{s,j}(t; x) - \pi_{s,j-1}(t; x) \right), \quad t \in I_i^+, \ \ i = 1, \ldots, n, \tag{3.25}$$

which is clearly valid for $s = 1$. Since $\pi_{s,j}^{(k)}(t; x) = \pi_{s-k,j}(t; x^{(k)})$, $k = 0, \ldots, s-1$, it follows that

$$\pi_{s,j}(t; x) - \pi_{s,j-1}(t; x) = \sum_{k=0}^{s-1} \left(x^{(k)}(t_j) - \pi_{s-k,j-1}\left(t_j; x^{(k)}\right) \right) \frac{(t - t_j)^k}{k!}.$$

Again, if $s > 1$ and $k \leqslant s-2$, then, applying the Taylor remainder formula, we get by the monotonicity of $x^{(s-1)}$,

$$\begin{aligned} x^{(k)}(t_j) - \pi_{s-k,j-1}\left(t_j; x^{(k)}\right) \\ = \frac{1}{(s-2-k)!} \int_{t_{j-1}}^{t_j} \left(x^{(s-1)}(\tau) - x^{(s-1)}(t_{j-1}) \right) (t - \tau)^{s-2-k}\, d\tau \geqslant 0. \end{aligned} \tag{3.26}$$

Hence,

$$\left(x^{(k)}(t_j) - \pi_{s-k,j-1}\left(t_j; x^{(k)}\right) \right) \frac{(t - t_j)^k}{k!} \geqslant 0, \quad t \in I_i, \ \ 1 \leqslant j \leqslant i,$$

so that it follows from (3.26) that

$$\begin{aligned} \pi_{s,j}(t; x) - \pi_{s,j-1}(t; x) &\geqslant \left(x^{(s-1)}(t_j) - \pi_{1,j-1}(t_j; x^{(s-1)}) \right) \frac{(t - t_j)^{s-1}}{(s-1)!} \\ &\geqslant \left(x^{(s-1)}(t_j) - x^{(s-1)}(t_{j-1}) \right) \frac{(t - t_j)^{s-1}}{(s-1)!} \\ &\geqslant \frac{(t - t_j)^{s-1}}{(s-1)!} \omega_{s,j-1}^{+}, \quad t \in I_i, \ \ 1 \leqslant j \leqslant i. \end{aligned} \tag{3.27}$$

Note that (3.27) is valid for $s = 1$, in fact with an equality sign in that case. Denote $\bar{t}_i := (t_i + t_{i+1})/2$. Then for $1 \leqslant j \leqslant i \leqslant n-1$ the inequalities hold

$$t - t_j \geqslant \frac{1}{2} \sum_{k=j}^{i} |I_k^+|, \quad t \in [\bar{t}_i, t_{i+1}].$$

Combining with (3.27) and substituting in (3.25), we get

$$x(t) \geqslant c \sum_{j=1}^{i} \left(\sum_{k=j}^{i} |I_k^+| \right)^{s-1} \omega_{s,j-1}^{+}, \quad t \in [\bar{t}_i, t_{i+1}),$$

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whence for $1 \leqslant p < \infty$,

$$\|x\|_{L_p(I_i^+)} \geqslant \|x|_{L_p[\bar{t}_i, t_{i+1}]} \geqslant c|I_i^+|^{1/p} \sum_{j=1}^{i} \left(\sum_{k=j}^{i} |I_k^+| \right)^{s-1} \omega_{s,j-1}^+.$$

Therefore,

$$\begin{aligned} \|x\|_{L_p(I^+)}^p &\geqslant c \sum_{i=1}^{n} \left|I_i^+\right| \left(\sum_{j=1}^{i} \left(\sum_{k=j}^{i} |I_k^+| \right)^{s-1} \omega_{s,j-1}^+ \right)^p \\ &\geqslant c \sum_{i=1}^{n} \left|I_i^+\right|^{(s-1)p+1} \left(\sum_{j=1}^{i} (i-j+1)^{s-1} \omega_{s,j-1}^+ \right)^p \\ &\geqslant \sum_{i=1}^{n} \left(\frac{c(n-i+1)^{s-1/p'}}{n^{2s-2/p'}} \sum_{j=1}^{i} (i-j+1)_s \omega_{s,j-1}^+ \right)^p, \end{aligned} \tag{3.28}$$

where for the second inequality we used the fact that

$$\left(\sum_{k=j}^{i} |I_k^+| \right)^{s-1} \geqslant (i-j+1)^{s-1} |I_i^+|^{s-1}, \quad 1 \leqslant j \leqslant i \leqslant n,$$

which readily follows from $|I_1^+| \geqslant |I_2^+| \geqslant \cdots \geqslant |I_n^+|$, and we applied $(m)_s \asymp m^{s-1}$, and the third inequality follows by (2.2).

Going back to (3.23), but limiting for a moment the discussion to I_+, we see that we should consider the extremal problem

$$\sum_{i=1}^{n} (n-i+2)^{s-1/q'} n^{-2s+2/q'} \omega_{s,i-1}^+ \to \sup, \tag{3.29}$$

with $\omega_{s,i}^+ \geqslant 0$, $i = 0, \ldots, n-1$, satisfying

$$\left(\sum_{i=1}^{n} \left(\frac{(n-i+1)^{s-1/p'}}{n^{2s-2/p'}} \sum_{j=1}^{i} (i-j+1)_s \omega_{s,j-1}^+ \right)^p \right)^{1/p} \leqslant \|x\|_{L_p(I^+)}, \tag{3.30}$$

and for $p = \infty$, a similar, appropriate inequality. We thus apply Lemma 7 for $1 \leqslant p \leqslant \infty$, with $\omega := (\omega_{s,0}^+, \ldots, \omega_{s,n-1}^+)$,

$$a_i := ((n-i+2)/n^2)^{s-1/q'}, \quad i = 1, \ldots, n$$

and

$$b_i := ((n-i+1)/n^2)^{s-1/p'}, \quad i = 1, \ldots, n.$$

That is, we have to estimate the $l_{p'}$-norm of $u := (u_1, \ldots, u_n)$ where

$$u_i := \left(n^2/(n-i+1)\right)^{s-1/p'} \sum_{k=0}^{n-i} (-1)^k \frac{s}{k} \left((n-i-k+2)/n^2\right)^{s-1/q'},$$

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and we note that for $i = 1, \ldots, n-s$,

$$\begin{aligned} u_i &= n^{2/p-2/q}(n-i+1)^{-s+1/p'} \sum_{k=0}^{s} (-1)^k \frac{s}{k} (n-i-k+2)^{s-1/q'} \\ &=: n^{2/p-2/q}(n-i+1)^{-s+1/p'} \Delta_{-1}^{s}(n-i+2)^{s-1/q'}, \end{aligned}$$

where we note that Δ_{-1}^{s} is the sth difference with the step $h = -1$. Now

$$\begin{aligned} \left|\Delta_{-1}^{s}(n-i+2)^{s-1/q'}\right| &= c\left|\int_0^1 \cdots \int_0^1 (n-i+2-\tau_1-\cdots-\tau_s)^{-1/q'}\, d\tau_1 \cdots d\tau_s\right| \\ &\leqslant c(n-i+2-s)^{-1/q'} \\ &= c(1-s/(n-i+2))^{-1/q'}(n-i+2)^{-1/q'} \\ &\leqslant c(1-s/(s+1))^{-1/q'}(n-i+2)^{-1/q'} \\ &= c(s+1)^{1/q'}(n-i+2)^{-1/q'}, \end{aligned}$$

so that for $1 < p \leqslant \infty$,

$$\begin{aligned} &\left(\sum_{i=1}^{n-s}\left(\left|\Delta_{-1}^{s}(n-i+2)^{s-1/q'}\right|(n-i+1)^{-s+1/p'}\right)^{p'}\right)^{1/p'} \\ &\quad\leqslant c\left(\sum_{i=1}^{n-s}(n-i+1)^{-(s-1/q+1/p)p'}\right)^{1/p'} \\ &\quad\leqslant c\left(\sum_{i=1}^{n} i^{-(s-1/q+1/p)p'}\right)^{1/p'}, \end{aligned} \tag{3.31}$$

where $c = c(s, q)$. In order to estimate the sum on the right-hand side of (3.31) we have to separate to various cases of s, p and q.

For $s = 1$ and $1 \leqslant q < p/2 \leqslant \infty$, it follows that $(s - 1/q + 1/p)p' = (1 - 1/q + 1/p)p' < 1$, so that

$$\left(\sum_{i=1}^{n} i^{-(1-1/q+1/p)p'}\right)^{1/p'} \leqslant c n^{1/q-2/p}, \quad 1 \leqslant q < p/2 \leqslant \infty. \tag{3.32}$$

If $1 \leqslant q = p/2 \leqslant \infty$, we have $(1 - 1/p)p' = p'/p' = 1$, so that

$$\left(\sum_{i=1}^{n} i^{-(1-1/p)p'}\right)^{1/p'} \leqslant c(\ln n)^{1/p'}, \quad 1 \leqslant q = p/2 \leqslant \infty. \tag{3.33}$$

And if $p/2 < q \leqslant p \leqslant \infty$ (note that this excludes $q = p = \infty$), we have $(1 - 1/q + 1/p)p' > 1$, so that

$$\left(\sum_{i=1}^{n} i^{-(1-1/q+1/p)p'}\right)^{1/p'} \leqslant c, \quad p/2 < q \leqslant p \leqslant \infty, \tag{3.34}$$

where, in all the above cases, $c = c(p, q)$.

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For $s>1$ and $1\leqslant q\leqslant p\leqslant\infty$, we have $(s-1/q+1/p)p'>1$, except when $s=2$, $q=1$ and $p=\infty$, so that

$$\left(\sum_{i=1}^{n} i^{-(s-1/q+1/p)p'}\right)^{1/p'} \leqslant \begin{cases} c\ln n, & s=2,\ q=1,\ p=\infty,\\ c & \text{all other cases of } s>1,\ 1\leqslant q\leqslant p\leqslant\infty, \end{cases} \tag{3.35}$$

where $c=c(s,p,q)$. Note that (3.34) and (3.35) are valid also for $p=q=1$, where the left-hand side is understood as the sup-norm. Thus, for a moment, we separate the case $s=2$, $q=1$, and $p=\infty$, and conclude that for all other cases it follows by (3.31)–(3.35) that

$$\left(\sum_{i=1}^{n-s} |u_i|^{p'}\right)^{1/p'} \leqslant \begin{cases} cn^{-1/q}, & s=1,\ 1\leqslant q<p/2\leqslant\infty,\\ cn^{-2/p}(\ln n)^{1/p'}, & s=1,\ 1\leqslant q=p/2\leqslant\infty,\\ cn^{-2/q+2/p}, & s=1,\ p/2<q\leqslant p\leqslant\infty,\\ cn^{-2/q+2/p}, & s>1,\ 1\leqslant q\leqslant p\leqslant\infty, \end{cases} \tag{3.36}$$

where $c=c(s,p,q)$, and that the last two inequalities in (3.36) are valid also for $p=q=1$, where the left-hand side is understood as the sup-norm.

For $i=n+1-s,\dots,n$ we take the crudest estimate

$$|u_i|\leqslant n^{2/p-2/q}(n-i+1)^{-s+1/p'}2^s(n-i+2)^{s-1/q'},$$

and we get

$$\begin{aligned}\left(\sum_{i=n+1-s}^{n} |u_i|^{p'}\right)^{1/p'} &\leqslant cn^{2/p-2/q}\left(\sum_{i=n+1-s}^{n}(n-i+1)^{(1/q-1/p)p'}\right)^{1/p'}\\ &\leqslant cn^{2/p-2/q},\end{aligned} \tag{3.37}$$

where $c=c(s,p,q)$.

Therefore, combining (3.36) and (3.37), we obtain by virtue of Lemma 7,

$$\begin{aligned} &n^{-2s+2/q'}\sum_{i=1}^{n}(n-i+2)^{s-1/q'}\omega^+_{s,i-1}\\ &\leqslant c\|x(\cdot)\|_{L_p(I^+)}\begin{cases} n^{-\min\{1/q,2/q-2/p\}}, & s=1,\ 1\leqslant q<p/2\leqslant\infty,\\ n^{-2/p}(\ln n)^{1/p'}, & s=1,\ 1\leqslant q=p/2\leqslant\infty,\\ n^{-2/q+2/p}, & s=1,\ p/2<q\leqslant p\leqslant\infty,\\ n^{-2/q+2/p}, & s>1,\ 1\leqslant q\leqslant p\leqslant\infty, \end{cases}\end{aligned} \tag{3.38}$$

where $c=c(s,p,q)$, and again, the last two inequalities in (3.38) are valid also for $p=q=1$, where the left-hand side is understood as the sup-norm.

Recall that (3.38) is yet to be established for the case $s=2$, $q=1$, and $p=\infty$. However, we observe that if $x\in\Delta^2_+L_\infty(I)$, $x'\in\Delta^1_+L_1(I)$ and $\|x'\|_{L_1(I^+)}=\|x\|_{L_\infty(I^+)}$. Furthermore, $\omega^+_{1,i}(x')=\omega^+_{2,i}(x)$, $i=0,\dots,n-1$. Hence, we know by the above proof that

$$n^{-2}\sum_{i=1}^{n}(n-i+1)\sum_{j=1}^{i}(i-j+1)_1\omega^+_{2,j-1}\leqslant c\|x'\|_{L_1(I^+)},$$

and we wish to estimate

$$n^{-4}\sum_{i=1}^{n}(n-i+2)^2\omega^+_{2,i-1}\to\sup.$$

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Thus, we apply Lemma 7 with $p = 1$,

$$a_i := (n - i + 2)^2/n^4, \quad i = 1, \dots, n,$$

$$b_i := (n - i + 1)/n^2, \quad i = 1, \dots, n,$$

and

$$\omega_i := \omega^+_{2,j-1}, \quad i = 1, \dots, n.$$

Therefore, we have to estimate the sup-norm of $u = (u_1, \dots, u_n)$, where

$$u_i := b_i^{-1}(a_{i+1} - a_i) \leqslant cn^{-2},$$

and we conclude that

$$n^{-4} \sum_{i=1}^{n} (n - i + 2)^2 \omega^+_{2,i-1} \leqslant c\|x'\|_{L_1(I^+)} n^{-2}$$
$$= c\|x\|_{L_\infty(I^+)} n^{-2}.$$

This is (3.38) for $s = 2$, $q = 1$, and $p = \infty$.

Similar estimates for the interval I^-.

We thus have established Theorem 1 for functions $x \in \Delta^s_+ L_p(I)$ which satisfy the conditions (3.2). In the general case, we consider the function

$$\tilde{x} := x - \pi_s(\cdot; x; 0), \quad t \in I,$$

where we recall that $\pi_s(\cdot; x; 0)$ is the Taylor polynomial of degree $\leqslant s - 1$ about $t_0 := 0$. Clearly, $\tilde{x} \in \Delta^s_+ L_p(I)$ and $\tilde{x}$ satisfies (3.2). Since

$$P_{s,n}(t; \tilde{x}) = P_{s,n}(t; x) - \pi_s(t; x; 0), \quad t \in I,$$

it follows that

$$x(t) - P_{s,n}(t; x) = \tilde{x}(t) - P_{s,n}(t; \tilde{x}), \quad t \in I,$$

and by Lemma 8,

$$\|\tilde{x}\|_{L_p(I)} \leqslant c\|x\|_{L_p(I)},$$

where $c = c(s, p)$. Hence, the upper estimates are valid for all $x \in \Delta^s_+ L_p(I)$.

The degree of the polynomials $P_{s,n}(\cdot; x)$ does not exceed $2\nu(2n - 1) + s$ where $\nu = \nu(s, p, q)$ is fixed. So the proof of the upper bounds of Theorem 1 is complete. □

4. Proof of Theorem 1—the lower bounds

Proof. Given $1 \leqslant q \leqslant p \leqslant \infty$, let $s \geqslant 1$ and $n > 1$. Set

$$\xi_{s,p,n}(t) := \lambda_{s,p,n}(t - t_n)^{s-1}_+, \quad t \in I = (-1, 1), \tag{4.1}$$

where

$$t_n := 1 - 1/(32n^2),$$

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and $\lambda_{s,p,n}$ is such that

$$\|\xi_{s,p,n}\|_{L_p(I)} = 1.$$

Clearly, $\xi_{s,p,n} \in \Delta_+^s B_p(I)$. It follows that

$$\begin{aligned}\lambda_{s,p,n}^{-1} &= \left(\int_{t_n}^{1} (t-t_n)^{(s-1)p}\, dt\right)^{1/p} \\ &= \left(\int_0^{1/(32n^2)} \tau^{(s-1)p}\, dt\right)^{1/p} \\ &= cn^{-2s+2/p'},\end{aligned}$$

where $c = c(s, p)$. Hence

$$\begin{aligned}\|\xi_{s,p,n}\|_{L_q(I)} &= cn^{2s-2/p'} \left(\int_{t_n}^{1} (t-t_n)^{(s-1)q}\, dt\right)^{1/q} \\ &=: c_* n^{-2/q+2/p},\end{aligned}$$

where $c_* := c(s, p, q)$.

Let $I_n := (-1 + 1/(4n)^2, 1 - 1/(4n)^2)$, and let $\bar{c} = \bar{c}(q) \geqslant 1$ be the constant from Lemma 9. Set $c^* := c_*/(2\bar{c})$, and assume to the contrary that there exists a polynomial $P_n \in \mathcal{P}_n(I)$ such that

$$\|\xi_{s,p,n} - P_n\|_{L_q(I)} < c^* n^{-2/q+2/p}. \tag{4.2}$$

Taking into account that $\xi_{s,p,n}(t) \equiv 0$, $t \in I_n$, we see that

$$\|P_n\|_{L_q(I_n)} < c^* n^{-2/q+2/p},$$

so that by virtue of Lemma 9 we obtain,

$$\|P_n\|_{L_q(I)} \leqslant \bar{c}\|P_n\|_{L_q(I_n)} < \bar{c}c^* n^{-2/q+2/p} = (c_*/2)n^{-2/q+2/p}.$$

Hence,

$$\begin{aligned}\|\xi_{s,p,n} - P_n\|_{L_q(I)} &\geqslant \|\xi_{s,p,n}\|_{L_q(I)} - \|P_n\|_{L_q(I)} \\ &> c_* n^{-2/q+2/p} - (c_*/2)n^{-2/q+2/p} \\ &= (c_*/2)n^{-2/q+2/p}.\end{aligned} \tag{4.3}$$

Since $c^* \leqslant c_*/2$, (4.2) implies

$$\|\xi_{s,p,n} - P_n\|_{L_q(I)} < (c_*/2)n^{-2/q+2/p},$$

a contradiction to (4.3).

Therefore, we have proved that for every $s \geqslant 1$ and $1 \leqslant q \leqslant p \leqslant \infty$, if $P_n \in \mathcal{P}_n(I)$, then

$$\|\xi_{s,p,n} - P_n\|_{L_q(I)} \geqslant c^* n^{-2/q+2/p}. \tag{4.4}$$

This proves the lower bounds in Theorem 1 for $s > 1$.

Now consider

$$\eta_p(t) := \begin{cases} 0, & t \in (-2, 0], \\ 1, & t \in (0, 2), \end{cases} \tag{4.5}$$

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which is clearly in $\Delta^1_+ B_p(I)$. Let $\hat{c} = \hat{c}(q) \geqslant 1$ be the constant from Lemma 10. Set $c^* := 2^{-2-2/q}\hat{c}^{-1}$, and assume to the contrary that there exists a polynomial $P_n \in \mathcal{P}_n(I)$, such that

$$\|\eta_p - P_n\|_{L_q(I)} < c^* n^{-1/q}.$$

Let $I_n := (-1/(8n), 1/(8n))$, and consider the function

$$\zeta_{p,n}(t) := \eta_p(t + 1/(8n)) - \eta_p(t - 1/(8n)), \quad t \in I.$$

Then, the polynomial

$$P_n^*(t) := P_n(t + 1/(8n)) - P_n(t - 1/(8n)), \quad t \in I,$$

obviously satisfies

$$\|\zeta_{p,n} - P_n^*\|_{L_q(I)} < 2c^* n^{-1/q}. \tag{4.6}$$

Since $\zeta_{p,n}(t) \equiv 0$, $t \in I \setminus I_n$, it follows that

$$\|\zeta_{p,n}\|_{L_q(I)} = \|\zeta_{p,n}\|_{L_q(I_n)} = 2^{-2/q} n^{-1/q}$$

and

$$\|P_n^*\|_{L_q(I\setminus I_n)} < 2c^* n^{-1/q}.$$

Thus, we conclude by virtue of Lemma 10 that

$$\|P_n^*\|_{L_q(I)} \leqslant \hat{c}\|P_n^*\|_{L_q(I\setminus I_n)} < 2\hat{c}c^* n^{-1/q} = 2^{-1-2/q} n^{-1/q}.$$

Hence,

$$\|\zeta_{p,n} - P_n^*\|_{L_q(I)} \geqslant \|\zeta_{p,n}\|_{L_q(I)} - \|P_n^*\|_{L_q(I)} > 2^{-1-2/q} n^{-1/q}. \tag{4.7}$$

On the other hand, $2c^* \leqslant 2^{-1-2/q}$, so that (4.6) yields

$$\|\zeta_{p,n}(\cdot) - P_n^*(\cdot)\|_{L_q(J)} < 2^{-1+1/p-2/q} n^{-1/q},$$

a contradiction to (4.7).

Therefore we proved that for $1 \leqslant q \leqslant p \leqslant \infty$, if $P_n \in \mathcal{P}_n(I)$, then

$$\|\eta_p(\cdot) - P_n(\cdot)\|_{L_q(I)} \geqslant c^* n^{-1/q}. \tag{4.8}$$

Combining (4.4) and (4.8) we obtain the lower bounds in Theorem 1 for $s = 1$. This completes the proof. □

5. Proof of Theorem 2

Proof. For $d = 1$, $b^1 = (-1, 1) =: I$. Given $x \in \Delta^{\circ,s} B_p(b^1)$, this is exactly $x \in \Delta^s_+ B_p(I)$ which, in addition, is an even function, satisfying $x^{(k)}(0) = 0$, $k = 0, \ldots, s-1$. Therefore this is covered by Theorem 1.

For $d > 1$ and $x \in \Delta^{\circ,s}_+ B_p(\mathbb{B}^d)$, we recall the function $y(|t|) = x(t)$, $t = (t_1, \ldots, t_n) \in \mathbb{B}^d$, which by Lemma 11 satisfies (2.6). We extend its definition to $I = (-1, 1)$ by symmetry, so that

$$\|y\|_{L_p(I)} \leqslant 2c^*(d, p)\|x\|_{L_p(\mathbb{B}^d)}. \tag{5.1}$$

From the proof of Theorem 1 we obtain the polynomials $P_{s,n}(\cdot; y)$, and we define $P_{s,n}(t; x) := P_{s,n}(|t|; y)$. Applying (2.7) (note that unlike (5.1), it does not require any properties of the function under the norm), we obtain

$$\|x - P_{s,n}(\cdot; x)\|_{L_p(\mathbb{B}^d)} \leqslant c(d, p)\|y - P_{s,n}(\cdot; y)\|_{L_p[0,1)}.$$

Then by Theorem 1 and (5.1), we establish the upper bounds in Theorem 2.

We turn to the lower bounds. Let $\xi_{s,p,n}(\tau)$ be the function of one variable defined by (4.1). Set $\xi^\circ_{s,p,n}(t) := \alpha\xi_{s,p,n}(|t|)$, $t \in \mathbb{B}^d$, where $\alpha = \alpha(d, p)$ is a normalizing factor such that $\|\xi^\circ_{s,p,n}\|_{L_p(\mathbb{B}^d)} = 1$. Let

$$\theta_p(\tau) := \begin{cases} 0, & \tau \in [0, 1/2], \\ 1, & \tau \in (1/2, 1]. \end{cases}$$

Set $\theta^\circ_p(t) := \gamma\theta_p(|t|), t \in \mathbb{B}^d$, where $\gamma = \gamma(d, p)$ is a normalizing factor such that $\|\theta^\circ_p\|_{L_p(\mathbb{B}^d)} = 1$. Evidently, $\xi^\circ_{s,p,n}, \theta^\circ_p \in \Delta^{\circ,1}_+ B_p(\mathbb{B}^d)$, and Lemma 7 together with (4.2) and (4.4) yield the required lower bounds. This completes the proof. □

6. Proof of Theorem 4

Proof. It is well known that the space $\mathcal{P}_n(\mathbb{B}^d)$ can be embedded in the manifold $\mathcal{R}_{m,q}(\mathbb{B}^d)$, where $1 \leqslant q \leqslant \infty$ and $m = \binom{n+d-1}{n}$. Since $m \asymp n^{d-1}$, the inequality

$$E\left(x, \mathcal{R}_{n^{d-1},2}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)} \leqslant E\left(x, \mathcal{P}_{\check{c}n}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)} \tag{6.1}$$

is obvious.

Thus we will prove the lower bounds. Fix any points $a_l \in \mathbb{S}^{d-1}$, $l = 1, \ldots, m$, on the sphere $\mathbb{S}^{d-1}$, and let r_l, $l = 1, \ldots, m$, be arbitrary univariate functions from $L_2(I)$ where $I := (-1, 1)$. Evidently, the function

$$R_m(t) = \sum_{l=1}^{m} r_l(a_l \cdot t), \quad t \in \mathbb{B}^d$$

belongs to $\mathcal{R}_{m,2}(\mathbb{B}^d)$. Given a radial function $x \in L_2(\mathbb{B}^d)$ we are going to estimate the norm $\|x - R_m\|_{L_2(\mathbb{B}^d)}$ from below.

Denote $\rho_l(t; a_l) := r_l(a_l \cdot t)$, $l = 1, \ldots, m$, $t \in \mathbb{B}^d$. Then by (2.10) we have

$$A_{d,k}(\xi; x - R_m) = A_{d,k}(\xi; x) - A_{d,k}(\xi; R_m) = A_{d,k}(\xi; x) - \sum_{l=1}^{m} A_{d,k}(\xi; \rho_l(\cdot; a_l)).$$

Since x is radial on $\mathbb{B}^d$, it follows that $A_{d,k}(\xi; x)$ does not depend on $\xi \in \mathbb{S}^{d-1}$ and we may write

$$A_{d,k}(x) := A_{d,k}(\xi; x), \quad \forall \xi \in \mathbb{S}^{d-1}. \tag{6.2}$$

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Thus,

$$A_{d,k}(\xi; x - R_m) = A_{d,k}(x) - \sum_{l=1}^{m} A_{d,k}(\xi; \rho_l(\cdot; a_l)), \quad \xi \in \mathbb{S}^{d-1}. \tag{6.3}$$

Note that (2.10) implies that $A_{d,2j-1}(x) = 0$. For the function x is radial and the polynomial $U_{d,2j-1}$ is odd on the interval I.

If we decompose each univariate function $r_l(t)$, $t \in I$, into its Fourier–Gegenbauer series

$$r_l(t) = \sum_{j=1}^{\infty} \hat{r}_{l,j} U_{d,j}(t),$$

where

$$\hat{r}_{l,j} := \int_I r_l(t) U_{d,j}(t) w_d(t)\, dt,$$

then we obtain

$$\begin{aligned} A_{d,k}(\xi; \rho_l(\cdot; a_l)) &= \int_{\mathbb{B}^d} \rho_l(\tau; a_l)(\tau) U_{d,k}(\xi \cdot \tau)\, d\tau \\ &= \int_{\mathbb{B}^d} r_l(a_l \cdot \tau) U_{d,k}(\xi \cdot \tau)\, d\tau \\ &= \sum_{j=1}^{\infty} \hat{r}_{l,j} \int_{\mathbb{B}^d} U_{d,j}(a_l \cdot \tau) U_{d,k}(\xi \cdot \tau)\, d\tau. \end{aligned} \tag{6.4}$$

Now, the following are well-known properties of $U_{d,n}(\xi \cdot t)$.

(i) For any fixed $\xi \in \mathbb{S}^{d-1}$ the function $U_{d,n}(\xi \cdot t)$, $t \in \mathbb{B}^d$, belongs to the space $\mathcal{P}_n(\mathbb{B}^d)$ and is orthogonal to the space $\mathcal{P}_{n-1}(\mathbb{B}^d)$ (see [19, p. 162, (3.4)]), i.e.,

$$\int_{\mathbb{B}^d} P(t) U_{d,k}(\xi \cdot t)\, dt = 0, \quad P \in \mathcal{P}_{n-1}(\mathbb{B}^d).$$

(ii) For any $\xi, \eta \in \mathbb{S}^{d-1}$ we have (see [19, p. 164, (3.10)])

$$\int_{\mathbb{B}^d} U_{d,n}(\xi \cdot t) U_{d,n}(\eta \cdot t)\, dt = \frac{U_{d,n}(\xi \cdot \eta)}{U_{d,n}(1)}.$$

(iii) Let $\mathcal{P}_n^h(\mathbb{R}^d)$ denote the space of all homogeneous polynomials of degree n on $\mathbb{R}^d$, and let $\mathcal{H}_n(\mathbb{S}^{d-1})$ denote the space of spherical harmonics of degree n on $\mathbb{S}^{d-1}$, i.e., $\mathcal{H}_n(\mathbb{S}^{d-1})$ is the set of those functions on $\mathbb{S}^{d-1}$ which are the restriction to $\mathbb{S}^{d-1}$ of a function from $\mathcal{P}_n^h(\mathbb{R}^d)$ which is harmonic in $\mathbb{R}^d$. Then for each $H \in \bigoplus_{i=0}^{[n/2]} \mathcal{H}_{n-2i}(\mathbb{S}^{d-1})$, and any fixed $\eta \in \mathbb{S}^{d-1}$, we have (see [19, p. 165, (3.17)])

$$\int_{\mathbb{S}^{d-1}} H(\xi) U_{d,n}(\xi \cdot \eta)\, dt = \frac{U_{d,n}(1)}{v_{d,n}} H(\eta).$$

It follows from (i) that for any $a_l, \xi \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{B}^d} U_{d,j}(a_l \cdot \tau) U_{d,k}(\xi \cdot \tau)\, d\tau = 0, \quad k \neq j.$$

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Hence, by (6.4),

$$A_{d,k}(\xi;\rho_l(\cdot;a_l)) = \hat{r}_{l,k}\int_{\mathbb{B}^d} U_{d,k}(a_l\cdot\tau)U_{d,k}(\xi\cdot\tau)\,d\tau = \hat{r}_{l,k}\frac{U_{d,k}(a_l\cdot\xi)}{U_{d,k}(1)},$$

which substituted into (6.3) yields

$$A_{d,k}(\xi;x-R_m) = A_{d,k}(x) - \sum_{l=1}^{m}\hat{r}_{l,k}\frac{U_{d,k}(a_l\cdot\xi)}{U_{d,k}(1)}. \tag{6.5}$$

Since as functions of ξ, $A_{d,k}(\xi;x-R_m)$ and $U_{d,k}(a_l\cdot\xi)$ belong to the space $H\in\bigoplus_{i=0}^{[n/2]}\mathcal{H}_{n-2i}(\mathbb{S}^{d-1})$ (see [19, p. 165], for explanation), we conclude that

$$\|A_{d,k}(\cdot;x-R_m)\|_{L_2(\mathbb{S}^{d-1})} = \sup_H \int_{\mathbb{S}^{d-1}} A_{d,k}(\xi;x-R_m)H(\xi)\,d\xi,$$

where the supremum is taken over all $H\in\bigoplus_{i=0}^{[n/2]}\mathcal{H}_{n-2i}(\mathbb{S}^{d-1})$, such that $\|H\|_{L_2(\mathbb{S}^{d-1})}\leqslant 1$. Therefore, by virtue of (6.5) and (iii), we get

$$\begin{aligned}\|A_{d,k}(\cdot;x-R_m)\|_{L_2(\mathbb{S}^{d-1})} &= \sup_H\left(A_{d,k}(x)\int_{\mathbb{S}^{d-1}}H(\xi)\,d\xi - \sum_{l=1}^{m}\hat{r}_{l,k}\int_{\mathbb{S}^{d-1}}\frac{U_{d,k}(a_l\cdot\xi)}{U_{d,k}(1)}H(\xi)\,d\xi\right)\\ &= \sup_H\left(A_{d,k}(x)\int_{\mathbb{S}^{d-1}}H(\xi)\,d\xi - \sum_{l=1}^{m}\frac{\hat{r}_{l,k}}{v_{d,k}}H(a_l)\right).\end{aligned} \tag{6.6}$$

We will prove that there exist constants $k(d,m)$ and $c=c(d)>0$, such that

$$\|A_{d,k}(\cdot;x-R_m)\|_{L_2(\mathbb{S}^{d-1})}\geqslant c|A_{d,k}(x)|, \quad k>k(d,m). \tag{6.7}$$

To this end we note that the inequality is trivial for $k=2j-1$ since $A_{d,2j-1}(x)=0$. Hence we restrict ourselves to even k's.

Let $\mathcal{P}_n^h(\mathbb{S}^{d-1})$ be the restriction to $\mathbb{S}^{d-1}$ of the space $\mathcal{P}_n^h(\mathbb{R}^d)$. It is well known (see, e.g., [22]) that

$$\mathcal{P}_{n-2j}^h(\mathbb{S}^{d-1})\subseteq\bigoplus_{i=0}^{[n/2]}\mathcal{H}_{n-2i}(\mathbb{S}^{d-1}), \quad j=0,\ldots,[n/2],$$

so that for $n=2k$ we have

$$\mathcal{P}_{2j}^h(\mathbb{S}^{d-1})\subseteq\bigoplus_{i=0}^{k}\mathcal{H}_{2i}(\mathbb{S}^{d-1}), \quad j=0,\ldots,k.$$

Let $\mathcal{S}_{2j}^h(\mathbb{S}^{d-1})$ denote the subspace of $\mathcal{P}_{2j}^h(\mathbb{S}^{d-1})$, of all spherical polynomials of the form

$$S_{2j}^h(\xi) = \sum_{j_1+\cdots+j_d=j}\alpha_{j_1,\ldots,j_d}\xi_1^{2j_1}\cdots\xi_d^{2j_d}, \quad \xi=(\xi_1,\ldots,\xi_d)\in\mathbb{S}^{d-1}, \tag{6.8}$$

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where $(j_1, \dots, j_d) \in \mathbb{Z}_+^d$ and $\alpha_{j_1,\dots,j_d} \in \mathbb{R}$, and let $\mathcal{S}_{2j}^h(\mathbb{S}^{d-1}; \{a_l\})$ be its subspace of all spherical polynomials $S_{2j}^h \in \mathcal{S}_{2j}^h(\mathbb{S}^{d-1})$ which satisfy

$$S_{2j}^h(a_l) = 0, \quad l = 1, \dots, m. \tag{6.9}$$

By virtue of (6.6) we conclude that

$$\|A_{d,2k}(\cdot; x - R_m)\|_{L_2(\mathbb{S}^{d-1})} \geqslant A_{d,2k}(x) \max_{0 \leqslant j \leqslant k} \sup_{S_{2j}^h} \int_{\mathbb{S}^{d-1}} S_{2j}^h(\xi)\, d\xi,$$

where the supremum is taken over the spherical polynomials $S_{2j}^h \in \mathcal{S}_{2j}^h(\mathbb{S}^{d-1}; \{a_l\})$, $0 \leqslant j \leqslant k$, such that $\|S_{2j}^h\|_{L_2(\mathbb{S}^{d-1})} \leqslant 1$.

Now,

$$\dim \mathcal{S}_{2j}^h(\mathbb{S}^{d-1}) = \binom{j+d-1}{j},$$

since the collection of all monomials $\xi_1^{2j_1} \cdots \xi_d^{2j_d}$, $j_1 + \cdots + j_d = j$, is linearly independent on $\mathbb{S}^{d-1}$. We impose in (6.9) at most m linear restrictions on the coefficients of the spherical polynomials in $\mathcal{S}_{2j}^h(\mathbb{S}^{d-1}; \{a_l\})$. Hence,

$$\dim \mathcal{S}_{2j}^h(\mathbb{S}^{d-1}; \{a_l\}) \geqslant \binom{j+d-1}{j} - m, \quad j = 0, \dots, k. \tag{6.10}$$

Let $\varphi := (\varphi_1, \dots, \varphi_{d-1})$ be the spherical coordinates on $\mathbb{S}^{d-1}$ defined by $\xi_1 = \cos\varphi_1$, $\xi_2 = \sin\varphi_1 \cos\varphi_2, \dots, \xi_{d-2} = \sin\varphi_1 \dots \sin\varphi_{d-3} \cos\varphi_{d-2}$, $\xi_{d-1} = \sin\varphi_1 \dots \sin\varphi_{d-2} \cos\varphi_{d-1}$, $\xi_d = \sin\varphi_1 \dots \sin\varphi_{d-2} \sin\varphi_{d-1}$, where $0 \leqslant \varphi_i \leqslant \pi$, $1 \leqslant i \leqslant d-2$, and $0 \leqslant \varphi_{d-1} < 2\pi$. With $\xi = \xi(\varphi)$, the surface element $d\xi$ of $\mathbb{S}^{d-1}$, becomes

$$d\xi = J(\varphi)\, d\varphi,$$

where the Jacobian is given by

$$J(\varphi) := (\sin\varphi_1)^{d-2} (\sin\varphi_2)^{d-3} \dots \sin\varphi_{d-2}. \tag{6.11}$$

It is easy to verify that for each $S_{2j}^h \in \mathcal{S}_{2j}^h(\mathbb{S}^{d-1}; \{a_l\})$ the function $T_{2j}^h(\varphi) := S_{2j}^h(\xi(\varphi))$, $\varphi \in \mathbb{T}^{d-1}$, belongs to the space $\mathcal{T}_{2k}(\mathbb{T}^{d-1})$ of trigonometric polynomials on the torus $\mathbb{T}^{d-1}$. We denote the collection of these functions by $\mathcal{T}_{2j}^h(\mathbb{T}^{d-1}; \{a_l\})$. Clearly,

$$\begin{aligned} \dim \mathcal{T}_{2j}^h(\mathbb{T}^{d-1}; \{a_l\}) &= \dim \mathcal{S}_{2j}^h(\mathbb{S}^{d-1}; \{a_l\}) \\ &\geqslant \binom{j+d-1}{j} - m, \quad j = 0, \dots, k, \end{aligned} \tag{6.12}$$

where we applied (6.10). It follows from (6.8) that $T_{2j}^h(\varphi)$ is even with respect to each variable φ_i, $i = 1, \dots, d-2$. Hence,

$$\int_{\mathbb{S}^{d-1}} S_{2j}^h(\xi)\, d\xi = \frac{1}{2^{d-2}} \int_{\mathbb{T}^{d-1}} T_{2j}^h(\varphi) |J(\varphi)|\, d\varphi \tag{6.13}$$

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and

$$\int_{\mathbb{S}^{d-1}} (S^h_{2j}(\xi))^2 \, d\xi = \frac{1}{2^{d-2}} \int_{\mathbb{T}^{d-1}} (T^h_{2j}(\varphi))^2 |J(\varphi)| \, d\varphi. \tag{6.14}$$

By virtue of (6.11) and (6.13) we obtain

$$\|A_{d,2k}(\cdot; x - R_m)\|_{L_2(\mathbb{S}^{d-1})} \geqslant \frac{A_{d,2k}(x)}{2^{d-2}} \max_{0 \leqslant j \leqslant k} \sup_{T^h_{2j}} \int_{\mathbb{T}^{d-1}} T^h_{2j}(\varphi) |J(\varphi)| \, d\varphi, \tag{6.15}$$

where, by (6.14), the supremum is taken over all $T^h_{2j} \in \mathcal{T}^h_{2j}(\mathbb{T}^{d-1}; \{a_l\})$ such that

$$\frac{1}{2^{d-2}} \int_{\mathbb{T}^{d-1}} (T^h_{2j}(\varphi))^2 |J(\varphi)| \, d\varphi \leqslant 1, \quad j = 0, \ldots, k. \tag{6.16}$$

Let $k(d,m) \in \mathbb{N}$ be the smallest k satisfying $k^{d-1} \geqslant 2^{2d-1}(d-1)!m$, and take $k \geqslant k(d,m)$. Denote $k' := 2[k/2]$, and consider the subspace

$$\mathcal{T}_* := \mathcal{T}^h_{k'}(\mathbb{T}^{d-1}; \{a_l\}).$$

Since

$$\binom{k'/2 + d - 1}{k'/2} \geqslant \frac{k^{d-1}}{4^{d-1}(d-1)!},$$

it follows by (6.12) that

$$\dim \mathcal{T}^h_{k'}(\mathbb{T}^{d-1}; \{a_l\}) \geqslant \frac{k^{d-1}}{2^{2d-1}(d-1)!}.$$

However,

$$\dim \mathcal{T}_{2k}(\mathbb{T}^{d-1}) \leqslant 5^{d-1} k^{d-1}.$$

Hence,

$$\dim \mathcal{T}_* = \dim \mathcal{T}^h_{k'}(\mathbb{T}^{d-1}; \{a_l\}) \geqslant c_* \dim \mathcal{T}_{2k}(\mathbb{T}^{d-1}),$$

where $c_* := 1/(5^{d-1} 2^{2d-1}(d-1)!)$.

Applying Lemma 13, it follows that there exists a trigonometric polynomial $T_* \in \mathcal{T}_*$ such that

$$\|T_*\|_{L_\infty(\mathbb{T}^{d-1})} = 1 \quad \text{and} \quad \|T_*\|_{L_2(\mathbb{T}^{d-1})} \geqslant c^*, \tag{6.17}$$

where $0 < c^* = c^*(d, c_*) < 1$.

Let $|\Phi|$ denote the Lebesgue measure of the (measurable) subset $\Phi \subseteq \mathbb{T}^{d-1}$, and let $T^*(\varphi) := (T_*(\varphi))^2$, $\varphi \in \mathbb{T}^{d-1}$. Then $T^* \in \mathcal{T}^h_{2k}(\mathbb{T}^{d-1}; \{a_l\})$, and by (6.17) we have

$$\|T^*\|_{L_\infty(\mathbb{T}^{d-1})} = 1 \quad \text{and} \quad \int_{\mathbb{T}^{d-1}} T^*(\varphi) \, d\varphi \geqslant c_\circ |\mathbb{T}^{d-1}|, \tag{6.18}$$

where $c_\circ := (c^*)^2 / |\mathbb{T}^{d-1}|$.

If Φ^* is the subset in $\mathbb{T}^{d-1}$ of all points φ such that $T^*(\varphi) \geqslant c_\circ/4$, then it follows by (6.18) that

$$|\Phi^*| \geqslant \int_{\Phi^*} T^*(\varphi)\, d\varphi = \int_{\mathbb{T}^{d-1}} T^*(\varphi)\, d\varphi - \int_{\mathbb{T}^{d-1}\setminus\Phi^*} T^*(\varphi)\, d\varphi \geqslant 3c_\circ|\mathbb{T}^{d-1}|/4. \tag{6.19}$$

For $\alpha \in [0, 1]$, let $\Phi(\alpha) \subseteq \mathbb{T}^{d-1}$ be the subset of all points φ such that $|J(\varphi)| \geqslant \alpha$, where the Jacobian $J(\varphi)$ was given in (6.11). If $d = 2$, then $J(\varphi) \equiv 1$, and if $d > 2$, then $|\Phi(\alpha)|$ is a continuous nonincreasing function in α assuming all values from $|\mathbb{T}^{d-1}|$ to 0. Hence, there exists $\alpha_* \in (0, 1)$, such that

$$|\Phi(\alpha_*)| \geqslant (1 - c_\circ/2)\, |\mathbb{T}^{d-1}|,$$

which combined with (6.19) implies that $|\Phi^* \cap \Phi(\alpha_*)| \geqslant c_\circ|\mathbb{T}^{d-1}|/4$.

Now,

$$\int_{\mathbb{T}^{d-1}} T^*(\varphi)|J(\varphi)|\, d\varphi \geqslant \int_{\Phi^*\cap\Phi(\alpha_*)} T^*(\varphi)|J(\varphi)|\, d\varphi \geqslant c_\circ\alpha_*|\Phi^* \cap \Phi(\alpha_*)|/4,$$

so that

$$\int_{\mathbb{T}^{d-1}} T^*(\varphi)|J(\varphi)|\, d\varphi \geqslant c_\circ^2|\mathbb{T}^{d-1}|\alpha_*/16. \tag{6.20}$$

Set $T_\circ(\varphi) := c^\circ T^*(\varphi)$, $\varphi \in \mathbb{T}^{d-1}$, where

$$c^\circ := \left(2^{d-2}\int_{\mathbb{T}^{d-1}} |J(\varphi)|\, d\varphi\right)^{-1/2}.$$

Then $T_\circ \in \mathcal{T}^h_{2k}(\mathbb{T}^{d-1}; \{a_l\})$, and by (6.18),

$$\frac{1}{2^{d-2}}\int_{\mathbb{T}^{d-1}} (T_\circ(\varphi))^2|J(\varphi)|\, d\varphi \leqslant 1.$$

Hence $T_\circ$ satisfies (6.16). Moreover, it follows by (6.20) that

$$\int_{\mathbb{T}^{d-1}} T_\circ(\varphi)|J(\varphi)|\, d\varphi \geqslant c^\circ c_\circ^2|\mathbb{T}^{d-1}|\alpha_*/16 > 0.$$

Thus, substituting in (6.15) we obtain

$$|A_{d,2k}(x)| \leqslant \breve{c}\|A_{d,2k}(\cdot; x - R_m)\|_{L_2(\mathbb{S}^{d-1})}, \tag{6.21}$$

where $\breve{c} := (c^\circ c_\circ^2|\mathbb{T}^{d-1}|\alpha_*/(2^{d+2}))^{-1} > 0$. This proves (6.7).

We are ready to conclude the proof of the lower bound in Theorem 4. By virtue of Lemma 12 we have

$$\begin{aligned}\left\|x(\cdot) - \sum_{0\leqslant k\leqslant 2k(d,m)} Q_{d,k}(\cdot; x)\right\|^2_{L_2(\mathbb{B}^d)} &= \sum_{k>2k(d,m)} \|Q_{d,k}(\cdot; x)\|^2_{L_2(\mathbb{B}^d)} \\ &= \sum_{k>2k(d,m)} v_{d,k}\|A_{d,k}(\cdot; x)\|^2_{L_2(\mathbb{S}^{d-1})}.\end{aligned}$$

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Recall that $A_{d,k}(\cdot;x) = A_{d,k}(x)$, and that $A_{d,2k-1}(x) = 0$. Thus, we have

$$\left\| x(\cdot) - \sum_{0 \leqslant k \leqslant 2k(d,m)} Q_{d,k}(\cdot;x) \right\|^2_{L_2(\mathbb{B}^d)} = |\mathbb{S}^{d-1}| \sum_{k > k(d,m)} v_{d,2k} |A_{d,2k}(x)|^2,$$

where $|\mathbb{S}^{d-1}|$ is the Lebesgue measure of the sphere $\mathbb{S}^{d-1}$. By virtue of (6.21) we get

$$\begin{aligned} \left\| x(\cdot) - \sum_{0 \leqslant k \leqslant 2k(d,m)} Q_{d,k}(\cdot;x) \right\|^2_{L_2(\mathbb{B}^d)} &\leqslant \tilde{c} \sum_{k > k(d,m)} v_{d,2k} \| A_{d,2k}(\cdot; x - R_m) \|^2_{L_2(\mathbb{S}^{d-1})} \\ &\leqslant \tilde{c} \sum_{n=0}^{\infty} v_{d,n} \| A_{d,n}(\cdot; x - R_m) \|^2_{L_2(\mathbb{S}^{d-1})} \\ &= \tilde{c} \sum_{n=0}^{\infty} \| Q_{d,n}(\cdot; x - R_m) \|^2_{L_2(\mathbb{B}^d)} \\ &= \tilde{c} \| x(\cdot) - R_m(\cdot) \|^2_{L_2(\mathbb{B}^d)}, \end{aligned} \tag{6.22}$$

where $\tilde{c} := \breve{c} |\mathbb{S}^{d-1}|$.

It is known (see, e.g., [19, p. 163, Remark (i)]) that $Q_{d,k}(\cdot;x) \in \mathcal{P}_k(\mathbb{B}^d)$, $k \in \mathbb{Z}_+$. Therefore,

$$P(t;x) := \sum_{0 \leqslant k \leqslant 2k(d,m)} Q_{d,k}(t;x), \quad t \in \mathbb{B}^d$$

is a polynomial of the degree $\leqslant 2k(d,m) \leqslant \lceil 4(d-1) m^{1/(d-1)} \rceil$. Take $m = n^{d-1}$ so that it is a polynomial of the degree $\leqslant 4(d-1)n$, and (6.22) yields

$$\bar{c} E\left(x, \mathcal{P}_{\hat{c}n}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)} \leqslant \| x(\cdot) - R_{n^{d-1}}(\cdot) \|_{L_2(\mathbb{B}^d)},$$

where $\bar{c} := \tilde{c}^{-1/2}$ and $\hat{c} := 4(d-1)$. Since this is valid for every $R_{n^{d-1}} \in \mathcal{R}_{n^{d-1},2}(\mathbb{B}^d)$, and we obtain

$$\bar{c} E\left(x, \mathcal{P}_{\hat{c}n}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)} \leqslant E\left(x, \mathcal{R}_{n^{d-1},2}\left(\mathbb{B}^d\right)\right)_{L_2(\mathbb{B}^d)},$$

which concludes the proof of the lower bound in Theorem 4. This completes our proof. □

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