SHAPE PRESERVING WIDTHS OF WEIGHTED SOBOLEV-TYPE CLASSES OF POSITIVE, MONOTONE AND CONVEX FUNCTIONS ON A FINITE INTERVAL

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Abstract. Let I be a finite interval, $r \in \mathbb{N}$ and $\rho(t) = \operatorname{dist}\{t, \partial I\}$, $t \in I$. Denote by $\Delta^s_+ L_q$ the subset of all functions $y \in L_q$ such that the s-difference $\Delta^s_\tau y(t)$ is nonnegative on I, $\forall \tau > 0$. Further, denote by $\Delta^s_+ W^r_{p,\alpha}$, $0 \le \alpha < \infty$ the classes of functions x on I with the seminorm $\|x^{(r)} \rho^\alpha\|_{L_p} \le 1$, such that $\Delta^s_\tau x \ge 0$, $\tau > 0$. For $s = 0, 1, 2$, we obtain two-sided estimates of the shape preserving widths

$$d_n \;\; \Delta^s_+ W^r_{p,\alpha}, \Delta^s_+ L_q \;\; {}_{L_q} := \inf_{M^n \in \mathcal{M}^n} \sup_{x \in \Delta^s_+ W^r_{p,\alpha}} \inf_{y \in M^n \cap \Delta^s_+ L_q} \|x - y\|_{L_q},$$

where $\mathcal{M}^n$ is the set of all linear manifolds M^n in L_q, such that $\dim M^n \le n$, and satisfying $M^n \cap \Delta^s_+ L_q \ne \emptyset$.

§1. Introduction and statement of the main results

Let X be a real linear space of vectors x with a norm $\|x\|_X$, $W \subset X$, $W \ne \emptyset$ and $V \subset X$, $V \ne \emptyset$. Let L^n be a subspace in X of dimension $\dim L^n \le n$, $n \ge 0$ and $M^n = M^n(z) := z + L^n$ be a shift of the subspace L^n by an arbitrary vector $z \in X$. If $M^n \cap V \ne \emptyset$, then we denote by

$$E(x, M^n \cap V)_X := \inf_{y \in M^n \cap V} \|x - y\|_X,$$

the best approximation of the vector $x \in X$ by $M^n \cap V$, and by

$$E(W, M^n \cap V)_X := \sup_{x \in W} E(x, M^n \cap V)_X,$$

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the deviation of W from $M^n \cap V$.

Let $\mathcal{M}^n = \mathcal{M}^n(X, V)$ be the set of all linear manifolds M^n, $\dim M^n \leq n$ such that $M^n \cap V \neq \emptyset$. The quantity

$$d_n(W, V)_X := \inf_{M^n \in \mathcal{M}^n} E(W, M^n \cap V)_X, \quad n \geq 0$$

is called the relative n-width of W with the constraint V in X. These widths were introduced by the first author in [9].

Evidently, if $V = X$, then the relative n-width $d_n(W, V)_X$ coincides with the Kolmogorov n-width $d_n(W)_X$. Clearly, $d_n(W, V)_X \geq d_n(W)_X$.

Let I be a finite interval in $\mathbb{R}$, and let $r \in \mathbb{N}$ and $0 \leq \alpha < \infty$. For $1 \leq p \leq \infty$, and $\rho(t) := \operatorname{dist}\{t, \partial I\}$, $t \in I$, we denote

$$W^r_{p,\alpha} := W^r_{p,\alpha}(I) := \{x : I \to \mathbb{R} \mid x^{(r-1)} \in AC_{loc}(I), \|x^{(r)}\rho^\alpha\|_{L_p(I)} \leq 1\}.$$

Let

$$\Delta^s_\tau x(t) := \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} x(t + k\tau), \quad \{t, t + s\tau\} \subset I, \quad s = 0, 1, \dots,$$

be the s-th difference of the function x, with step $\tau > 0$, and denote by $\Delta^s_+ W^r_{p,\alpha} = \Delta^s_+ W^r_{p,\alpha}(I)$, $s = 0, 1, \dots$, the subclasses of functions $x \in W^r_{p,\alpha}$ for which $\Delta^s_\tau x(t) \geq 0$, for all $\tau > 0$ such that $[t, t + s\tau] \subseteq I$. By $\Delta^s_+ L_q = \Delta^s_+ L_q(I)$ we denote the subclass of all functions $y \in L_q(I)$ such that $\Delta^s_\tau y(t) \geq 0$, $\tau > 0$. If $\alpha = 0$, then we write $W^r_p := W^r_{p,0}$ and $\Delta^s_+ W^r_p := \Delta^s_+ W^r_{p,0}(I)$. Throughout this paper we will work with the generic finite interval $I = [-1, 1]$.

The behavior of the Kolmogorov and linear widths in the case $\alpha = 0$, i.e., for the classes $W^r_{r,0} = W^r_p$, has been thoroughly investigated. We refer the reader to the list of references for earlier results. Recently, in [10], we have obtained two-sided estimates of the Kolmogorov widths $d_n\left(W^r_{p,\alpha}\right)_{L_q}$ and of the linear widths $d_n\left(W^r_{p,\alpha}\right)^{lin}_{L_q}$ in the case $0 < \alpha < \infty$, and in [11] we have investigated the behaviour of the Kolmogorov widths $d_n\left(\Delta^s_+ W^r_{p,\alpha}\right)_{L_q}$ and of the linear widths $d_n\left(\Delta^s_+ W^r_{p,\alpha}\right)^{lin}_{L_q}$, $s = 0, 1, \dots, r + 1$, $O \leq \alpha < \infty$. In particular, in [11] we have obtained the following results.

Theorem KL1. *Let $r \in \mathbb{N}$, $1 \le p, q \le \infty$ and $0 \le \alpha < \infty$, be such that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. If $(r,p) \ne (1,1)$ and if $(r,p) = (1,1)$ and $1 \le q \le 2$, then for each $s = 0, 1, \dots, r$,*

$$d_n(\Delta_+^s W_{p,\alpha}^r)_{L_q} \asymp n^{-r + (\max\{\frac{1}{p},\frac{1}{2}\} - \max\{\frac{1}{q},\frac{1}{2}\})_+}, \quad n \ge r,$$

where $(u)_+ := \max\{u, 0\}$ and $a_n \asymp b_n$ means that there exist two constants $0 < C_1 < C_2$, such that $C_1 a_n \le b_n \le C_2 a_n$, $\forall n$. If on the other hand, $(r,p) = (1,1)$ and $2 < q < \infty$, then for $s = 0, 1$,

$$c_1 n^{-\frac{1}{2}} \le d_n(\Delta_+^s W_{1,\alpha}^1)_{L_q} \le c_2 n^{-\frac{1}{2}} \big(\log(n+1)\big)^{\frac{3}{2}}, \quad n \ge 1,$$

where $c_1 > 0$ and c_2 do not depend on n.

Theorem KL2. *Let $r \in \mathbb{N}$, $1 \le p, q \le \infty$ and $0 \le \alpha < \infty$, be such that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. Then*

$$d_n(\Delta_+^{r+1} W_{p,\alpha}^r)_{L_q} \asymp n^{-r - \max\{\frac{1}{q},\frac{1}{2}\}}. \quad n > r.$$

For $X = L_q$, $W = \Delta_+^s W_{p,\alpha}^r$ and $V = \Delta_+^s L_q$, we call $d_n\left(\Delta_+^s W_{p,\alpha}^r, \Delta_+^s L_q\right)_{L_q}$, the relative n-width, the shape preserving n-width of the class $\Delta_+^s W_{p,\alpha}^r$ in L_q. In recent years shape preserving approximation has become a central subject especially in application. This is due to the fact that in CAGD and especially in questions of design, shape preservation is one of the main considerations. Our results below show what one may expect to achieve and what is beyond reach of any approximation process which involves approximation from linear n dimensional manifolds, when we preserve the most important shape features of the approximants, namely, positivity, monotonicity and convexity. We are aware of only one previous attempt to consider such widths. The question of the behavior of the widths $d_n\left(\Delta_+^1 W_\infty^r, \Delta_+^1 L_\infty\right)_{L_\infty}$, was considered in [18]. We are indebted to A. Pinkus for bringing [18] to our attention.

The main results of this paper are the following three theorems. For positivity preserving widths we have

Theorem 1. *Let $r \in \mathbb{N}$, $1 \le p, q \le \infty$ and $0 \le \alpha < \infty$, be such that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. Then*

$$(1.1) \quad c_1 n^{-r+(\max\{\frac{1}{p},\frac{1}{2}\}-\max\{\frac{1}{q},\frac{1}{2}\})_+} \le d_n(\Delta_+^0 W_{p,\alpha}^r, \Delta_+^0 L_q)_{L_q} \le c_2 n^{-r+(\frac{1}{p}-\frac{1}{q})_+}, \quad n \ge r,$$

and in particular if $1 \le q \le p \le \infty$, and if $1 \le p \le q \le 2$, then this implies

$$(1.2) \qquad d_n(\Delta_+^0 W_{p,\alpha}^r, \Delta_+^0 L_q)_{L_q} \asymp n^{-r+(\max\{\frac{1}{p},\frac{1}{2}\}-\max\{\frac{1}{q},\frac{1}{2}\})_+}, \quad n \ge r.$$

Furthermore, (1.2) *holds for all other cases of p and q, if we actually have the (stronger) inequality $r - \alpha - \frac{1}{p} > 0$. (Note that under our assumptions, the latter always holds when $q = \infty$.) Finally, if $(r, \alpha, p) = (1, 0, 1)$ and $2 < q < \infty$, then*

$$(1.3) \qquad c_1 n^{-\frac{1}{2}} \le d_n(\Delta_+^0 W_{1,0}^1, \Delta_+^0 L_q)_{L_q} \le c_2 n^{-\frac{1}{2}} \left(\ln(n+1)\right)^{\frac{3}{2}}, \quad n \ge 1,$$

where $c_1 > 0$ and c_2 do not depend on n.

Remarks. i. In view of (1.2) one might be tempted to conjecture that in (1.1) the left-hand quantity is the correct asymptotic order of the positivity preserving widths in all the remaining cases as well. However, this is not supported by the asymptotics we have obtained for the monotonicity and the convexity preserving widths (see Theorems 2 and 3 below). We don't know whether the left-hand quantity always provides the exact asymptotics for positivity preserving widths.

ii. An upper bound in (1.1) can be had if one knew the one-sided width of $W_{p,\alpha}^r$ in L_q, that is, when the width is measured by approximation of the elements in $W_{p,\alpha}^r$, from above. For then if one approximates a nonnegative element, then the approximant from above is nonnegative too. We are aware of very few estimates for one-sided widths. In fact the only result we are aware of is the asymptotics of the one-sided width $d_n^+(\tilde{W}_p^r)_{L_p}$, $1 \le p \le \infty$ of the periodic Sobolev class $\tilde{W}_p^r$ in L_p. From this one can easily obtain the asymptotics of $d_n^+(\tilde{W}_p^r)_{L_q}$ for $1 \le q \le p \le \infty$ (see [1]). The asymptotics $d_n^+(\tilde{W}_p^r)_{L_q} \asymp n^{-r}$, is exactly the upper bound in (1.1) for $1 \le q \le p \le \infty$. (In fact it is exactly the asymptotics in (1.1) for $1 \le q \le p \le \infty$, but even in the periodic case we could conclude nothing from it on the lower bound in (1.1).) It should be emphasized that the proof of this estimate relies heavily on the periodicity of the functions.

For monotonicity preserving widths we show

Theorem 2. *Let $r \in \mathbb{N}$, $1 \le p, q \le \infty$ and $0 \le \alpha < \infty$, be such that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. Then*

$$d_n(\Delta_+^1 W_{p,\alpha}^r, \Delta_+^1 L_q)_{L_q} \asymp n^{-r+(\frac{1}{p}-\frac{1}{q})_+}, \quad n \ge r. \tag{1.4}$$

And for convexity preserving widths we obtain

Theorem 3. *Let $r \in \mathbb{N}$, $1 \le p, q \le \infty$ and $0 \le \alpha < \infty$, be such that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$. If $r > 1$, then*

$$d_n(\Delta_+^2 W_{p,\alpha}^r, \Delta_+^2 L_q)_{L_q} \asymp n^{-r+(\frac{1}{p}-\frac{1}{q})_+}, \quad n \ge r, \tag{1.5}$$

and if $r = 1$, then

$$d_n(\Delta_+^2 W_{p,\alpha}^1, \Delta_+^2 L_q)_{L_q} \asymp n^{-1-\frac{1}{q}}, \quad n \ge 1. \tag{1.6}$$

§2. Positivity preserving widths of the classes $\Delta_+^0 W_{p,\alpha}^r$ in L_q

For $n \in \mathbb{N}$ and $1 \le p \le \infty$, let l_p^n denote, as usual, the spaces of vectors $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with the norms

$$\|x\|_{l_p^n} := \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \max_{1 \le i \le n} |x_i|, & p = \infty, \end{cases}$$

and let B_p^n be its unit ball. For the proof of (1.3) we need the following lemma (see [5]).

Lemma K. *Let $1 < \lambda < \infty$ and $m, n \in \mathbb{N}$ be such that $m < n \le m^\lambda$. Then*

$$d_m\left(B_1^n\right)_{l_\infty^n} \le c m^{-\frac{1}{2}},$$

where $c = c(\lambda)$.

Proof of Theorem 1. The lower bounds in (1.1) through (1.3) follow from Theorem KL1 since

$$d_n(\Delta^0_+ W^r_{p,\alpha}, \Delta^0_+ L_q)_{L_q} \ge d_n(\Delta^0_+ W^r_{p,\alpha})_{L_q}.$$

Thus we only have to prove the upper bounds. First we show that

$$d_n(\Delta^0_+ W^r_{p,\alpha}, \Delta^0_+ L_q)_{L_q} \le cn^{-r+(\frac{1}{p}-\frac{1}{q})_+}, \quad n \ge r, \tag{2.1}$$

where $c = c(r, \alpha, p, q)$.

To this end we recall the construction of the continuous piecewise polynomials we had in [10]. We take the generic interval $I = (-1, 1)$, so that

$$\rho(t) = \operatorname{dist}(t, \{-1, 1\}) = \min\{|1 + t|, |1 - t|\}, \quad t \in I.$$

Fix $r \in \mathbb{N}$, $0 \le \alpha < \infty$, $1 \le p, q \le \infty$ such that $r - \alpha - \frac{1}{p} + \frac{1}{q} > 0$, and write

$$\beta := \beta(r, \alpha, p, q) := \left(r - \tfrac{1}{p} + \tfrac{1}{q}\right)\left(r - \alpha - \tfrac{1}{p} + \tfrac{1}{q}\right)^{-1}. \tag{2.2}$$

Given $n \in \mathbb{N}$, let

$$t_{ni} := t_{ni}(r, \alpha, p, q) := \begin{cases} 1 - \left(\frac{n-i}{n}\right)^{\beta}, & i = 0, 1, \dots, n, \\ -1 + \left(\frac{n+i}{n}\right)^{\beta}, & i = -n, \dots, -1, \end{cases} \tag{2.3}$$

be a partition of I. Denote by

$$I_{ni} := I_{ni}(r, \alpha, p, q) := \begin{cases} [t_{n,i-1}, t_{ni}], & i = 1, \dots, n, \\ [t_{ni}, t_{n,i+1}], & i = -n, \dots . -1, \end{cases}$$

the intervals of the partition, and let

$$\bar{t}_{ni} := \begin{cases} t_{2n,2i-1}, & i = 1, \dots, n, \\ t_{2n,2i+1}, & i = -n, \dots, -1. \end{cases}$$

On each interval I_{ni}, we have defined two complementary splines φ_{*ni} and φ^*_{ni}, with the following properties. The functions are piecewise quadratic polynomials on the respective intervals,

$$\begin{aligned} &\varphi_{*ni}(t_{n,i-1}) = \varphi^*_{ni}(t_{ni}) = 1, \quad \varphi_{*ni}(t_{ni}) = \varphi^*_{ni}(t_{n,i-1}) = 0, \quad i = 1, \dots, n, \\ &\varphi_{*ni}(t_{n,i+1}) = \varphi^*_{ni}(t_{ni}) = 1, \quad \varphi_{*ni}(t_{ni}) = \varphi^*_{ni}(t_{n,i+1}) = 0, \quad i = -n, \dots, -1, \end{aligned} \tag{2.4}$$

and for all $-n \le i \le n$,

$$0 \le \varphi_{*ni}(t) \le 1, \quad 0 \le \varphi^*_{ni}(t) \le 1, \quad \text{and} \quad \varphi_{*ni}(t) + \varphi^*_{ni}(t) \equiv 1, \quad t \in I_{ni}. \tag{2.5}$$

Thus in particular,

$$\|\varphi_{*ni}\|_{L_\infty(I_{ni})} = \|\varphi^*_{ni}\|_{L_\infty(I_{ni})} = 1, \quad i = \pm 1, \dots, \pm n.$$

Also their derivatives satisfy

$$\begin{aligned} {\varphi^*_{ni}}' &= -\varphi'_{*ni} \quad \text{and} \\ {\varphi^*_{ni}}'' &= -\varphi''_{*ni}, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\|\varphi'_{*ni}\|_{L_\infty(I_{ni})} = \|{\varphi^*_{ni}}'\|_{L_\infty(I_{ni})} = 2|I_{ni}|^{-1}, \quad \text{and} \\ &\|\varphi''_{*ni}\|_{L_\infty(I_{ni})}, \|{\varphi^*_{ni}}''\|_{L_\infty(I_{ni})} \le 2^{\beta+1}|I_{ni}|^{-2}, \quad i = \pm 1, \dots, \pm n. \end{aligned} \tag{2.7}$$

For $x \in W^r_{p,\alpha}$ and $1 \le i \le n$, let $\pi_{*,r-1}(x;i;t)$ and $\pi^*_{r-1}(x;i;t)$, be the Taylor polynomials of degree $r-1$ of x, expanded respectively, about the left-hand and the right-hand endpoints of the interval I_{ni}, that is,

$$\begin{aligned} \pi_{*,r-1}(x;i;t) &:= \sum_{s=0}^{r-1} \frac{1}{s!} x^{(s)}(t_{n,i-1})(t - t_{n,i-1})^s, i = 1, \dots, n, \\ \pi^*_{r-1}(x;i;t) &:= \sum_{s=0}^{r-1} \frac{1}{s!} x^{(s)}(t_{ni})(t - t_{ni})^s, \quad i = 1, \dots, n-1. \end{aligned}$$

Symmetrically, for $-n \le i \le -1$, let $\pi_{*,r-1}(x;i;t)$, $i = -n, \dots, -1$, and $\pi^*_{r-1}(x;i;t)$, $i = -n+1, \dots, -1$ denote the Taylor polynomials of degree $r-1$ of x, expanded respectively, about the right-hand and the left-hand endpoints of the interval I_{ni}.

Then the function

$$\sigma_{r,n}(x;t) := \begin{cases} \pi_{*,r-1}(x;i;t)\varphi_{*ni}(t) + \pi^*_{r-1}(x;i;t)\varphi^*_{ni}(t), \quad t \in I_{ni}, \\ \quad i = \pm 1, \dots, \pm(n-1), \\ \pi_{*,r-1}(x;\pm n;t), \quad t \in I_{n,\pm n}, \end{cases} \tag{2.8}$$

is in $C^1(I)$, and it is a polynomial of degree $\le r+1$ on each interval of the refined partition (in fact on the two end intervals it is a polynomial of degree $\le r-1$). Moreover, it was proved in [10] (see [10, (2.23) and (2.9)]) that

$$\|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q(I_{ni})} \le c\|x^{(r)}\rho^\alpha\|_{L_p(I_{ni})} n^{-r+\frac{1}{p}-\frac{1}{q}}, \quad i=\pm 1,\dots,\pm n, \tag{2.9}$$

where $c=c(r,\alpha,p,q)$ and

$$\sup_{x\in W^r_{p,\alpha}} \|x(\cdot)-\sigma_{r,n}(x;\cdot)\|_{L_q} \le cn^{-r+(\frac{1}{p}-\frac{1}{q})_+}, \tag{2.10}$$

where $c=c(r,\alpha,p,q)$.

If $r=1$, then clearly $\sigma_{1,n}(x;\cdot)\ge 0$ on I, for each $x\in\Delta^0_+W^r_{p,\alpha}$. But for $r>1$ we have to somewhat modify $\sigma_{r,n}(x;\cdot)$. Thus for $i=1,\dots,n-1$, we set

$$\eta_{n,i}(t) := \begin{cases} 0, & -1\le t\le t_{n,i-2}, \\ (t-t_{n,i-2})\left(t_{n,i-1}-t_{n,i-2}\right)^{-1}, & t_{n,i-2}<t<t_{n,i-1}, \\ 1, & t_{n,i-1}\le t\le t_{n,i}, \\ (t_{n,i+1}-t)\left(t_{n,i+1}-t_{n,i}\right)^{-1}, & t_{ni}<t<t_{n,i+1}, \\ 0, & t_{n,i+1}\le t\le 1, \end{cases},$$

and

$$\eta_{n,n}(t) := \begin{cases} 0, & -1\le t\le t_{n,n-1} \\ \left(\int_{t_{n,n-1}}^t (\rho(\tau))^{(r-\alpha-1)p'}d\tau\right)^{\frac{1}{p'}}, & t_{n,n-1}<t<1, \end{cases}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. For $i=-n,\dots,-1$ we set

$$\eta_{n,i}(t) := \eta_{n,-i}(-t).$$

Now define the correcting splines by

$$\kappa_{r,n}(x;t) := \frac{1}{(r-1)!}\sum_{i=\pm 1}^{\pm(n-1)} \left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}(t_{n,i})\left|I_{n,i}\right|^{r-\frac{1}{p}}\eta_{n,i}(t)$$
$$+\frac{1}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{n,-n})}\eta_{n,-n}(t)+\frac{1}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{nn})}\eta_{n,n}(t).$$

And finally set

$$\dot{\sigma}_{1,n}(x;t) := \sigma_{1,n}(x;t), \quad \dot{\sigma}_{r,n}(x;t) := \sigma_{r,n}(x;t) + \kappa_{r,n}(x;t), \quad r > 1, \quad t \in I.$$

It is easy to see that the spline $\dot{\sigma}_{r,n}(x;\cdot)$ is continuous on I, and it is a polynomial of degree $\le r+1$ in each interval $[t_{n,i-1}, \bar{t}_{ni}]$ and $[\bar{t}_{ni}, t_{ni}]$, $1 \le i \le n-1$, and in each interval $[\bar{t}_{ni}, t_{n,i+1}]$ and $[t_{ni}, \bar{t}_{ni}]$, $-n+1 \le i \le -1$. Also, in the end intervals $I_{n,\pm n}$, it is the sum of a polynomial of degree $\le r$ and the function $\eta_{n,\pm n}$. Hence if we denote the collection of such functions by $\dot{\Sigma}_{r,n}$, then $\dim \dot{\Sigma}_{r,n} \le 4(r+1)n$.

We will show that $\dot{\sigma}_{rn}(x;t) \ge 0$, $t \in I$ and that

$$\sup_{x \in \Delta^0_+ W^r_{p,\alpha}(I)} \|x(\cdot) - \dot{\sigma}_{r,n}(x;\cdot)\|_{L_q(I)} \le c n^{-r+(\frac{1}{p}-\frac{1}{q})_+}, \tag{2.11}$$

where $c = c(r,\alpha,p,q)$.

Indeed, on each interval I_{ni}, $i = \pm 1, \dots, \pm(n-1)$

$$\begin{aligned}\dot{\sigma}_{r,n}(x;t) &= \pi_{*,r-1}(x;i;t)\varphi_{*ni}(t) + \pi^*_{r-1}(x;i;t)\varphi^*_{ni}(t) + \kappa_{r,n}(x;t)\\ &\ge \pi_{*,r-1}(x;i;t)\varphi_{*ni}(t) + \pi^*_{r-1}(x;i;t)\varphi^*_{ni}(t)\\ &+ \frac{1}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}(t_{n,i})\left|I_{n,i}\right|^{r-\frac{1}{p}}\eta_{n,i}(t)\\ &\ge 0,\end{aligned}$$

since by (2.5), Taylor's formula and Hölder's inequality we get for $t \in I_{ni}$,

$$\begin{aligned}&\pi_{*,r-1}(x;i;t)\varphi_{*ni}(t) + \pi^*_{r-1}(x;i;t)\varphi^*_{ni}(t)\\ &= x(t) - (x(t) - \pi_{*,r-1}(x;i;t))\,\varphi_{*ni}(t) - \left(x(t) - \pi^*_{r-1}(x;i;t)\right)\varphi^*_{ni}(t)\\ &\ge -\left|x(t) - \pi_{*,r-1}(x;i;t)\right|\varphi_{*ni}(t) - \left|x(t) - \pi^*_{r-1}(x;i;t)\right|\varphi^*_{ni}(t)\\ &\ge -\frac{1}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}(t_{n,i})\left|I_{n,i}\right|^{r-\frac{1}{p}}.\end{aligned}$$

Here we have used the fact that $x(t) \ge 0$, $t \in I$.

Similarly, on the interval I_{nn} we have

$$\begin{aligned}\dot{\sigma}_{r,n}(x;t) &= \pi_{*,r-1}(x;i;t) + \kappa_{r,n}(x;t)\\ &\ge \pi_{*,r-1}(x;i;t) + \frac{1}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{nn})}\left(\int_{t_{n,n-1}}^t (\rho(\tau))^{(r-\alpha-1)p'}d\tau\right)^{\frac{1}{p'}}\\ &\ge 0,\end{aligned}$$

since by Taylor's formula and Hölder's inequality we get for $t \in I_{nn}$,

$$\pi_{*,r-1}(x;i;t) = x(t) - (x(t) - \pi_{*,r-1}(x;i;t)) \geq -|x(t) - \pi_{*,r-1}(x;i;t)|$$
$$\geq -\frac{1}{(r-1)!}\|x^{(r)}\rho^{\alpha}\|_{L_p(I_{nn})}\left(\int_{t_{n,n-1}}^{t}(\rho(\tau))^{(r-\alpha-1)p'}d\tau\right)^{\frac{1}{p'}}.$$

The proof for $I_{n,-n}$ is the same.

The proof of (2.10) (see the proof of [10, (2.9)]) readily yields

$$\sup_{x\in\Delta_+^0 W_{p,\alpha}^r(I)}\|\kappa_{r,n}(x;\cdot)\|_{L_q(I)} \leq cn^{-r+(\frac{1}{p}-\frac{1}{q})_+},$$

and this in turn together with (2.9) implies (2.11). This completes the proof of (1.1).

If the inequality $r-\alpha-\frac{1}{p} > 0$ is valid, then we can improve (2.1) for the cases $2 \leq p < q \leq \infty$ and $1 \leq p < 2 < q \leq \infty$. Indeed, under this condition $W_{p,\alpha}^r \subset L_\infty$, so given $x \in \Delta_+^0 W_{p,\alpha}^r$, let $M^n = M^n(I)$ be any linear manifold in L_∞ such that $\dim M^n \leq n$ and

$$\inf_{y\in M^n}\|x-y\|_{L_\infty} \leq cn^{-r+(\frac{1}{p}-\frac{1}{2})_+},$$

where $c = c(r,\alpha,p)$. Such a linear manifold is guaranteed by [10, Theorem 1] (and we actually know that it may be taken as a subspace of continuous splines). Then there exists $C = C(r,\alpha,p)$ and $y_x \in M^n$ such that

$$\|x-y_x\|_{L_\infty} \leq Cn^{-r+(\frac{1}{p}-\frac{1}{2})_+}.$$

If we set $\dot{y}_x(t) := y_x(t) + Cn^{-r+(\frac{1}{p}-\frac{1}{2})_+}$, $t \in I$, then clearly, $\dot{y}_x(t) \geq 0$, $t \in I$, and

$$\|x-\dot{y}_x\|_{L_\infty} \leq 2Cn^{-r+(\frac{1}{p}-\frac{1}{2})_+}.$$

Hence we have proved the existence of a linear manifold M^{n+1} in L_∞ such that $\dim M^{n+1} \leq n+1$, $M^{n+1}\cap\Delta_+^0 L_\infty \neq \emptyset$ and

$$E\left(\Delta_+^0 W_{p,\alpha}^r, M^{n+1}\cap\Delta_+^0 L_\infty\right)_{L_\infty} \leq 2Cn^{-r+(\frac{1}{p}-\frac{1}{2})_+}.$$

This completes the proof of (1.2).

In order to conclude the proof of Theorem 1 we take $(r, \alpha, p) = (1, 0, 1)$ and $2 < q < \infty$. If $x \in \Delta_+^0 W_1^1$, let $\sigma_{1,n}(x; \cdot)$ be the spline defined in (2.7) which, as we recall, is nonnegative and satisfies (2.10), namely,

$$\|x(\cdot) - \sigma_{1,n}(x; \cdot)\|_{L_q} \le cn^{-\frac{1}{q}}, \quad 2 < q < \infty \tag{2.12}$$

where $c = c(q)$.

For $n > 1$ let $\Sigma_{1,n}^0$ be the space of continuous piecewise quadratic polynomials $\zeta \in C(I)$, on the refined partition. Then $\dim \Sigma_{1,n}^0 = 8n + 1$. For $n = 1$ we take $\Sigma_{1,1}^0$ to be the space of constants. We are going to prove that for each $n \ge 1$ and $2 < q < \infty$ there is an integer $a = a(q) > 0$ such that a subspace $\Sigma_{1,a2^n} \subseteq \Sigma^0_{1,2^{\lceil \frac{q}{2} n \rceil}}$, of dimension $\dim \Sigma_{1,a2^n} \le a2^n$, exists, for which

$$\sup_{x \in \Delta_+^0 W_{1,0}^1} \inf_{\sigma \in \Sigma_{1,a2^n}} \|\sigma_{1,2^{\lceil \frac{q}{2} n \rceil}}(x; \cdot) - \sigma(\cdot)\|_{L_\infty(I)} \le cn^{\frac{3}{2}} 2^{-\frac{n}{2}},$$

where $c = c(q)$ and $\lceil u \rceil$ denotes the integer ceiling of u.

The space $\Sigma_{1,n}^0$ was considered in [10] as one of the spaces of splines $\Sigma_{r,n}^0$, $r \in \mathbb{N}$. A one-to-one correspondence between the spaces $\Sigma_{r,n}^0$ and $\mathbb{R}^{2n(r)+1}$, $n(r) := 2n(r + 1)$ was given by the invertible discretization operator

$$A_{r,\beta,q,n} : \Sigma_{r,n}^0 \ni \zeta \to y = \big(y_{-n(r)}, \ldots, y_{-1}, y_0, y_1, \ldots, y_{n(r)}\big) \in \mathbb{R}^{2n(r)+1},$$

where

$$y_j = n(r)^{-\frac{\beta}{q}} (n(r) - |j| + 1)^{\frac{\beta - 1}{q}} \zeta(t_{n(r),j}), \quad j = 0, \pm 1, \ldots, \pm n(r). \tag{2.13}$$

The inverse operator is

$$A_{r,\beta,q,n}^{-1} : \mathbb{R}^{2n(r)+1} \ni y = \big(y_{-n(r)}, \ldots, y_{-1}, y_0, y_1, \ldots, y_{n(r)}\big) \to \zeta \in \Sigma_{r,n}^0,$$

where ζ is uniquely defined by the interpolation equations

$$\zeta(t_{n(r),j}) = n(r)^{\frac{\beta}{q}} (n(r) - |j| + 1)^{-\frac{\beta - 1}{q}} y_j, \quad j = 0, \pm 1, \ldots, \pm n(r).$$

It was proved that the norms $\|A_{r,\beta,q,n}\zeta\|_{l_q^{2n(r)+1}}$ and $\|\zeta\|_{L_q}$ are equivalent, the equivalence constants depending only on p, q, r and α.

If $r=1$, $\alpha=0$ and $p=1$, then $n(1)=4n$ and $\beta=1$, so that (2.13) becomes the much simpler

$$y_j = n(r)^{-\frac{1}{q}}\zeta(t_{n(r),j}), \quad j=0,\pm1,\dots,\pm n(r),$$

and following the above mentioned proof, it is readily seen that there exist absolute constants $c_1>0$ and c_2 such that

$$c_1 n^{\frac{1}{q}}\|A_{1,1,q,n}\zeta\|_{l_\infty^{8n+1}} \le \|\zeta\|_{L_\infty} \le c_2 n^{\frac{1}{q}}\|A_{1,1,q,n}\zeta\|_{l_\infty^{8n+1}} \tag{2.14}$$

for all $\zeta\in\Sigma_{1,n}^0$.

Fix $n\in\mathbb{N}$. Then each $\sigma_{1,2^N}(x;t)$ can be written as

$$\sigma_{1,2^N}(x;t) = \sigma_{r,1}(x;t) + \sum_{\nu=1}^{N}\big(\sigma_{1,2^\nu}(x;t) - \sigma_{r,2^{\nu-1}}(x;t)\big), \quad t\in I. \tag{2.15}$$

We proved in [10] that for every $x\in W_1^1$, the mapping $A_{1,1,q,2^\nu}$ maps $\big(\sigma_{1,2^\nu}(x;\cdot) - \sigma_{1,2^{\nu-1}}(x;\cdot)\big)$ into the ball $c2^{-\frac{1}{q}\nu}B_1^{82^\nu+1}$.

Let $m_0:=1$ and the integers $m_\nu\le 82^\nu+1$, $\nu=1,2,\dots,N$, be prescribed and let $L^{m_\nu}, \nu=1,2,\dots,N$ be any subspaces of $\mathbb{R}^{82^\nu+1}$, $\dim L^{m_\nu}{=}m_\nu$. Set

$$\Sigma^{m_0} := \Sigma_{1,1}^0, \quad \Sigma^{m_\nu} := A_{1,1,q,2^\nu}^{-1}L^{m_\nu}, \quad \nu=1,2,\dots,N.$$

Then clearly $\Sigma^{m_\nu}\subset\Sigma_{1,2^\nu}^0$ and $\dim\Sigma^{m_\nu}=m_\nu$, $\nu=0,1,2,\dots,N$. Denote

$$\Sigma^{m_0,\dots,m_N} := \operatorname{span}\big(\cup_{\nu=0}^N\Sigma^{m_\nu}\big).$$

Then $\Sigma^{m_0,\dots,m_N}\subset\Sigma_{1,2^N}^0$ and $\dim\Sigma^{m_0,\dots,m_N}\le m_0+\dots+m_N$.

Now take L^{m_ν} to be such that

$$E\big(B_1^{82^\nu+1}, L^{m_\nu}\big)_{l_\infty^{82^\nu+1}} \le 2d_{m_\nu}\big(B_1^{82^\nu+1}\big)_{l_\infty^{82^\nu+1}}, \quad \nu=1,\dots,N.$$

Then by (2.14) and (2.15),

$$\sup_{x\in\Delta_+^0 W_1^1} E\big(\sigma_{1,2^N}(x;\cdot),\Sigma^{m_0,\dots,m_N}\big)_{L_\infty} \le c\sum_{\nu=1}^{N} d_{m_\nu}\big(B_1^{82^\nu+1}\big)_{l_\infty^{82^\nu+1}},$$

where $c=c(q)$.

If we put $N:=\lceil \frac{q}{2}n\rceil$, and set

$$m_\nu := 82^\nu+1,\quad \nu=1,\dots,n-1,$$

$$m_\nu := \lceil n^{-1}2^n\rceil,\quad \nu=n,\dots,N,$$

then $m_0+m_1+\cdots+m_N\le a2^n$, where $a=a(q)\in\mathbb{N}$. We apply Lemma K and obtain

$$\begin{aligned}\sup_{x\in\Delta_+^0 W_1^1} E\big(\sigma_{1,2^{\lceil\frac{q}{2}n\rceil}}(x;\cdot),\Sigma^{m_0,\dots,m_N}\big)_{L_\infty} &\le \sum_{\nu=1}^{N} d_{m_\nu}\big(B_1^{82^\nu+1}\big)_{l_\infty^{82^\nu+1}}\\ &= \sum_{\nu=n}^{N} d_{m_\nu}\big(B_1^{82^\nu+1}\big)_{l_\infty^{82^\nu+1}}\\ &\le c\sum_{\nu=n}^{N} n^{\frac12}2^{-\frac n2} \le cn^{\frac32}2^{-\frac n2},\end{aligned}$$

where $c=c(q)$.

Given $x\in\Delta_+^0W_1^1$, let $\sigma(x;\cdot)\in\Sigma^{m_0,\dots,m_N}$ be such that

$$\Big\|\sigma_{1,2^{\lceil\frac{q}{2}n\rceil}}(x;\cdot)-\sigma(x;\cdot)\Big\|_{L_\infty}\le 2cn^{\frac32}2^{-\frac n2},$$

and set

$$\dot\sigma(x;t):=\sigma(x;t)+2cn^{\frac32}2^{-\frac n2}.$$

Then we have $\dot\sigma(x;t)\ge\sigma_{1,2^{\lceil\frac{q}{2}n\rceil}}(x;t)\ge 0$, $t\in I$, and

$$\Big\|\sigma_{1,2^{\lceil\frac{q}{2}n\rceil}}(x;\cdot)-\dot\sigma(x;\cdot)\Big\|_{L_\infty}\le 4cn^{\frac32}2^{-\frac n2}.$$

Combining this with (2.12), yields

$$\begin{aligned}\|x(\cdot)-\dot\sigma(x;\cdot)\|_{L_q} &\le \|x(\cdot)-\sigma_{1,2^{\lceil\frac{q}{2}n\rceil}}(x;\cdot)\|_{L_q}+\|\sigma_{1,2^{\lceil\frac{q}{2}n\rceil}}(x;\cdot)-\dot\sigma(x;\cdot)\|_{L_q}\\ &\le \|x(\cdot)-\sigma_{1,2^{\lceil\frac{q}{2}n\rceil}}(x;\cdot)\|_{L_q}+\|\sigma_{1,2^{\lceil\frac{q}{2}n\rceil}}(x;\cdot)-\dot\sigma(x;\cdot)\|_{L_\infty}\\ &\le c2^{-\frac n2}+cn^{\frac32}2^{-\frac n2}\le cn^{\frac32}2^{-\frac n2},\end{aligned}$$

where $c=c(q)$. Now, the upper bound in (1.3) follows by standard technique. This concludes the proof of Theorem 1. □

§3. Monotonicity preserving widths of the classes $\Delta_+^1 W_{p,\alpha}^r$ in L_q

We begin with

Lemma 1. *Let J be a finite interval, and let $\{t_i\}_{i=1}^r$ be a collection of $r \in \mathbb{N}$ disjoint points in J. Set $\delta_1 := 1$ and $\delta_r := \min\{|t_i - t_j|, i \neq j\}$, if $r > 1$. Then for any function x such that $x^{(r)} \in L_1(J)$,*

$$\|x\|_{L_\infty(J)} \leq \frac{r}{(r-1)!}\left(\frac{|J|}{\delta_r}\right)^{\frac{r(r-1)}{2}}\left(\max_{1\leq i\leq r}\big|x(t_i)\big| + \frac{|J|^{r-1}}{(r-1)!}\big\|x^{(r)}\big\|_{L_1(J)}\right). \tag{3.1}$$

Proof. Fix $t \in J$. Then integration by parts yields the system of r equations for the r unknowns $x^{(s)}(t)$, $s = 0, \dots, r-1$,

$$\sum_{s=0}^{r-1} x^{(s)}(t)\big(t_i - t\big)^s = x\big(t_i\big) - \int_t^{t_i} x^{(r)}(\tau)\big(t_i - \tau\big)^{r-1} d\tau, \quad i = 1, \dots, r,$$

which readily yields (3.1) for $r = 1$. For $r > 1$, we are interested in the solution of the system only for $s = 0$, that is,

$$x(t) = W_r^{-1}\sum_{i=1}^r W_{r,i}\left(x\big(t_i\big) - \int_t^{t_i} x^{(r)}(\tau)\big(t_i - \tau\big)^{r-1} d\tau\right), \tag{3.2}$$

where W_r is the determinant of this system and $W_{r,i}$ are the co-factors. Evidently, W_r is the Vandermonde determinant,

$$W_r = \prod_{1\leq i<j\leq r}\big(t_j - t_i\big),$$

and

$$W_{r,i} = (-1)^{i+1}\prod_{\substack{1\leq j\leq r\\ j\neq i}}\big(t_j - t\big)\prod_{\substack{1\leq k<l\leq r\\ k\neq i, l\neq i}}\big(t_l - t_k\big).$$

Therefore,

$$\big|W_r\big| \geq (r-1)!\,\delta_r^{\frac{r(r-1)}{2}}, \quad \text{and} \quad \big|W_{r,i}\big| \leq |J|^{\frac{r(r-1)}{2}},$$

so that (3.1) readily follows from (3.2). □

It is well known (see, e.g., [24]) that the distance $E(x,L)_X$, between a vector $x \in X$ and a linear subspace $L \subset X$, is given by

$$E(x,L)_X = \sup_{x^* \in X^*, \|x^*\|_{X^*}=1, x^* \perp L} < x^*, x >,$$

where X^* denotes the dual of X. Also the distance $E(x^*, L^*)_{X^*}$, between $x^* \in X^*$ and a linear subspace $L^* \subset X^*$ is given by

$$E(x^*, L^*)_{X^*} = \sup_{x \in X, \|x\|_X = 1, x \perp L^*} < x^*, x > .$$

This immediately implies the following well known result which we quote for the sake of reference later on.

Lemma 2. *Let v be a nonzero vector in $\mathbb{R}^n$, $n > 1$ and let $\mathbb{R}^{n-1}(v)$ denote the $(n-1)$-dimensional hyperplane, perpendicular to v. If $M^{n-1}(v;z) := z + \mathbb{R}^{n-1}(v)$, then for each $x \in \mathbb{R}^n$ and any $1 \le q \le \infty$,*

$$E\big(x; M^{n-1}(v;z)\big)_{l_q^n} = \|v\|_{l_{q'}^n}^{-1} | < x - z, v > |,$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

In the sequel we need the standard notation for the unit vectors along the axes, namely,

$$E^n := \big\{e^{(i)}\big\}_{i=1}^n, \quad e^{(i)} := (0, \dots, 1, \dots, 0), \tag{3.3}$$

where the 1 is standing in the ith entry, and

$$\tilde{E}^n := \big\{\tilde{e}^{(i)}\big\}_{i=1}^n, \quad \tilde{e}^{(1)} := (1,1,\dots,1), \tilde{e}^{(2)} := (0,1,\dots,1), \dots, \tilde{e}^{(n)} := (0,\dots,0,1). \tag{3.4}$$

Finally, we denote

$$e^{(0)} = \tilde{e}^{(0)} := \bar{0} := (0, \dots, 0).$$

We need the following lemma of Tikhomirov [23] (see also [12] or [19]).

Lemma T. *Let $n \in \mathbb{N}$, and let X be a real linear normed space of dimension* $\dim X > n$ *and $B \subset X$ its unit ball. Then $d_n(B)_X = 1$.*

Lemma 3. *Let $n > 1$ and denote $\delta B_1^n := \left\{x \mid x \in \mathbb{R}^n, \|x\|_{l_1^n} \le \delta\right\}$. Then for any $\delta_*, \delta^* >$ 0 one has*

$$d_{n-1}\left(\delta_* B_1^n, \delta^* B_1^n\right)_{l_\infty^n} = \max\left\{\delta_* - \frac{\delta^*}{2}, \frac{\delta_*}{n}\right\}. \tag{3.5}$$

Proof. The sets $\delta_* B_1^n$ and $\delta^* B_1^n$ are centrally symmetric convex sets. Therefore

$$d_{n-1}\left(\delta_* B_1^n, \delta^* B_1^n\right)_{l_\infty^n} = \inf_{L^{n-1} \subset l_\infty^n} \sup_{x \in \delta_* B_1^n} \inf_{y \in L^{n-1} \cap \delta^* B_1^n} \|x - y\|_{l_\infty^n},$$

where L^{n-1} is a subspace of dimension $n-1$, in l_∞^n.

We begin with the lower bound. Suppose to the contrary, that for some nonzero vector v,

$$E\left(\delta_* B_1^n, \mathbb{R}^{n-1}(v) \cap \delta^* B_1^n\right)_{l_\infty^n} < \delta_* - \frac{\delta^*}{2}.$$

Let $|v_{i_0}| := \max_{1 \le i \le n} |v_i|$ and let $x^* = (x_1^*, \ldots, x_n^*)$, be the element of best approximation of $\delta_* e^{(i_0)}$ from the set $\mathbb{R}^{n-1}(v) \cap \delta^* B_1^n$, that is, $\|\delta_* e^{(i_0)} - x^*\|_{l_\infty^n} = E\left(\delta_* e^{(i_0)}, \mathbb{R}^{n-1}(v) \cap \delta^* B_1^n\right)_{l_\infty^n}$. Since v is the normal, then we have $-x_{i_0}^* v_{i_0} = \sum_{i \ne i_0} x_i^* v_i$, so that

$$\left|x_{i_0}^* v_{i_0}\right| = \left|\sum_{i \ne i_0} x_i^* v_i\right| \le |v_{i_0}| \sum_{i \ne i_0} |x_i^*|,$$

and it follows that $\left|x_{i_0}^*\right| \le \sum_{i \ne i_0} |x_i^*|$. At the same time

$$\delta_* - \left|x_{i_0}^*\right| \le \left|\delta_* - x_{i_0}^*\right| \le \|\delta_* e^{(i_0)} - x^*\|_{l_\infty^n} < \delta_* - \frac{\delta^*}{2}.$$

Hence

$$\sum_{i \ne i_0} |x_i^*| \ge \left|x_{i_0}^*\right| > \frac{\delta^*}{2},$$

implying

$$\|x^*\|_{l_1^n} = \sum_{i=1}^n |x_i^*| > \delta^*,$$

thus contradicting $x^* \in \delta^* B_1^n$. Therefore,

$$d_{n-1}\left(\delta_* B_1^n, \delta^* B_1^n\right)_{l_\infty^n} \geq \delta_* - \frac{\delta^*}{2}. \tag{3.6}$$

Evidently, the polytope $\delta_* B_1^n$ contains the cube $\frac{\delta_*}{n} B_\infty^n$. Thus, applying Lemma T to $X = l_\infty^n$, we obtain

$$d_{n-1}\left(\delta_* B_1^n, \delta^* B_1^n\right)_{l_\infty^n} \geq d_{n-1}\left(\delta_* B_1^n\right)_{l_\infty^n} \geq \frac{\delta_*}{n} d_{n-1}\left(B_\infty^n\right)_{l_\infty^n} = \frac{\delta_*}{n},$$

which combined with (3.6) completes the proof of the lower bound in (3.5).

In order to prove the upper bound in (3.5), we first assume that $\delta_* - \frac{\delta^*}{2} > \frac{\delta_*}{n}$, and note that this implies that $\frac{\delta_*}{n} > \frac{\delta^*}{2(n-1)}$. Let $\pm\epsilon^{(i)}$, $1 \leq i \leq n$, denote the vertices of $\delta_* B_1^n$ and take $x^{(i)} := \left(x_1^{(i)}, \ldots, x_n^{(i)}\right) \in \mathbb{R}^{n-1}(\tilde{e}^{(1)}) \cap \delta^* B_1^n$, so that $x_i^{(i)} := \frac{\delta^*}{2}$, $x_j^{(i)} := -\frac{\delta^*}{2(n-1)}$, $j \neq i$. Then clearly

$$\left\|\pm\epsilon^{(i)} - \pm x^{(i)}\right\|_{l_\infty^n} \leq \max\left\{\delta_* - \frac{\delta^*}{2}, \frac{\delta^*}{2(n-1)}\right\} = \delta_* - \frac{\delta^*}{2}. \tag{3.7}$$

Otherwise, take $x_i^{(i)} := \delta_* - \frac{\delta_*}{n}$, and $x_j^{(i)} := -\frac{\delta_*}{n}$, $j \neq i$, $1 \leq j \leq n$. Since in this case $2\frac{n-1}{n}\delta_* \leq \delta^*$, it follows that $\pm x^{(i)} \in \mathbb{R}^{n-1}(\tilde{e}^{(1)}) \cap \delta^* B_1^n$, and

$$\left\|\pm\epsilon^{(i)} - \pm x^{(i)}\right\|_{l_\infty^n} = \frac{\delta_*}{n}.$$

Combining with (3.7), we conclude that

$$E\left(\delta_* B_1^n, \mathbb{R}^{n-1}(\tilde{e}^{(1)}) \cap \delta^* B_1^n\right)_{l_\infty^n} \leq \max\left\{\delta_* - \frac{\delta^*}{2}, \frac{\delta_*}{n}\right\}.$$

This establishes the upper bound in (3.5) and concludes the proof of Lemma 3. $\square$

Finally, for $Y := \left\{y^{(i)}\right\}_{i=1}^n$, a system of vectors in the space X, and for $1 \leq p \leq \infty$, the set

$$S_p^+(Y) := \left\{y \mid y := \sum_{i=1}^n a_i y^{(i)},\ a = (a_1, \ldots, a_n) \in \mathbb{R}^n,\ a_i \geq 0,\ i = 1, \ldots, n,\ \|a\|_{l_p^n} \leq 1\right\},$$

is called the positive p-sector over the system Y in X, and

$$B_p(Y) := \left\{y \mid y := \sum_{i=1}^n a_i y^{(i)},\ a = (a_1, \ldots, a_n) \in \mathbb{R}^n, \quad \|a\|_{l_p^n} \leq 1\right\},$$

is called the p-ball over the system Y in X.

Lemma 4. *Let $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, be so that $m+1 < n$, and let $1 \le p \le q \le \infty$. Let $\tilde{E}^n$ be the system from (3.4), and denote by*

$$\Delta_+^1 := \{x = (x_1, \dots, x_n) \mid x_1 \le \cdots \le x_n\},$$

the cone of vectors x with nondecreasing coordinates in $\mathbb{R}^n$. Then

$$d_m\big(S_p^+(\tilde{E}^n), \Delta_+^1\big)_{l_q^n} \ge \frac{1}{8}.$$

Proof. First note that

$$\begin{aligned} S_1^+(\tilde{E}^n) &= \{x = (x_1, \dots, x_n) \mid 0 \le x_1, 0 \le x_2 - x_1, \dots, 0 \le x_n - x_{n-1}, x_n \le 1\} \\ &= \{x = (x_1, \dots, x_n) \mid 0 \le x_1 \le x_2 \le \cdots \le x_n \le 1\}, \end{aligned}$$

and that the vectors $\tilde{e}^{(i)}$, $i = 0, \dots, n$, are the vertices of this n-dimensional pyramid. Evidently $S_1^+(\tilde{E}^n) \subset \Delta_+^1$. Also, since for $1 \le p \le q \le \infty$, $\|x\|_{l_1^n} \ge \|x\|_{l_p^n} \ge \|x\|_{l_q^n} \ge \|x\|_{l_\infty^n}$, it follows that $S_1^+(\tilde{E}^n) \subseteq S_p^+(\tilde{E}^n)$. Hence

$$d_m\big(S_p^+(\tilde{E}^n), \Delta_+^1\big)_{l_q^n} \ge d_m\big(S_1^+(\tilde{E}^n), \Delta_+^1\big)_{l_\infty^n},$$

and it suffices to consider $S_1^+(\tilde{E}^n)$.

Let M^m be an arbitrary m-dimensional linear manifold and let $L^{m+1} \supseteq M^m$ be a subspace of dimension $\dim L^{m+1} \le m+1$ in $\mathbb{R}^n$. Then clearly

$$E\big(S_1^+(\tilde{E}^n), M^m \cap \Delta_+^1\big)_{l_\infty^n} \ge E\big(S_1^+(\tilde{E}^n), L^{m+1} \cap \Delta_+^1\big)_{l_\infty^n}. \tag{3.8}$$

Fix $0 < \epsilon < \frac{1}{2}$ and let

$$S_{\epsilon,1}^+(\tilde{E}^n) := \{x = (x_1, \dots, x_n) \mid \epsilon \le x_1 \le x_2 \le \cdots \le x_n \le 1 - \epsilon\} \subset S_1^+(\tilde{E}^n).$$

Then

$$E\big(S_1^+(\tilde{E}^n), L^{m+1} \cap \Delta_+^1\big)_{l_\infty^n} \ge E\big(S_{\epsilon,1}^+(\tilde{E}^n), L^{m+1} \cap \Delta_+^1\big)_{l_\infty^n}. \tag{3.9}$$

Also,

$$S^{+}_{\epsilon,1}\big(\tilde{E}^n\big) = (1-2\epsilon)S^{+}_{1}\big(\tilde{E}^n\big) + \epsilon\tilde{e}^{(1)}, \tag{3.10}$$

and the vertices of the n-dimensional pyramid $S^{+}_{\epsilon,1}\big(\tilde{E}^n\big)$ are $\tilde{e}^{(i)}_{\epsilon} := \epsilon\tilde{e}^{(1)} + (1-2\epsilon)\tilde{e}^{(i)}$, $i = 0, 1, \ldots, n$.

For $x^0 \in S^{+}_{\epsilon,1}\big(\tilde{E}^n\big)$, we have

$$\begin{aligned} &E\big(x^0, L^{m+1} \cap \Delta^1_+\big)_{l^n_\infty} \\ &= \min\Big\{E\big(x^0, L^{m+1} \cap \big(\Delta^1_+ \setminus S^{+}_1\big(\tilde{E}^n\big)\big)\big)_{l^n_\infty}, E\big(x^0, L^{m+1} \cap S^{+}_1\big(\tilde{E}^n\big)\big)\big)_{l^n_\infty}\Big\}. \end{aligned} \tag{3.11}$$

Therefore we may deal separately with each term on the right. We begin with the left-hand term. By Lemma 2 with $q = \infty$, we obtain that

$$E\big(x^0, \mathbb{R}^{n-1}(e^{(1)})\big)_{l^n_\infty} = x^0_1 \geq \epsilon \quad \text{and} \quad E\big(x^0, M^{n-1}(e^{(n)}, e^{(n)})\big)_{l^n_\infty} = |x^0_n - e^{(n)}_n| \geq \epsilon, \tag{3.12}$$

where the $e^{(i)}$'s are from (3.3), and

$$\begin{aligned} \mathbb{R}^{n-1}(e^{(1)}) &= \{x = (x_1, \ldots, x_n) \mid x_1 = 0\} \quad \text{and} \\ M^{n-1}(e^{(n)}, e^{(n)}) &= e^{(n)} + \mathbb{R}^{n-1}(e^{(n)}) = \{x = (x_1, \ldots, x_n) \mid x_n = 1\}. \end{aligned}$$

So, if we denote the half-spaces

$$\begin{aligned} \mathbb{R}^{n-1}_{-}(e^{(1)}) &:= \{x = (x_1, \ldots, x_n) \mid x_1 < 0\} \quad \text{and} \\ \mathbb{R}^{n-1}_{-}(e^{(n)}; e^{(n)}) &:= \{x = (x_1, \ldots, x_n) \mid x_n > 1\}, \end{aligned}$$

then by virtue of (3.12) we have

$$E\big(x^0, \mathbb{R}^{n-1}_{-}(e^{(1)}) \cup \mathbb{R}^{n-1}_{-}(e^{(n)}; e^{(n)})\big)_{l^n_\infty} \geq \epsilon.$$

Since $\Delta^1_+ \setminus S^{+}_1\big(\tilde{E}^n\big) = \Delta^1_+ \cap \big(\mathbb{R}^{n-1}_{-}(e^{(1)}) \cup \mathbb{R}^{n-1}_{-}(e^{(n)}; e^{(n)})\big)$, this implies

$$\begin{aligned} &E\big(x^0, L^{m+1} \cap \big(\Delta^1_+ \setminus S^{+}_1\big(\tilde{E}^n\big)\big)\big)_{l^n_\infty} \\ &= E\big(x^0, L^{m+1} \cap \big(\Delta^1_+ \cap \big(\mathbb{R}^{n-1}_{-}(e^{(1)}) \cup \mathbb{R}^{n-1}_{-}(e^{(n)}; e^{(n)})\big)\big)\big)_{l^n_\infty} \\ &\geq E\big(x^0, \mathbb{R}^{n-1}_{-}(e^{(1)}) \cup \mathbb{R}^{n-1}_{-}(e^{(n)}; e^{(n)})\big)_{l^n_\infty} \\ &\geq \epsilon. \end{aligned}$$

Hence by (3.11)

$$E\big(x^0, L^{m+1} \cap \Delta^1_+\big)_{l^n_\infty} \geq \min\Big\{\epsilon, E\big(x^0, L^{m+1} \cap S^+_1\big(\tilde{E}^n\big)\big)_{l^n_\infty}\Big\},$$

which in turn implies

$$E\big(S^+_{\epsilon,1}\big(\tilde{E}^n\big), L^{m+1} \cap \Delta^1_+\big)_{l^n_\infty} \geq \min\Big\{\epsilon, E\big(S^+_{\epsilon,1}\big(\tilde{E}^n\big), L^{m+1} \cap S^+_1\big(\tilde{E}^n\big)\big)\big)_{l^n_\infty}\Big\} \tag{3.13}$$

Now we consider the right-hand term in (3.13). Let the operator $\tilde{T}_n : \mathbb{R}^n \ni x \to y \in \mathbb{R}^n$ be defined by

$$y_1 = x_1, \quad y_2 = x_2 - x_1, \dots, y_n = x_n - x_{n-1},$$

so that it is invertible and its inverse is given by

$$x_i = \sum_{j=1}^{i} y_j, \quad i = 1, \dots, n.$$

It follows that $\tilde{T}_n \tilde{e}^{(i)} = e^{(i)}$, and $\tilde{T}_n \tilde{e}^{(i)}_\epsilon = \epsilon e^{(1)} + (1-2\epsilon)e^{(i)} =: e^{(i)}_\epsilon$, $i = 0, 1, \dots, n$. Therefore $\tilde{T}_n S^+_1\big(\tilde{E}^n\big) = S^+_1\big(E^n\big) =: S^+_1$, where E^n is from (3.3), and by (3.10), $\tilde{T}_n S^+_{\epsilon,1}\big(\tilde{E}^n\big) = \epsilon e^{(1)} + (1-2\epsilon)S^+_1 =: S^+_{\epsilon,1}\big(E^n\big)$.

Denote by $\tilde{T}_n l^n_\infty$ the space $\mathbb{R}^n$ with the norm

$$\|y\|_{\tilde{T}_n l^n_\infty} := \max\big\{|y_1|, |y_1 + y_2|, \dots, |y_1 + \cdots + y_n|\big\}.$$

Then

$$\begin{aligned} E\big(S^+_{\epsilon,1}\big(\tilde{E}^n\big), L^{m+1} \cap S^+_1\big(\tilde{E}^n\big)\big)_{l^n_\infty} &= E\big(S^+_{\epsilon,1}\big(E^n\big), \tilde{T}_n L^{m+1} \cap S^+_1\big)_{\tilde{T}_n l^n_\infty} \\ &\geq \frac{1}{2} E\big(S^+_{\epsilon,1}\big(E^n\big), \tilde{T}_n L^{m+1} \cap S^+_1\big)_{l^n_\infty} \\ &\geq \frac{1}{2} E\big(S^+_1, \tilde{T}_n L^{m+1} \cap S^+_1\big)_{l^n_\infty} - \epsilon, \end{aligned} \tag{3.14}$$

since the unit ball of $\tilde{T}_n l^n_\infty$ is contained in the cube $2B^n_\infty$ and $\max_{1\leq i\leq n} \|e^{(i)} - e^{(i)}_\epsilon\|_{l^n_\infty} = 2\epsilon$.

Now,

$$
\begin{aligned}
E\big(S_1^+, \tilde{T}_n L^{m+1} \cap S_1^+\big)_{l_\infty^n} &= E\big(-S_1^+ \cup S_1^+, \tilde{T}_n L^{m+1} \cap \big(-S_1^+ \cup S_1^+\big)\big)_{l_\infty^n} \\
&= E\big(B_1^n, \tilde{T}_n L^{m+1} \cap \big(-S_1^+ \cup S_1^+\big)\big)_{l_\infty^n} \\
&\geq E\big(B_1^n, \tilde{T}_n L^{m+1} \cap B_1^n\big)_{l_\infty^n} \qquad (3.15)\\
&\geq d_{n-1}\big(B_1^n, B_1^n\big)_{l_\infty^n} \\
&= \max\left\{\frac{1}{2}, \frac{1}{n}\right\} = \frac{1}{2},
\end{aligned}
$$

where for the last equation we applied Lemma 3 with $\delta_* = \delta^* = 1$. Taking $\epsilon = \frac{1}{8}$ and combining with (3.14) we conclude that

$$E\big(S_{\epsilon,1}^+\big(\tilde{E}^n\big), L^{m+1} \cap S_1^+\big(\tilde{E}^n\big)\big)_{l_\infty^n} \geq \frac{1}{8},$$

which together with (3.8), (3.9) and (3.13), yields

$$E\big(S_1^+\big(\tilde{E}^n\big), M^m \cap \Delta_+^1\big)_{l_\infty^n} \geq \frac{1}{8}.$$

Since M^m is an arbitrary linear manifold of dimension m, it follows that

$$d_m\big(S_1^+\big(\tilde{E}^n\big), \Delta_+^1\big)_{l_\infty^n} \geq \frac{1}{8}.$$

This completes the proof of Lemma 4. $\square$

We are ready for the proof of Theorem 2.

Proof of Theorem 2. We begin by proving the upper bound. Let $\sigma_{r,n}(x;\cdot)$ be the spline defined in (2.8). If $r = 1$, then for each $x \in \Delta_+^1 W_{p,\alpha}^1$, clearly $\sigma_{1,n}(x;\cdot)$ is nondecreasing and there is nothing to prove. If $r > 1$ and $x \in \Delta_+^1 W_{p,\alpha}^r$, then we have to modify $\sigma_{r,n}(x;\cdot)$. Let

$$m(r) = m(r, \alpha, p, q) := \lceil (r-1)2^{\beta+1} \rceil, \qquad (3.16)$$

and set

$$t_{n,i,k} := \begin{cases} 1 - \left(\frac{m(r)n - m(r)(i-1) - k}{m(r)n}\right)^\beta, & k = 0, 1, \ldots, m(r), \quad i = 1, \ldots, n, \\ -1 + \left(\frac{m(r)n + m(r)(i+1) - k}{m(r)n}\right)^\beta, & k = 0, 1, \ldots, m(r), \quad i = -1, \ldots, -n. \end{cases} \qquad (3.17)$$

Then

$$t_{n,i,0} = \begin{cases} t_{n,i-1}, & i = 1, \dots, n-1, \\ t_{n,i+1}, & i = -1, \dots, -n+1, \end{cases}$$

and

$$t_{n,i,m(r)} = t_{n,i}, \quad i = \pm 1, \dots, \pm n,$$

where the points t_{ni} are from (2.3). That is, the points $t_{n,i,0}$ and $t_{n,i,m(r)}$ are the endpoints of the intervals I_{ni}. Set

$$I_{n,i,k} := \begin{cases} \big[t_{n,i,k-1}, t_{n,i,k}\big], & k = 1, \dots, m(r), \quad i = 1, \dots, n-1 \\ \big[t_{n,i,k}, t_{n,i,k-1}\big], & k = 1, \dots, m(r), \quad i = -1, \dots, -n+1. \end{cases} \tag{3.18}$$

Thus the intervals $I_{n,i,k}$, $k = 1, \dots, m(r)$, form a partition of the interval I_{ni}, and it is readily seen that

$$\frac{1}{m(r)2^{\beta-1}}\big|I_{ni}\big| \le \big|I_{n,i,k}\big| \le \frac{2^{\beta-1}}{m(r)}\big|I_{ni}\big|, \quad i = \pm 1, \dots, \pm(n-1), \quad k = 1, \dots, m(r). \tag{3.19}$$

The first derivative x' is called *small* on I_{ni}, $1 \le |i| \le n-1$ if there exist at least $2r-3(\le m(r))$ subintervals I_{n,i,k_j}, each of which contains a point $t_{i,k_j} \in I_{n,i,k_j}$, such that

$$x'\big(t_{i,k_j}\big) \le \frac{2(r+3)}{(r-1)!}\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-1-\frac{1}{p}}. \tag{3.20}$$

Otherwise the first derivative is called *big* on that interval.

Let $1 \le i \le n-1$ and assume that the first derivative x' is small on I_{ni}. Then we replace $\sigma_{r,n}(x;\cdot)$ on I_{ni} by the linear function

$$\tilde{\sigma}_{r,n}(x;t) := \big[x(t_{n,i-1})\big(t_{ni} - t\big) + x(t_{n,i})\big(t - t_{n,i-1}\big)\big]\,\big|I_{ni}\big|^{-1}, \quad t \in I_{ni},$$

which interpolates x at the endpoints of I_{ni}.

If on the other hand, x' is big on I_{ni}, then there exist at most $2r-4$ subintervals I_{n,i,k_j}, $j = 1, \dots, m \le 2r-4$ (possibly none, then $m = 0$), and points t_{i,k_j} in them, for which (3.20) holds. We have to modify $\sigma_{r,n}(x;\cdot)$ on I_{ni}. Let

$$\xi_{ni}(t) := \begin{cases} \frac{2(r+3)}{(r-1)!}\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-1-\frac{1}{p}}, & t \in I_{n,i,k_j}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.21}$$

and set

$$(3.22)\qquad \tilde\kappa_{r,n,i}(x;t) := \int_{t_{n,i-1}}^{t} \xi_{ni}(\tau)d\tau - \int_{t_{n,i-1}}^{t_{ni}} \xi_{ni}(\tau)d\tau\big(t-t_{n,i-1}\big)\big|I_{ni}\big|^{-1}, \quad t\in I_{ni}.$$

It readily follows that for each $1\le q\le\infty$,

$$(3.23)\qquad \|\tilde\kappa_{r,n,i}(x;t)\|_{L_q(I_{ni})} \le \frac{2(r+3)}{(r-1)!}\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-\frac1p+\frac1q}.$$

Now put

$$\tilde\sigma_{r,n}(x;t) := \sigma_{r,n}(x;t) + \tilde\kappa_{r,n,i}(x;t),$$

and clearly $\tilde\sigma_{r,n}(x;t_{n,i-1}) = \sigma_{r,n}(x;t_{n,i-1})$ and $\tilde\sigma_{r,n}(x;t_{ni}) = \sigma_{r,n}(x;t_{ni})$.

Finally for $t\in I_{nn}$, let

$$(3.24)\qquad \tilde\kappa_{r,n,n}(x;t) := \frac{\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{nn})}}{(r-2)!}\int_{t_{n,n-1}}^{t}\left(\int_{t_{n,n-1}}^{\tau}(\rho(\theta))^{(r-\alpha-2)p'}d\theta\right)^{\frac{1}{p'}}d\tau,$$

where $\frac1p+\frac1{p'}=1$, and again put

$$\tilde\sigma_{r,n}(x;t) := \sigma_{r,n}(x;t) + \tilde\kappa_{r,n,n}(x;t), \quad t\in I_{nn}.$$

Similarly we define $\tilde\sigma_{r,n}(x;\cdot)$ on I_{ni}, $i=-n,\dots,-1$. The spline $\tilde\sigma_{r,n}(x;t)$ is then defined on I and it is continuous there. Moreover $\tilde\sigma_{r,n}(x;\cdot)$ is nondecreasing on I. Indeed, all we have to show is that $\tilde\sigma'_{r,n}(x;t)\ge 0$, $t\in I_{ni}$, for an arbitrary $-n\le i\le n$.

Assume that x' is small on I_{ni} for some $1\le i<n$. Then $\tilde\sigma'_{r,n}(x;t) = \big(x(t_{ni}) - x(t_{n,i-1})\big)|I_{ni}|^{-1}\ge 0$, $t\in I_{ni}$, since x is nondecreasing.

Otherwise x' is big on I_{ni}. By (2.5) and (2.6) we rewrite

$$\begin{aligned}
\sigma'_{r,n}(x;t) &= \pi_{*,r-2}(x';i;t)\varphi_{*ni}(t) + \pi^*_{r-2}(x';i;t)\varphi^*_{ni}(t)\\
&\quad + \pi_{*,r-1}(x;i;t)\varphi'_{*ni}(t) + \pi^*_{r-1}(x;i;t){\varphi^*_{ni}}'(t)\\
&= x'(t) - \big(x'(t)-\pi_{*,r-2}(x';i;t)\big)\varphi_{*ni}(t) - \big(x'(t)-\pi^*_{r-2}(x';i;t)\big)\varphi^*_{ni}(t)\\
&\quad - \big(x(t)-\pi_{*,r-1}(x;i;t)\big)\varphi'_{*ni}(t) - \big(x(t)-\pi^*_{r-1}(x;i;t)\big){\varphi^*_{ni}}'(t).
\end{aligned}$$

Now Taylor's formula and Hölder's inequality yield,

$$\left\|x'(\cdot)-\pi_{*,r-2}(x';i;\cdot)\right\|_{L_\infty(I_{ni})}\le\frac{1}{(r-2)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-1-\frac{1}{p}},$$

$$\left\|x'(\cdot)-\pi^*_{r-2}(x';i;\cdot)\right\|_{L_\infty(I_{ni})}\le\frac{1}{(r-2)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-1-\frac{1}{p}},$$

$$\left\|x(\cdot)-\pi_{*,r-1}(x;i;\cdot)\right\|_{L_\infty(I_{ni})}\le\frac{1}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-\frac{1}{p}},$$

$$\left\|x(\cdot)-\pi^*_{r-1}(x;i;\cdot)\right\|_{L_\infty(I_{ni})}\le\frac{1}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-\frac{1}{p}}.$$

Therefore by (2.7) and (4.11) we obtain,

$$\begin{aligned}\sigma'_{r,n}(x;t)\ge x'(t)-\frac{r+3}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-1-\frac{1}{p}},\\ t\in I_{ni}.\end{aligned}\tag{3.25}$$

Since x' is big on I_{ni}, there are only for $0\le m=m\big(I_{ni}\big)\le 2r-4$ subintervals I_{n,i,k_j}, $j=1,\dots,m$, containing points $t_{i,k_j}\in I_{n,i,k_j}$, for which (3.20) holds. On these subintervals, it readily follows by (3.16), (3.19), (3.21) and (3.22), that

$$\begin{aligned}\tilde\kappa'_{r,n,i}(x;t)&=\frac{2(r+3)}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-1-\frac{1}{p}}-\int_{t_{n,i-1}}^{t_{ni}}\xi_{ni}(\tau)d\tau\left|I_{ni}\right|^{-1}\\&=\left(1-\sum_{j=1}^{m}\frac{\left|I_{n,i,k_j}\right|}{\left|I_{ni}\right|}\right)\frac{2(r+3)}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-1-\frac{1}{p}}\\&\ge\left(1-\frac{(r-2)2^\beta}{m(r)}\right)\frac{2(r+3)}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-1-\frac{1}{p}}\\&\ge\frac{(r+3)}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-1-\frac{1}{p}},\end{aligned}$$

which together with (3.25) implies

$$\begin{aligned}\tilde\sigma'_{r,n}(x;t)&\ge x'(t)+\tilde\kappa'_{r,n,i}(x;t)-\frac{r+3}{(r-1)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\left|I_{ni}\right|^{r-1-\frac{1}{p}}\\&\ge x'(t)\ge 0.\end{aligned}$$

On the other subintervals $I_{n,i,k}$, $k \neq k_j$, $j = 1, \ldots, m$, $1 \leq k \leq m(r)$, we have

$$\begin{aligned}
\tilde{\kappa}'_{r,n,i}(x;t) &= -\int_{t_{n,i-1}}^{t_{ni}} \xi_{ni}(\tau) d\tau \big|I_{ni}\big|^{-1} \\
&= -\left(\sum_{j=1}^{m} \frac{\big|I_{n,i,k_j}\big|}{\big|I_{ni}\big|}\right) \frac{2(r+3)}{(r-1)!} \big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})} \rho^{-\alpha}\big(t_{ni}\big) \big|I_{ni}\big|^{r-1-\frac{1}{p}} \\
&\geq -\frac{(r-2)2^{\beta}}{m(r)} \frac{2(r+3)}{(r-1)!} \big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})} \rho^{-\alpha}\big(t_{ni}\big) \big|I_{ni}\big|^{r-1-\frac{1}{p}} \\
&\geq -\frac{(r+3)}{(r-1)!} \big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})} \rho^{-\alpha}\big(t_{ni}\big) \big|I_{ni}\big|^{r-1-\frac{1}{p}},
\end{aligned}$$

which together with (3.25) implies

$$\begin{aligned}
\tilde{\sigma}'_{r,n}(x;t) &\geq x'(t) + \tilde{\kappa}'_{r,n,i}(x;t) - \frac{r+3}{(r-1)!} \big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})} \rho^{-\alpha}\big(t_{ni}\big) \big|I_{ni}\big|^{r-1-\frac{1}{p}} \\
&\geq x'(t) - \frac{2(r+3)}{(r-1)!} \big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})} \rho^{-\alpha}\big(t_{ni}\big) \big|I_{ni}\big|^{r-1-\frac{1}{p}} > 0,
\end{aligned}$$

since (3.20) fails there.

On I_{nn} we recall the definition of $\sigma_{r,n}(x;t)$ from (2.8) and apply Taylor's formula and Hölder's inequality to obtain

$$\begin{aligned}
\big|x'(t) - \sigma'_{r,n}(x;t)\big| &= \big|x'(t) - \pi_{*,r-2}(x';n;t)\big| \\
&\leq \frac{\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{nn})}}{(r-2)!} \left(\int_{t_{n,n-1}}^{t} (\rho(\tau))^{(r-\alpha-2)p'} d\tau\right)^{\frac{1}{p'}}.
\end{aligned}$$

So, together with (3.24) this yields,

$$\begin{aligned}
\tilde{\sigma}'_{r,n}(x;t) :&= x'(t) - \big(x'(t) - \sigma'_{r,n}(x;t)\big) + \tilde{\kappa}'_{r,n,n}(x;t) \\
&\geq \tilde{\kappa}'_{r,n,n}(x;t) - \big|x'(t) - \sigma'_{r,n}(x;t)\big| \\
&= \tilde{\kappa}'_{r,n,n}(x;t) - \big|x'(t) - \pi_{*,r-2}(x';n;t)\big| \\
&\geq 0.
\end{aligned}$$

For the intervals $I_{n,i}$, $i = -1, \ldots, -n$ the proof is similar.

Thus we conclude that the spline $\tilde{\sigma}_{r,n}(x;\cdot)$, indeed is nondecreasing in I, and what is left is to show that it approximates well x.

If x' is small on I_{ni}, then there are $r-1$ subintervals $I_{n,i,k_j}\subset I_{ni}$, $j=1,\dots,r-1$, such that $I_{n,i,k_{j'}}\cap I_{n,i,k_{j''}}=\emptyset$, $j'\neq j''$, and points $t_{i,k_j}\in I_{n,i,k_j}$, $j=1,\dots,r-1$, for which (3.20) holds. Hence by (3.19)

$$\begin{aligned}\min\{\left|t_{k_{j'}}-t_{k_{j''}}\right|, j'\neq j''\}&\geq \min_{k=1,\dots,m(r)}\left|I_{n,i,k}\right|\\&\geq \big(m(r)\big)^{-1}2^{-\beta+1}\left|I_{ni}\right|.\end{aligned}$$

By virtue of Lemma 1 and Hölder's inequality we get

$$\begin{aligned}\|x'\|_{L_\infty(I_{ni})}&\leq \frac{r-1}{(r-2)!}\big(m(r)2^{\beta-1}\big)^{\frac{(r-1)(r-2)}{2}}\left(\max_{1\leq j\leq r-1}\left|x'(t_{i,k_j})\right|+\frac{|I_{ni}|^{r-2}}{(r-2)!}\left\|x^{(r)}\right\|_{L_1(I_{ni})}\right)\\&\leq c\left(\max_{1\leq j\leq r-1}\left|x'(t_{i,k_j})\right|+\frac{1}{(r-2)!}\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-1-\frac{1}{p}}\right),\end{aligned}$$

which in turn, by (3.20), implies that

$$\|x'\|_{L_\infty(I_{ni})}\leq c\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-1-\frac{1}{p}}, \tag{3.26}$$

where $c=c(r,\beta)$. Since $\tilde{\sigma}_{r,n}(x;\cdot)$ is linear and interpolates x at the endpoints of I_{ni}, (3.26) yields

$$\begin{aligned}\left\|x(\cdot)-\tilde{\sigma}_{r,n}(x;\cdot)\right\|_{L_q(I_{ni})}&\leq c\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-\frac{1}{p}+\frac{1}{q}}\\&\leq c\left\|x^{(r)}\rho^\alpha\right\|_{L_p(I_{ni})}n^{-r+\frac{1}{p}-\frac{1}{q}},\end{aligned} \tag{3.27}$$

where $c=c(r,\alpha,p,q)$, and where we have applied the readily seen inequalities

$$\rho\big(t_{ni}\big)=n^{-\beta}(n-|i|)^\beta,\quad |I_{ni}|\leq cn^{-\beta}(n-|i|)^{\beta-1},\quad 1\leq|i|\leq n-1,$$

which by the definition of β (see (2.2)), yield

$$\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-\frac{1}{p}+\frac{1}{q}}\leq cn^{-r+\frac{1}{p}-\frac{1}{q}}.$$

If x' is big on I_{ni}, then by (2.9) and (3.23),

$$
\begin{aligned}
&\big\|x(\cdot)-\tilde\sigma_{r,n}(x;\cdot)\big\|_{L_q(I_{ni})}\\
&\le \big\|x(\cdot)-\sigma_{r,n}(x;\cdot)\big\|_{L_q(I_{ni})}+\big\|\tilde\kappa_{r,n,i}(x;\cdot)\big\|_{L_q(I_{ni})}\\
&\le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}n^{-r+\frac1p-\frac1q}+\frac{2(r+3)}{(r-1)!}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-\frac1p+\frac1q}\\
&\le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}n^{-r+\frac1p-\frac1q},
\end{aligned}
\tag{3.28}
$$

where $c=c(r,\alpha,p,q)$.

Finally for $i=n$, by (3.24) we have,

$$
\begin{aligned}
&\big\|\tilde\kappa_{r,n,n}(x;\cdot)\big\|_{L_q(I_{nn})}\\
&\le \frac{\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{nn})}}{(r-2)!}\left(\int_{t_{n,n-1}}^1\left(\int_{t_{n,n-1}}^t\left(\int_{t_{n,n-1}}^\tau(1-\theta)^{(r-\alpha-2)p'}d\theta\right)^{\frac{1}{p'}}d\tau\right)^q dt\right)^{\frac1q}\\
&\le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{nn})}n^{-r+\frac1p-\frac1q},
\end{aligned}
\tag{3.29}
$$

where $c=c(r,\alpha,p,q)$. Indeed, we fix $\epsilon_1=\epsilon_1(r,\alpha,p)\ge 0$, $\epsilon_2=\epsilon_2(r,\alpha,p)\ge 0$ and $\epsilon_3=\epsilon_3(r,\alpha,p,q)\ge 0$ so small that $(r-\alpha-2-\epsilon_1)p'\ne -1$, $r-\alpha-1-\frac1p-\epsilon_1-\epsilon_2\ne-1$, $(r-\alpha-\frac1p-\epsilon_1-\epsilon_2-\epsilon_3)q\ne -1$, and $r-\alpha-\frac1p+\frac1q-\epsilon_1-\epsilon_2-\epsilon_3>0$. Then

$$
\begin{aligned}
&\left(\int_{t_{n,n-1}}^\tau(1-\theta)^{(r-\alpha-2)p'}d\theta\right)^{\frac{1}{p'}}\\
&\le c_1(1-t_{n,n-1})^{\epsilon_1}\max\left\{(1-t_{n,n-1})^{r-\alpha-1-\frac1p-\epsilon_1},(1-\tau)^{r-\alpha-1-\frac1p-\epsilon_1}\right\},
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\int_{t_{n,n-1}}^t\left(\int_{t_{n,n-1}}^\tau(1-\theta)^{(r-\alpha-2)p'}d\theta\right)^{\frac{1}{p'}}d\tau\\
&\le c_1c_2(1-t_{n,n-1})^{\epsilon_1+\epsilon_2}\max\left\{(1-t_{n,n-1})^{r-\alpha-\frac1p-\epsilon_1-\epsilon_2},(1-t)^{r-\alpha-\frac1p-\epsilon_1-\epsilon_2}\right\},
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\left(\int_{t_{n,n-1}}^1\left(\int_{t_{n,n-1}}^t\left(\int_{t_{n,n-1}}^\tau(1-\theta)^{(r-\alpha-2)p'}d\theta\right)^{\frac{1}{p'}}d\tau\right)^q dt\right)^{\frac1q}\\
&\le c_1c_2c_3(1-t_{n,n-1})^{\epsilon_1+\epsilon_2+\epsilon_2}(1-t_{n,n-1})^{r-\alpha-\frac1p+\frac1q-\epsilon_1-\epsilon_2-\epsilon_3}\\
&=c_1c_2c_3(1-t_{n,n-1})^{r-\alpha-\frac1p+\frac1q},
\end{aligned}
$$

where $c_1 = c_1(r,\alpha,p)$,$c_2 = c_2(r,\alpha,p,)$ and $c_3 = c_3(r,\alpha,p,q,)$. Now (3.29) follows since

$$(1-t_{n,n-1})^{r-\alpha-\frac{1}{p}+\frac{1}{q}} = n^{-\beta(r-\alpha-\frac{1}{p}+\frac{1}{q})} = n^{-r+\frac{1}{p}-\frac{1}{q}}.$$

The proof for $i = -n$ is similar.

Combining (3.27), (3.28) and (3.29), we obtain

$$\left\|x(\cdot) - \tilde{\sigma}_{r,n}(x;\cdot)\right\|_{L_q(I)} \le cn^{-r+(\frac{1}{p}-\frac{1}{q})_+}. \tag{3.30}$$

The functions $\tilde{\sigma}_{r,n}(x;\cdot)$ belong to the space $\tilde{\Sigma}_{r,n}(I)$ of continuous splines that are polynomials of degree $\le r+1$ on each interval $I_{n,i,k}$, $i = \pm 1,\dots,\pm(n-1)$, $k = 1,\dots,m(r)$, and that on $I_{n,\pm n}$ are sums of polynomials of degree $\le r-1$ and the functions $\tilde{\kappa}_{r,n,\pm n}(x;\cdot)$, defined in (3.24) (and analogously for $i = -n$). Evidently, $\dim \tilde{\Sigma}_{r,n}(I) \le cn$, where $c = c(r,\alpha,p,q)$. Hence, (3.30) yields the upper bound in (1.4) for $r > 1$.

We turn now to proving the lower bound in (1.4). It suffices to establish it for the classes $\Delta^1_+ W^r_p \subseteq \Delta^1_+ W^r_{p,\alpha}$, $0 \le \alpha < \infty$. Also since by Theorem KL1, for $1 \le q \le p \le \infty$ (and actually for $1 \le p \le q \le 2$),

$$d_n(\Delta^1_+ W^r_p, \Delta_+ L_q)_{L_q} \ge d_n(\Delta^1_+ W^r_p)_{L_q} \asymp n^{-r+(\frac{1}{p}-\frac{1}{q})_+},$$

the lower bounds in these cases follow. Thus we only have to consider $1 \le p \le q \le \infty$, (in fact only for $q > 2$). To this end let

$$\phi_0(t) := \begin{cases} 1, & t \in [-1,1] \\ 0, & t \in \mathbb{R} \setminus [-1,1] \end{cases},$$

and define by induction

$$\phi_s(t) := \int_{t-1}^{t} \phi_{s-1}(2\tau+1)\,d\tau, \quad t \in \mathbb{R}, \quad s \in \mathbb{N}.$$

It follows that for all $s \in \mathbb{Z}_+$, ϕ_s is even, $\phi_s \ge 0$, $\phi_s(t) = 0, t \in \mathbb{R} \setminus [-1,1]$, $|\phi^{(s)}_s(t)| = 2^{s-1}$, in $[-1,1]$ except for a few dyadic points with denominator 2^{-s+1}, and

$$\phi_s(0) = \|\phi_s\|_{L_\infty} = \int_{-1}^{1} \phi_s(t)\,dt = 2^{-s+1}, \quad s \in \mathbb{N}.$$

For $N \in \mathbb{N}$, write $\phi_{s,N}(t) := N^{-s}\phi_s(Nt)$, and for

$$\tau_{N,i} := -1 + \frac{2i}{N}, \quad i = 0, 1, \dots, N,$$
$$\bar\tau_{N,i} := -1 + \frac{2i-1}{N}, \quad i = 1, \dots, N,$$

let

$$\phi_{s,N,i}(t) := \phi_{s,N}\big(t - \bar\tau_{N,i}\big), \quad i = 1, \dots, N, \quad s \in \mathbb{Z}_+.$$

Finally for $1 \le p \le \infty$, set

$$\phi_{p,s,N,i}(t) := 2^{-s+1-\frac{1}{p}} N^{\frac{1}{p}} \phi_{s,N,i}(t), \quad s \in \mathbb{Z}_+.$$

Clearly, $\phi_{p,s,N,i}(t)$ is symmetric about $\bar\tau_{N,i}$, $\phi_{p,s,N,i}(t) = 0$, for $t \notin J_{N,i} := [\tau_{N,i-1}, \tau_{N,i}]$, and

$$\phi_{p,0,N,i}(\bar\tau_{N,i}) = 2^{1-\frac{1}{p}} N^{\frac{1}{p}}, \quad \phi_{p,s,N,i}(\bar\tau_{N,i}) = 2^{-2s+2-\frac{1}{p}} N^{-s+\frac{1}{p}}, \quad s \in \mathbb{N}.$$

Also

$$\big\|\phi^{(s)}_{p,s,N,i}\big\|_{L_p} = 1, \quad s \in \mathbb{Z}_+. \tag{3.31}$$

For later use we want to record the fact that by the symmetry,

$$\int_{-1}^{1} (t - \bar\tau_{N,i})\phi_{p,s,N,i}(t)\, dt = \int_{\tau_{N,i-1}}^{\tau_{N,i}} (t - \bar\tau_{N,i})\phi_{p,s,N,i}(t)\, dt = 0. \tag{3.32}$$

We are ready to construct the system of vectors that will give us the lower bound. Denote

$$\psi_{p,r,N,i}(t) := \int_{-1}^{t} \phi_{p,r-1,2N,2i-1}(\tau) d\tau, \quad i = 1, \dots, N, \quad t \in [-1, 1].$$

Then it is nondecreasing and, by (3.31), belongs to $\Delta^1_+ W^r_p$. It follows that

$$\psi_{p,r,N,i}(t) = \begin{cases} 0, & t \le \tau_{2N,2i-2}, \\ 2^{-3r+4} N^{-r+\frac{1}{p}}, & t \ge \tau_{2N,2i-1}, \end{cases} \tag{3.33}$$

so that, in particular, it is also in L_q, $1 \le q \le \infty$. Denote the system $\Psi^N_{p,r} := \{\psi_{p,r,N,i}(\cdot)\}_{i=1}^N$, and let $S_p^+(\Psi^N_{p,r})$ be the positive p-sector over this system. Then $S_p^+(\Psi^N_{p,r}) \subset \Delta^1_+ W^r_p$, which implies

$$(3.34) \qquad d_m\big(\Delta^1_+ W^r_p, \Delta^1_+ L_q\big)_{L_q} \ge d_m\big(S_p^+(\Psi^N_{p,r}), \Delta^1_+ L_q\big)_{L_q}.$$

Define the discretization operator $A_{N,q} : L_q \ni x \to A_{N,q}x \in l_q^N$ by

$$A_{N,q}x := \left(|J_{2N,2}|^{-1+\frac{1}{q}} \int_{J_{2N,2}} x(t)dt, \dots, |J_{2N,2N}|^{-1+\frac{1}{q}} \int_{J_{2N,2N}} x(t)dt \right).$$

Then it is easy to see that

$$\|A_{N,q}x\|_{l_q^N} \le \|x\|_{L_q}, \quad x \in L_q.$$

If M^m is an arbitrary subspace in L_q of dimension $\le m$, then the set $A_{N,q}(M^m \cap \Delta^1_+ L_q)$ consists of vectors with nondecreasing coordinates, i.e.,

$$A_{N,q}(M^m \cap \Delta^1_+ L_q) \subseteq \Delta^1_+ \subset \mathbb{R}^N,$$

where Δ^1_+ was defined in Lemma 4. Hence

$$(3.35) \qquad d_m\big(S_p^+(\Psi^N_{p,r}), \Delta^1_+ L_q\big)_{L_q} \ge d_m\big(A_{N,q} S_p^+(\Psi^N_{p,r}), \Delta^1_+\big)_{l_q^N}.$$

Now by (3.33)

$$A_{N,q}\psi_{p,r,N,i} = c(r,p)N^{-r+\frac{1}{p}-\frac{1}{q}}\tilde{e}^{(i)}, \quad i = 1, \dots, N,$$

where $\tilde{e}^{(i)}$ are the N-tuples from (3.4) (with n replaced by N). Hence

$$A_{N,q} S_p^+(\Psi^N_{p,r}) = c(r,p)N^{-r+\frac{1}{p}-\frac{1}{q}} S_p^+(\tilde{E}^N),$$

where $\tilde{E}^N := \{\tilde{e}^{(i)}\}_{i=1}^N$. Therefore

$$(3.36) \qquad d_m\big(A_{N,q} S_p^+(\Psi^N_{p,r}), \Delta^1_+\big)_{l_q^N} = c(r,p)N^{-r+\frac{1}{p}-\frac{1}{q}} d_m\big(S_p^+(\tilde{E}^N)), \Delta^1_+\big)_{l_q^N}.$$

Taking $m = n$ and $N = n+2$, we obtain by Lemma 4,

$$d_n\big(S_p^+(\tilde{E}^{n+2})), \Delta^1_+\big)_{l_q^{n+2}} \ge c > 0,$$

where c is an absolute constant. So finally combining (3.34), (3.35) and (3.36) we conclude

$$d_n\big(\Delta^1_+ W^r_p, \Delta^1_+ L_q\big)_{L_q} \ge cn^{-r+\frac{1}{p}-\frac{1}{q}},$$

where $c = c(r,p,q)$. This proves the lower bounds for $1 \le p \le q \le \infty$ and completes the proof of Theorem 2. $\square$

§4. Convexity preserving widths of the classes $\Delta_+^2 W_{p,\alpha}^r$ in L_q

We begin by denoting

(4.1)
$$\check{E}^n := \{\check{e}^{(i)}\}_{i=1}^n, \quad \check{e}^{(1)} := (1,2,\dots,n), \check{e}^{(2)} := (0,1,\dots,n-1),\dots,\check{e}^{(n)} := (0,\dots,0,1).$$

We need the following result the proof of which is similar to that of Lemma 4.

Lemma 5. *Let $m,n\in\mathbb{N}$, be so that $m<n+1$, and let $1\le p\le q\le\infty$. Denote by*

$$\Delta_+^2 := \{x=(x_1,\dots,x_n) \mid x_2-x_1\le\cdots\le x_n-x_{n-1}\},$$

the cone of vectors $x\in\mathbb{R}^n$, with convex coordinates. Then

$$d_m\big(S_p^+(\check{E}^n),\Delta_+^2\big)_{l_q^n} \ge \frac{1}{26}. \tag{4.2}$$

Proof. First note that

$$\begin{aligned} S_1^+(\check{E}^n) &= \{x=(x_1,\dots,x_n) \mid x_1\ge 0, x_2-2x_1\ge 0, x_3-2x_2+x_1\ge 0,\dots,\\ &\qquad x_n-2x_{n-1}+x_{n-2}\ge 0, x_n-x_{n-1}\le 1\}\\ &= \{x=(x_1,\dots,x_n) \mid 0\le x_1\le x_2-x_1\le x_3-x_2\le\cdots\le x_n-x_{n-1}\le 1\}, \end{aligned}$$

and that the vectors $\check{e}^{(0)} := \bar{0}$, $\check{e}^{(i)}$, $i=1,\dots,n$ are the vertices of this n-dimensional pyramid. Evidently $S_1^+(\check{E}^n)\subset\Delta_+^2$, and $S_p^+(\check{E}^n)\supseteq S_1^+(\check{E}^n)$, so that

$$d_m\big(S_p^+(\check{E}^n),\Delta_+^2\big)_{l_q^n} \ge d_m\big(S_1^+(\check{E}^n),\Delta_+^2\big)_{l_\infty^n}.$$

Thus again, we may consider just $S_1^+(\check{E}^n)$. Let M^m be an arbitrary m-dimensional linear manifold and let L^{m+1} be a subspace of $\mathbb{R}^n$, of dimension $\dim L^{m+1}\le m+1$ so that $L^{m+1}\supseteq M^m$. Then we have

$$E\big(S_1^+(\check{E}^n),M^m\cap\Delta_+^2\big)_{l_\infty^n} \ge E\big(S_1^+(\check{E}^n),L^{m+1}\cap\Delta_+^2\big)_{l_\infty^n}. \tag{4.3}$$

Fix $\epsilon : 0 < \epsilon < \frac{1}{3}$ and denote

$$S^+_{\epsilon,1}\big(\check{E}^n\big) := \{x = (x_1, \ldots, x_n) \mid 2\epsilon \le x_1 + \epsilon \le x_2 - x_1 \le x_3 - x_2 \le \cdots \le x_n - x_{n-1} \le 1 - \epsilon\}.$$

Then clearly $S^+_{\epsilon,1}\big(\check{E}^n\big) \subset S^+_1\big(\check{E}^n\big)$, and its vertices are the vectors $\check{e}^{(i)}_\epsilon := \epsilon\check{e}^{(1)} + \epsilon\check{e}^{(2)} + (1-3\epsilon)\check{e}^{(i)}$, $i = 0, 1, \ldots, n$. Hence,

$$S^+_{\epsilon,1}\big(\check{E}^n\big) = \epsilon\check{e}^{(1)} + \epsilon\check{e}^{(2)} + (1-3\epsilon)S^+_{\epsilon,1}\big(\check{E}^n\big).$$

Also

$$E\big(S^+_1\big(\check{E}^n\big), L^{m+1} \cap \Delta^2_+\big)_{l^n_\infty} \ge E\big(S^+_{\epsilon,1}\big(\check{E}^n\big), L^{m+1} \cap \Delta^2_+\big)_{l^n_\infty}. \tag{4.4}$$

For $x^0 \in S^+_{\epsilon,1}\big(\check{E}^n\big)$, we have

$$\begin{aligned} &E\big(x^0, L^{m+1} \cap \Delta^2_+\big)_{l^n_\infty} \\ &= \min\Big\{E\big(x^0, L^{m+1} \cap \big(\Delta^2_+ \setminus S^+_1\big(\check{E}^n\big)\big)\big)_{l^n_\infty}, E\big(x^0, L^{m+1} \cap S^+_1\big(\check{E}^n\big)\big)\big)_{l^n_\infty}\Big\}, \end{aligned} \tag{4.5}$$

and we are going to treat separately each of the terms on the right. We begin with the left-hand term and denote $\check{e} := (-2, 1, 0, \ldots, 0)$ and $\check{\epsilon} := (0, \ldots, 0, -1, 1)$. By Lemma 2 with $q = \infty$, we obtain

$$\begin{aligned} &E\big(x^0, \mathbb{R}^{n-1}(e^{(1)})\big)_{l^n_\infty} \ge \epsilon, \\ &E\big(x^0, \mathbb{R}^{n-1}(\check{e})\big)_{l^n_\infty} \ge \frac{\epsilon}{3}, \\ &E\big(x^0, M^{n-1}(\check{\epsilon}, e^{(n)})\big)_{l^n_\infty} \ge \frac{\epsilon}{2}, \end{aligned} \tag{4.6}$$

where the $e^{(i)}$'s are from (3.3), and

$$\begin{aligned} \mathbb{R}^{n-1}(e^{(1)}) &= \{x = (x_1, \ldots, x_n) \mid x_1 = 0\}, \\ \mathbb{R}^{n-1}(\check{e}) &= \{x = (x_1, \ldots, x_n) \mid x_2 - 2x_1 = 0\}, \quad \text{and} \\ M^{n-1}(\check{\epsilon}, e^{(n)}) &= e^{(n)} + \mathbb{R}^{n-1}(\check{\epsilon}) = \{x = (x_1, \ldots, x_n) \mid x_n - x_{n-1} = 1\}. \end{aligned}$$

So, if we (again) denote the half-spaces

$$\begin{aligned} &\mathbb{R}^{n-1}_-(e^{(1)}) = \{x = (x_1, \ldots, x_n) \mid x_1 < 0\}, \\ &\mathbb{R}^{n-1}_-(\check{e}) := \{x = (x_1, \ldots, x_n) \mid x_2 - 2x_1 < 0\}, \quad \text{and} \\ &\mathbb{R}^{n-1}_-(\check{\epsilon}; e^{(n)}) := \{x = (x_1, \ldots, x_n) \mid x_n - x_{n-1} > 1\}, \end{aligned}$$

then we get by virtue of (4.6),

$$E\big(x^0, \mathbb{R}_-^{n-1}(e^{(1)}) \cup \mathbb{R}_-^{n-1}(\check{e}) \cup \mathbb{R}_-^{n-1}(\check{\epsilon}; e^{(n)})\big)_{l_\infty^n} \ge \frac{\epsilon}{3}.$$

Observing that $\Delta_+^2 \setminus S_1^+\big(\check{E}^n\big) = \Delta_+^2 \cap \big(\mathbb{R}_-^{n-1}(e^{(1)}) \cup \mathbb{R}_-^{n-1}(\check{e}) \cup \mathbb{R}_-^{n-1}(\check{\epsilon}; e^{(n)})\big)$, we conclude that

$$\begin{aligned}
&E\big(x^0, L^{m+1} \cap \big(\Delta_+^2 \setminus S_1^+\big(\check{E}^n\big)\big)\big)_{l_\infty^n} \\
&= E\big(x^0, L^{m+1} \cap \big(\Delta_+^2 \cap \big(\mathbb{R}_-^{n-1}(e^{(1)}) \cup \mathbb{R}_-^{n-1}(\check{e}) \cup \mathbb{R}_-^{n-1}(\check{\epsilon}; e^{(n)})\big)\big)\big)_{l_\infty^n} \\
&\ge E\big(x^0, \mathbb{R}_-^{n-1}(e^{(1)}) \cup \mathbb{R}_-^{n-1}(\check{e}) \cup \mathbb{R}_-^{n-1}(\check{\epsilon}; e^{(n)})\big)_{l_\infty^n} \\
&\ge \frac{\epsilon}{3}.
\end{aligned}$$

Therefore by (4.5),

$$E\big(x^0, L^{m+1} \cap \Delta_+^2\big)_{l_\infty^n} \ge \min\left\{\frac{\epsilon}{3}, E\big(x^0, L^{m+1} \cap S_1^+\big(\check{E}^n\big)\big)_{l_\infty^n}\right\},$$

which becomes

$$E\big(S_{\epsilon,1}^+\big(\check{E}^n\big), L^{m+1} \cap \Delta_+^2\big)_{l_\infty^n} \ge \min\left\{\frac{\epsilon}{3}, E\big(S_{\epsilon,1}^+\big(\check{E}^n\big), L^{m+1} \cap S_1^+\big(\check{E}^n\big)\big)_{l_\infty^n}\right\}. \tag{4.7}$$

Now we have to take care of the right-hand term in (4.7). Let $\check{T}_n : \mathbb{R}^n \ni x \to y \in \mathbb{R}^n$, be defined by

$$y_1 = x_1, \quad y_2 = x_2 - 2x_1, \quad y_3 = x_3 - 2x_2 + x_1, \dots, y_n = x_n - 2x_{n-1} + x_{n-2},$$

so that it is invertible and its inverse is given by

$$x_i = \sum_{j=1}^{i} (i - j + 1) y_j, \quad i = 1, \dots, n.$$

It is readily seen that $\check{T}_n \check{e}^{(i)} = e^{(i)}$ and $\check{T}_n \check{e}_\epsilon^{(i)} = \epsilon e^{(1)} + \epsilon e^{(2)} + (1 - 3\epsilon) e^{(i)} =: \dot{e}_\epsilon^{(i)}$, $i = 0, 1, \dots, n$. Hence $\check{T}_n S_1^+\big(\check{E}^n\big) = S_1^+\big(E^n\big) = S_1^+$, and $\check{T}_n S_{\epsilon,1}^+\big(\check{E}^n\big) = \epsilon e^{(1)} + \epsilon e^{(2)} + (1 - 3\epsilon) S_1^+ =: \dot{S}_{\epsilon,1}^+\big(E^n\big)$.

Denote by $\check{T}_n l_\infty^n$ the space $\mathbb{R}^n$ with the norm

$$\|y\|_{\check{T}_n l_\infty^n} := \max_{1 \le i \le n} \Big|\sum_{j=1}^{i} (i - j + 1) y_j\Big|.$$

Then

$$E\big(S^+_{\epsilon,1}(\check{E}^n), L^{m+1}\cap S^+_1(\check{E}^n)\big)_{l^n_\infty} = E\big(\dot{S}^+_{\epsilon,1}(E^n), \check{T}_nL^{m+1}\cap S^+_1(E^n)\big)_{\check{T}_nl^n_\infty}$$

$$\text{(4.8)}\qquad \geq \frac{1}{4}E\big(\dot{S}^+_{\epsilon,1}(E^n), \check{T}_nL^{m+1}\cap S^+_1(E^n)\big)_{l^n_\infty}$$

$$\geq \frac{1}{4}\big(E\big(S^+_1, \check{T}_nL^{m+1}\cap S^+_1\big)_{l^n_\infty} - 3\epsilon\big),$$

since the unit ball of $\check{T}_nl^n_\infty$ is contained in the cube $4B^n_\infty$ and $\max_{1\leq i\leq n}\|e^{(i)}-\dot{e}^{(i)}_\epsilon\|_{l^n_\infty} = 3\epsilon$.

Now, as in (3.15)

$$E\big(S^+_1, \check{T}_nL^{m+1}\cap S^+_1\big)_{l^n_\infty} \geq \frac{1}{2},$$

which by virtue of (4.7) and (4.8) implies

$$E\big(S^+_{\epsilon,1}(\check{E}^n), L^{m+1}\cap S^+_1(\check{E}^n)\big)_{l^n_\infty} \geq \min\left\{\frac{\epsilon}{3}, \frac{1}{4}\big(\frac{1}{2}-3\epsilon\big)\right\}.$$

Taking $\epsilon = \frac{3}{26}$ we obtain,

$$E\big(S^+_{\epsilon,1}(\check{E}^n), L^{m+1}\cap S^+_1(\check{E}^n)\big)_{l^n_\infty} \geq \frac{1}{26},$$

which combined with (4.3) and (4.4) yields

$$E\big(S^+_1(\check{E}^n), M^m\cap \Delta^2_+\big)_{l^n_\infty} \geq \frac{1}{26}.$$

Since M^m was an arbitrary linear manifold of dimension m, we conclude that that (4.2) is valid, and the proof of Lemma 5 is complete. □

We are ready to prove Theorem 3.

Proof of Theorem 3. We begin with the upper bounds. First we observe that the upper bound in (1.6) follows by the proof of Theorem KL2 in [11]. Indeed, we note that in that proof, $\sigma_{1,n}(x;\cdot)$ is piecewise linear, and it is convex whenever x is, thus the upper in (1.6) follows by [11, (4.6)]. Therefore only the upper bound in (1.5) has to be proved. To this end, we first take $r=2$, $\beta=\beta(2,\alpha,p,q)$ from (2.2) and the points t_{ni} defined by (2.3). Let

$$\check{\sigma}_{2,n}(x;t) := \pm\left(x(t_{n,\pm(i-1)})\big(t_{n,\pm i}-t\big) + x(t_{n,i})\big(t-t_{n,\pm(i-1)}\big)\right)|I_{ni}|^{-1},$$

$$t\in I_{n,\pm i},\quad 1\leq i\leq n-1,$$

and let

$$\check{\sigma}_{2,n}(x;t) := \pi_{*,1}(x;\pm n;t), \quad t \in I_{n,\pm n},$$

where we recall that $\pi_{*,1}(x;\pm n;\cdot)$ are the Taylor polynomials of degree 1 of x, expanded about the points $t_{n,\pm(n-1)}$. Evidently, $\check{\sigma}_{2,n}(x;\cdot)$ is piecewise linear and interpolates x at the points $\{t_{ni}\}$, $1 \le |i| \le n-1$. So obviously it is convex and it follows that for $i = \pm 1, \dots, \pm(n-1)$,

$$\big\|x(\cdot) - \check{\sigma}_{2,n}(x;\cdot)\big\|_{L_\infty(I_{ni})} \le c\big\|x''\big\|_{L_1(I_{ni})}\big|I_{ni}\big| \le c\big\|x''\rho^\alpha\big\|_{L_p(I_{ni})}(\rho(t_{ni}))^{-\alpha}\big|I_{ni}\big|^{2-\frac{1}{p}},$$

where $c = c(\alpha, p, q)$, whence

$$\big\|x(\cdot) - \check{\sigma}_{2,n}(x;\cdot)\big\|_{L_q(I_{ni})} \le c\big\|x''\rho^\alpha\big\|_{L_p(I_{ni})}n^{-2+\frac{1}{p}-\frac{1}{q}}. \tag{4.9}$$

Also, as in the proof of (3.29), we obtain

$$\big\|x(\cdot) - \pi_{*,1}(x;\pm n;\cdot)\big\|_{L_q(I_{n,\pm n})} \le c\big\|x''\rho^\alpha\big\|_{L_p(I_{n,\pm n})}n^{-2+\frac{1}{p}-\frac{1}{q}}, \tag{4.10}$$

where $c = c(\alpha, p, q)$. Combining (4.9) and (4.10), it now follows that

$$\big\|x(\cdot) - \check{\sigma}_{2,n}(x;\cdot)\big\|_{L_q} \le cn^{-2+(\frac{1}{p}-\frac{1}{q})_+},$$

proving (1.5) for $r = 2$.

For $r \ge 3$, let $\sigma_{r,n}(x;\cdot)$ be the spline defined in (2.8). We have to modify it so that it be convex whenever $x \in \Delta^2_+ W^r_{p,\alpha}$, but stay close to x in the L_p-norm. Let β be defined in (2.2) and set

$$m(r) = m(r, \alpha, p, q) := \big\lceil (r-2)2^{\beta+1}\big(2^\beta + 1\big)\big\rceil\,. \tag{4.11}$$

Let the points $t_{n,i,k}$ and the subintervals $I_{n,i,k}$ be respectively defined, by (3.17) and (3.18), for this $m(r)$, and finally write

$$C(r, \beta) := \frac{1}{(r-3)!} + \frac{8}{(r-2)!} + \frac{2^{\beta+2}}{(r-1)!}. \tag{4.12}$$

The second derivative x'' is called *small* on I_{ni}, $1 \leq |i| \leq n-1$, if there exist at least $2r-5(\leq m(r))$ subintervals I_{n,i,k_j}, and points $t_{i,k_j} \in I_{n,i,k_j}$, such that

$$x''\big(t_{i,k_j}\big) \leq 2C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-2-\frac{1}{p}}. \tag{4.13}$$

Otherwise x'' is called *big* on the interval I_{ni}.

If x'' is small on I_{ni}, $1 \leq i \leq n-1$, then replace $\sigma_{r,n}(x;\cdot)$ on that interval by the linear interpolant

$$\check{\sigma}_{r,n}(x;t) := \big(x\big(t_{n,i-1}\big)\big(t_{ni}-t\big) + x\big(t_{n,i}\big)\big(t-t_{n,i-1}\big)\big)\,\big|I_{ni}\big|^{-1}, \quad t \in I_{ni}.$$

If on the other hand, x'' is big on I_{ni}, $1 \leq i \leq n-1$, then there are at most $2r-6$ subintervals I_{n,i,k_j}, $j=1,\dots,m$ $(0 \leq m \leq 2r-6)$, such that each contains a point t_{i,k_j}, for which (4.13) holds. Let

$$\check{\xi}_{ni}(t) := \begin{cases} 2C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-2-\frac{1}{p}}, & t \in I_{n,i,k_j}, \\ 0, & \text{otherwise}, \end{cases} \tag{4.14}$$

and define

$$\begin{aligned} \check{\kappa}_{r,n,i}(x;t) : = & \int_{t_{n,i-1}}^{t}\int_{t_{n,i-1}}^{\tau}\check{\xi}_{ni}(\theta)d\theta d\tau - \frac{1}{2}\int_{t_{n,i-1}}^{t_{ni}}\check{\xi}_{ni}(\theta)d\theta\big(t-t_{n,i-1}\big)^2\big|I_{ni}\big|^{-1} \\ & - \left(\int_{t_{n,i-1}}^{t_{ni}}\int_{t_{n,i-1}}^{\tau}\check{\xi}_{ni}(\theta)d\theta d\tau - \frac{1}{2}\int_{t_{n,i-1}}^{t_{ni}}\check{\xi}_{ni}(\theta)d\theta\big|I_{ni}\big|\right)\varphi^*_{ni}(t), \end{aligned} \tag{4.15}$$

where $\varphi^*_{ni}(\cdot)$ is from (2.4). We immediately obtain,

$$\big\|\check{\kappa}_{r,n,i}(x;\cdot)\big\|_{L_q(I_{ni})} \leq c\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-\frac{1}{p}+\frac{1}{q}}, \tag{4.16}$$

where $c = c(r,\alpha,p,q)$. Finally, for $t \in I_{nn}$, let

$$\check{\kappa}_{r,n,n}(x;t) := \frac{\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{nn})}}{(r-3)!}\int_{t_{n,n-1}}^{t}\int_{t_{n,n-1}}^{\tau}\left(\int_{t_{n,n-1}}^{\theta}(\rho(u))^{(r-\alpha-3)p'}du\right)^{\frac{1}{p'}}d\theta d\tau, \tag{4.17}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Now set

$$\check{\sigma}_{r,n}(x;t) := \sigma_{r,n}(x;t) + \check{\kappa}_{r,n,i}(x;t), \quad t \in I_{ni}, \quad i=1,\dots,n. \tag{4.18}$$

Similarly we define $\check{\sigma}_{r,n}(x;\cdot)$ on I_{ni}, $i=-1,\dots,-n$.

It is readily seen that $\check{\sigma}_{r,n}(x;t)$ is continuous on I, and in order to prove that it is convex, it suffices to show that $\check{\sigma}''_{r,n}(x;\cdot)\ge 0$ in I_{ni} for all $-n\le i\le n$, and that $\check{\sigma}'_{r,n}(x;t_{ni}-)\le \check{\sigma}'_{r,n}(x;t_{ni}+)$, for all $-n+1\le i\le n-1$.

If x'' is small on I_{ni}, then for all $t\in I_{ni}$, $\check{\sigma}''_{r,n}(x;t)=0$, and $\check{\sigma}'_{r,n}(x;t)=x'(\theta_{ni})$, where $\theta_{ni}\in I_{ni}$, hence $x'(t_{n,i-1})\le x'(\theta_{ni})\le x'(t_{ni})$, thus satisfying the requirements.

Suppose that x'' is big on I_{ni}. First by (2.5), (2.6) and (2.8), if $t\in I_{ni}$, then

$$\begin{aligned}
\sigma''_{r,n}(x;t) &= \pi_{*,r-3}(x'';i;t)\varphi_{*ni}(t)+\pi^*_{r-3}(x'';i;t)\varphi^*_{ni}(t)\\
&\quad+2\big(\pi_{*,r-2}(x';i;t)\varphi'_{*ni}(t)+\pi^*_{r-2}(x';i;t){\varphi^*_{ni}}'(t)\big)\\
&\quad+\pi_{*,r-1}(x;i;t)\varphi''_{*ni}(t)+\pi^*_{r-1}(x;i;t){\varphi^*_{ni}}''(t)\\
&= x''(t)-\big(x''(t)-\pi_{*,r-3}(x'';i;t)\big)\varphi_{*ni}(t)-\big(x''(t)-\pi^*_{r-3}(x'';i;t)\big)\varphi^*_{ni}(t)\\
&\quad-2\Big(\big(x'(t)-\pi_{*,r-2}(x';i;t)\big)\varphi'_{*ni}(t)+\big(x'(t)-\pi^*_{r-2}(x';i;t)\big){\varphi^*_{ni}}'(t)\Big)\\
&\quad-\Big(\big(x(t)-\pi_{*,r-1}(x;i;t)\big)\varphi''_{*ni}(t)+\big(x(t)-\pi^*_{r-1}(x;i;t)\big){\varphi^*_{ni}}''(t)\Big).
\end{aligned}$$

Now by the Taylor remainder formula and Hölder's inequality we get

$$\begin{aligned}
\big\|x''(\cdot)-\pi_{*,r-3}(x'';i;\cdot)\big\|_{L_\infty(I_{ni})} &\le \frac{1}{(r-3)!}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-2-\frac1p},\\
\big\|x''(\cdot)-\pi^*_{r-3}(x'';i;\cdot)\big\|_{L_\infty(I_{ni})} &\le \frac{1}{(r-2)!}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-2-\frac1p},\\
\big\|x'(\cdot)-\pi_{*,r-2}(x';i;\cdot)\big\|_{L_\infty(I_{ni})} &\le \frac{1}{(r-2)!}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-1-\frac1p},\\
\big\|x'(\cdot)-\pi^*_{r-2}(x';i;\cdot)\big\|_{L_\infty(I_{ni})} &\le \frac{1}{(r-2)!}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-1-\frac1p},\\
\big\|x(\cdot)-\pi_{*,r-1}(x;i;\cdot)\big\|_{L_\infty(I_{ni})} &\le \frac{1}{(r-1)!}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-\frac1p},\\
\big\|x(\cdot)-\pi^*_{r-1}(x;i;\cdot)\big\|_{L_\infty(I_{ni})} &\le \frac{1}{(r-1)!}\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-\frac1p}.
\end{aligned}$$

Therefore by (2.7) and (4.12),

$$\check{\sigma}''_{r,n}(x;t) \geq x''(t) - C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-2-\frac{1}{p}}, \quad t \in I_{ni}. \tag{4.19}$$

Since x'' is big on I_{ni}, there exist $0 \leq m = m\big(I_{ni}\big) \leq 2r-6$, subintervals I_{n,i,k_j}, $j = 1,\ldots,m$, and points t_{i,k_j} in them, for which (4.13) holds. Then (4.18) and (4.19)imply

$$\begin{aligned} \check{\sigma}''_{r,n}(x;t) &= \sigma''_{r,n}(x;t) + \check{\kappa}''_{r,n,i}(x;t) \\ &\geq x''(t) + \check{\kappa}''_{r,n,i}(x;t) - C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-2-\frac{1}{p}}. \end{aligned} \tag{4.20}$$

Now, for $t \in I_{n,i,k_j}$, $j = 1,\ldots,m$, combining (2.6), (3.16), (4.11), (4.14) and (4.15), we obtain

$$\begin{aligned} &\check{\kappa}''_{r,n,i}(x;t) \\ &= 2C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-2-\frac{1}{p}} - \int_{t_{n,i-1}}^{t_{ni}}\check{\xi}_{ni}(\tau)d\tau\big|I_{ni}\big|^{-1} \\ &\quad - \left(\int_{t_{n,i-1}}^{t_{ni}}\int_{t_{n,i-1}}^{\tau}\check{\xi}_{ni}(\theta)d\theta d\tau - \frac{1}{2}\int_{t_{n,i-1}}^{t_{ni}}\check{\xi}_{ni}(\theta)d\theta|I_{ni}|\right)\phi''_{*ni}(t) \\ &\geq 2C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-2-\frac{1}{p}} - \int_{t_{n,i-1}}^{t_{ni}}\check{\xi}_{ni}(\tau)d\tau\big|I_{ni}\big|^{-1} \\ &\quad - \left(\frac{1}{2}\int_{t_{n,i-1}}^{t_{ni}}\check{\xi}_{ni}(\theta)d\theta|I_{ni}|\right)|\phi''_{*ni}(t)| \\ &\geq \left(1 - (1+2^{\beta})\sum_{j=1}^{m}\frac{\big|I_{n,i,k_j}\big|}{\big|I_{ni}\big|}\right)2C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-2-\frac{1}{p}} \\ &\geq \left(1 - \frac{(r-3)2^{\beta}(1+2^{\beta})}{m(r)}\right)2C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-2-\frac{1}{p}} \\ &\geq C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-2-\frac{1}{p}}. \end{aligned} \tag{4.21}$$

Similarly it follows for all other $t \in I_{ni}$, that

$$
\begin{aligned}
&\check{\kappa}''_{r,n,i}(x;t)\\
&= -\int_{t_{n,i-1}}^{t_{ni}} \check{\xi}_{ni}(\tau)d\tau|I_{ni}|^{-1}\\
&\quad -\left(\int_{t_{n,i-1}}^{t_{ni}}\int_{t_{n,i-1}}^{\tau}\check{\xi}_{ni}(\theta)d\theta d\tau - \frac{1}{2}\int_{t_{n,i-1}}^{t_{ni}}\check{\xi}_{ni}(\theta)d\theta\big|I_{ni}\big|\right)\phi''_{*ni}(t)\\
&\ge -(1+2^{\beta})\left(\sum_{j=1}^{m}\frac{\big|I_{n,i,k_j}\big|}{|I_{ni}|}\right)2C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)\big|I_{ni}\big|^{r-2-\frac{1}{p}}\\
&\ge -\frac{(r-3)2^{\beta}(1+2^{\beta})}{m(r)}2C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-2-\frac{1}{p}}\\
&\ge -C(r,\beta)\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-2-\frac{1}{p}}.
\end{aligned}
\tag{4.22}
$$

Combining (4.20) through (4.22), we conclude that

$$
\check{\sigma}''_{r,n}(x;t) \ge 0, \quad t \in I_{ni}.
$$

On I_{nn} we have by (2.8),

$$
\begin{aligned}
\check{\sigma}''_{r,n}(x;t) :&= \sigma''_{r,n}(x;t) + \check{\kappa}''_{r,n,n}(x;t)\\
&= x''(t) - \big(x''(t) - \sigma''_{r,n}(x;t)\big) + \check{\kappa}''_{r,n,n}(x;t)\\
&\ge \check{\kappa}''_{r,n,n}(x;t) - \big|x''(t) - \sigma''_{r,n}(x;t)\big|\\
&= \check{\kappa}''_{r,n,n}(x;t) - \big|x''(t) - \pi_{*,r-3}(x'';n;t)\big|.
\end{aligned}
\tag{4.23}
$$

We apply the Taylor remainder formula and Hölder's inequality to obtain

$$
\big|x''(t) - \pi_{*,r-3}(x'';n;t)\big| \le \frac{\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{nn})}}{(r-3)!}\left(\int_{t_{n,n-1}}^{t}(\rho(\tau))^{(r-\alpha-3)p'}d\tau\right)^{\frac{1}{p'}},
$$

while

$$
\check{\kappa}''_{r,n,n}(x;t) = \frac{\big\|x^{(r)}\rho^{\alpha}\big\|_{L_p(I_{nn})}}{(r-3)!}\left(\int_{t_{n,n-1}}^{t}(\rho(\tau))^{(r-\alpha-3)p'}d\tau\right)^{\frac{1}{p'}}.
$$

Together with (4.23) these imply that $\check{\sigma}''_{r,n}(x;t) \ge 0$ for $t \in I_{nn}$. For the intervals $I_{n,i}$, $i = -n, \ldots, -1$, the proof is similar.

Also if x'' is big on I_{ni}, $1 \le |i| \le n-1$ then we our construction guarantees that $\check{\sigma}'_{r,n}(x;\cdot)$ coincides with $x'(\cdot)$ at the endpoints of I_{ni}, and $\check{\sigma}'_{r,n}(x;t_{n,\pm(n-1)}\pm) = x'(t_{n,\pm(n-1)})$. Thus we have proved that $\check{\sigma}_{r,n}(x;\cdot)$ is convex on I.

We have to show that $\check{\sigma}_{r,n}(x;\cdot)$ approximates x well. To this end, if x'' is small on I_{ni}, then by Lemma 1 we obtain exactly as in the proof of Lemma 4, that

$$\|x''\|_{L_\infty(I_{ni})} \le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-2-\frac{1}{p}},$$

where $c = c(r,\alpha,p,q)$, which in turn implies

$$\begin{aligned}\big\|x(\cdot)-\check{\sigma}_{r,n}(x;\cdot)\big\|_{L_q(I_{ni})} &\le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-\frac{1}{p}+\frac{1}{q}}\\ &\le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}n^{-r+\frac{1}{p}-\frac{1}{q}},\end{aligned} \tag{4.24}$$

where $c = c(r,\alpha,p,q)$.

On the other hand, if x'' is big on I_{ni}, then there exist at most $2r-6$ subintervals I_{n,i,k_j} and points t_{i,k_j} in them for which (4.13) holds. It follows by (2.9) and (4.16) that

$$\begin{aligned}&\big\|x(\cdot)-\check{\sigma}_{r,n}(x;\cdot)\big\|_{L_q(I_{ni})}\\ &\le \big\|x(\cdot)-\sigma_{r,n}(x;\cdot)\big\|_{L_q(I_{ni})} + \big\|\check{\kappa}_{r,n,i}(x;\cdot)\big\|_{L_q(I_{ni})}\\ &\le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}n^{-r+\frac{1}{p}-\frac{1}{q}} + c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}\rho^{-\alpha}\big(t_{ni}\big)|I_{ni}|^{r-\frac{1}{p}+\frac{1}{q}}\\ &\le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{ni})}n^{-r+\frac{1}{p}-\frac{1}{q}},\end{aligned} \tag{4.25}$$

where $c = c(r,\alpha,p,q)$. Finally, for $t \in I_{nn}$, we apply the same computations as in the proof of (3.29) and obtain that

$$\big\|\check{\kappa}_{r,n,\pm n}(x;\cdot)\big\|_{L_q(I_{nn})} \le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{nn})}n^{-r+\frac{1}{p}-\frac{1}{q}},$$

where $c = c(r,\alpha,p,q)$, and a similar result for $t \in I_{n,-n}$. Therefore by (2.9),

$$\big\|x(\cdot)-\check{\sigma}_{r,n}(x;\cdot)\big\|_{L_q(I_{n,\pm n})} \le c\big\|x^{(r)}\rho^\alpha\big\|_{L_p(I_{n,\pm n})}n^{-r+\frac{1}{p}-\frac{1}{q}},$$

where $c = c(r,\alpha,p,q)$. Combining this with (4.24) and (4.25) we get

$$\big\|x(\cdot)-\check{\sigma}_{r,n}(x;\cdot)\big\|_{L_q(I)} \le cn^{-r+(\frac{1}{p}-\frac{1}{q})_+}. \tag{4.26}$$

Note that $\check{\sigma}_{r,n}(x;\cdot)$ belongs to the space $\check{\Sigma}_{r,n}$ of continuous splines that are polynomials of degree $\le r+1$ on each interval $I_{n,i,k}$, $i=\pm1,\dots,\pm(n-1)$, $k=1,\dots,m(r)$, while on $I_{n,\pm n}$ they are sums of polynomials of degree $\le r-1$ and functions $\check{\kappa}_{r,n,\pm n}(x;\cdot)$ defined in (4.17). Clearly $\dim\check{\Sigma}_{r,n}\le cn$, where $c=c(r,\alpha,p,q)$. Hence the upper bound in (1.5) follows by (4.26).

Next we prove the lower bound in (1.5). Since $r\ge 2$ and

$$d_n(\Delta_+^2 W_p^r,\Delta_+^2 L_q)_{L_q}\ge d_n(\Delta_+^2 W_p^r)_{L_q},$$

then (1.5) follows from Theorem KL1 for $1\le q\le p\le\infty$ and for $1\le p\le q\le 2$. Also since $\Delta_+^2 W_p^r\subseteq\Delta_+^2 W_{p,\alpha}^r$ for all $0\le\alpha<\infty$, it suffices to prove (1.5) for the former class and $1\le p\le q\le\infty$. To this end we take the points $\tau_{N,i}$, the intervals $J_{N,i}$ and the functions $\phi_{p,s,N,i}(\cdot)$ as defined in the proof of Theorem 2, and we fix some $k\in\mathbb{N}$, $k>1$ to be prescribed. Denote

$$\begin{aligned}\check{\psi}_{p,r,k,N,i}(t)&:=\int_{-1}^{t}\int_{-1}^{\tau}\phi_{p,r-2,kN,k(i-1)+1}(\theta)d\theta d\tau\\&=\int_{-1}^{t}\phi_{p,r-2,kN,k(i-1)+1}(\theta)(t-\theta)d\theta,\quad i=1,\dots,N,\quad t\in[-1,1].\end{aligned}$$

Then it is nondecreasing and convex and by (3.31) it follows that $\check{\psi}_{p,r,k,N,i}\in\Delta_+^2 W_p^r$. We note that

$$\check{\psi}_{p,r,k,N,i}(t)=0,\quad t\le\tau_{kN,k(i-1)},$$

and that like in (3.33), we have

$$\int_{J_{kN,k(i-1)+1}}\phi_{p,r-2,kN,k(i-1)+1}(\theta)d\theta=c(r,p)(kN)^{-r+1+\frac{1}{p}},\quad i=1,\dots,N,$$

where $c(r,p)>0$ depends only on r and p. Hence, for $t\ge\tau_{kN,k(i-1)+1}$, we obtain by (3.32),

$$\check{\psi}_{p,r,k,N,i}(t)=c(r,p)(kN)^{-r+1+\frac{1}{p}}\left(t-\bar{\tau}_{kN,k(i-1)+1}\right),\quad ,\tag{4.27}$$

so,in particular, $\check{\psi}_{p,r,k,N,i}\in L_q$.

Denote $\check{\Psi}^N_{p,r,k} := \{\check{\psi}_{p,r,k,N,i}\}_{i=1}^N$, and let $S_p^+(\check{\Psi}^N_{p,r,k})$, denote the positive p-sector over this system. Evidently, $S_p^+(\check{\Psi}^N_{p,r,k}) \subset \Delta^2_+ W^r_p$. Therefore

$$d_m\big(\Delta^2_+ W^r_p, \Delta^2_+ L_q\big)_{L_q} \ge d_m\big(S_p^+(\check{\Psi}^N_{p,r,k}), \Delta^2_+ L_q\big)_{L_q}. \tag{4.28}$$

Again define the discretization operator $A_{k,N,q} : L_q \ni x \to A_{k,N,q}x \in l_q^N$, by

$$A_{k,N,q}x := \left(|J_{kN,k}|^{-1+\frac{1}{q}} \int_{J_{kN,k}} x(t)dt, \dots, |J_{kN,kN}|^{-1+\frac{1}{q}} \int_{J_{kN,kN}} x(t)dt\right), \tag{4.29}$$

and it is easy to see that

$$\|x(\cdot)\|_{L_q} \ge \|A_{k,N,q}x\|_{l_q^N}, \quad x \in L_q.$$

Let M^m be an arbitrary linear manifold in L_q of dimension $\le m$. Then the set $A_{k,N,q}(M^m \cap \Delta^2_+ L_q)$ consists of vectors with convex coordinates, i.e., $A_{k,N,q}(M^m \cap \Delta^2_+ L_q) \subset \Delta^2_+$. By virtue of (4.28) we thus conclude that

$$\begin{aligned} d_m\big(S_p^+(\check{\Psi}^N_{p,r,k}), \Delta^2_+ L_q\big)_{L_q} &\ge d_m\big(A_{k,N,q} S_p^+(\check{\Psi}^N_{p,r,k}), \Delta^2_+\big)_{l_q^N} \\ &\ge d_m\big(A_{k,N,q} S_1^+(\check{\Psi}^N_{p,r,k}), \Delta^2_+\big)_{l_\infty^N}, \end{aligned} \tag{4.30}$$

since $1 \le p \le q \le \infty$. By (4.27), straightforward computations yield for $j \ge i$,

$$\begin{aligned} &|J_{kN,kj}|^{-1+\frac{1}{q}} \int_{J_{kN,kj}} \check{\psi}_{p,r,k,N,i}(t)dt \\ &= 2^{-1+\frac{1}{q}} c(r,p)(kN)^{-r+2+\frac{1}{p}-\frac{1}{q}} \int_{\tau_{kN,kj-1}}^{\tau_{kN,kj}} \big(t - \bar{\tau}_{kN,k(i-1)+1}\big)dt \\ &= 2^{1+\frac{1}{q}} c(r,p) k^{-r+\frac{1}{p}-\frac{1}{q}+1} N^{-r+\frac{1}{p}-\frac{1}{q}} \big((j-i+1) - \frac{1}{k}\big). \end{aligned} \tag{4.31}$$

Also, for $j < i$ we have

$$|J_{kN,kj}|^{-1+\frac{1}{q}} \int_{J_{kN,kj}} \psi_{p,r,k,N,i}(t)dt = 0. \tag{4.32}$$

Let $\check{E}^N$, be the system from (4.1), and recall $\{\tilde{e}^{(i)}\}_{i=1}^N$ from (3.4) (with N replacing n). Then by (4.31) and (4.32), it is readily seen that

$$A_{k,N,q}\check{\psi}_{p,r,k,N,i} = 2^{-1+\frac{1}{q}} c(r,p) k^{-r+\frac{1}{p}-\frac{1}{q}+1} N^{-r+\frac{1}{p}-\frac{1}{q}} (\check{e}^{(i)} - \frac{1}{k}\tilde{e}^{(i)}),$$

whence

$$d_m\big(A_{k,N,q}S_1^+\big(\check{\Psi}^N_{p,r,k}\big),\Delta^2_+\big)_{l^N_\infty}$$
$$\geq 2^{-1+\frac{1}{q}}c(r,p)k^{-r+\frac{1}{p}-\frac{1}{q}+1}N^{-r+\frac{1}{p}-\frac{1}{q}}\big(d_m\big(S_1^+\big(\check{E}^N\big),\Delta^2_+\big)_{l^N_\infty}-\frac{1}{k}\big).$$

Applying Lemma 5 with $m=n$ and $N=n+2$, we have

$$d_n\big(S_1^+\big(\check{E}^{n+2}\big),\Delta^2_+\big)_{l^{n+2}_\infty}\geq\frac{1}{26}.$$

So, prescribing $k=27$ yields,

$$d_n\big(A_{k,n+2,q}S_1^+\big(\check{\Psi}^{n+2}_{p,r,k}\big),\Delta^2_+\big)_{l^{n+2}_\infty}\geq cn^{-r+\frac{1}{p}-\frac{1}{q}},$$

where $c=c(r,p,q)>0$. Finally combining this with (4.28) and (4.30) completes the proof of the lower bound in (1.5).

We conclude with the proof of the lower bound in (1.6). In view of the inclusion $\Delta^2_+W^1_\infty\subseteq\Delta^2_+W^1_{p,\alpha}$, $1\leq p\leq\infty$, $0\leq\alpha<\infty$, it suffices to prove that

$$d_n\big(\Delta^2_+W^1_\infty,\Delta^2_+L_q\big)_{L_q}\geq cn^{-1-\frac{1}{q}},\quad 1\leq q\leq\infty. \tag{4.33}$$

Set

$$\check{\psi}_{\infty,1,k,N,i}(t):=(t-\tau_{kN,k(i-1)})_+,\quad i=1,\dots,N,\quad t\in I,$$

which clearly are convex and belong to $\Delta^2_+W^1_\infty\cap L_q$, $1\leq q\leq\infty$. Again denote $\check{\Psi}^N_{\infty,1,k}:=\big\{\check{\psi}_{\infty,1,k,N,i}\big\}^N_{i=1}$. Since $S_1^+\big(\check{\Psi}^N_{\infty,1,k}\big)\subset\Delta^2_+W^1_\infty$, we have

$$d_m\big(\Delta^2_+W^1_\infty,\Delta^2_+L_q\big)_{L_q}\geq d_m\big(S_1^+\big(\check{\Psi}^N_{\infty,1,k}\big),\Delta^2_+L_q\big)_{L_q}$$
$$\geq d_m\big(A_{k,N,q}S_1^+\big(\check{\Psi}^N_{\infty,1,k}\big),\Delta^2_+\big)_{l^N_\infty},$$

where $A_{k,N,q}$ was defined in (4.29). Now, for $j\geq i$ we have

$$|J_{kN,kj}|^{-1+\frac{1}{q}}\int_{J_{kN,kj}}\psi_{p,r,k,N,i}(t)dt$$
$$=2^{-1+\frac{1}{q}}(kN)^{1-\frac{1}{q}}\int_{\tau_{kN,kj-1)}}^{\tau_{kN,kj}}\big(t-\tau_{kN,k(i-1)}\big)dt$$
$$=2^{-1+\frac{1}{q}}(kN)^{1-\frac{1}{q}}\big(\tau_{kN,kj}-\tau_{kN,kj-1}\big)$$
$$\times\big(\tau_{kN,kj-1}+\tau_{kN,kj}-2\tau_{kN,k(i-1)}\big)$$
$$=2^{1+\frac{1}{q}}k^{-\frac{1}{q}}N^{-1-\frac{1}{q}}\big((j-i+1)-\frac{1}{2k}\big).$$

Also for $j < i$ we have

$$|J_{kN,kj}|^{-1+\frac{1}{q}} \int_{J_{kN,kj}} \psi_{p,r,k,N,i}(t)dt = 0,$$

and (4.33) follows as before, with the prescribed $k = 14$. This completes the proof of Theorem 3. $\square$

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