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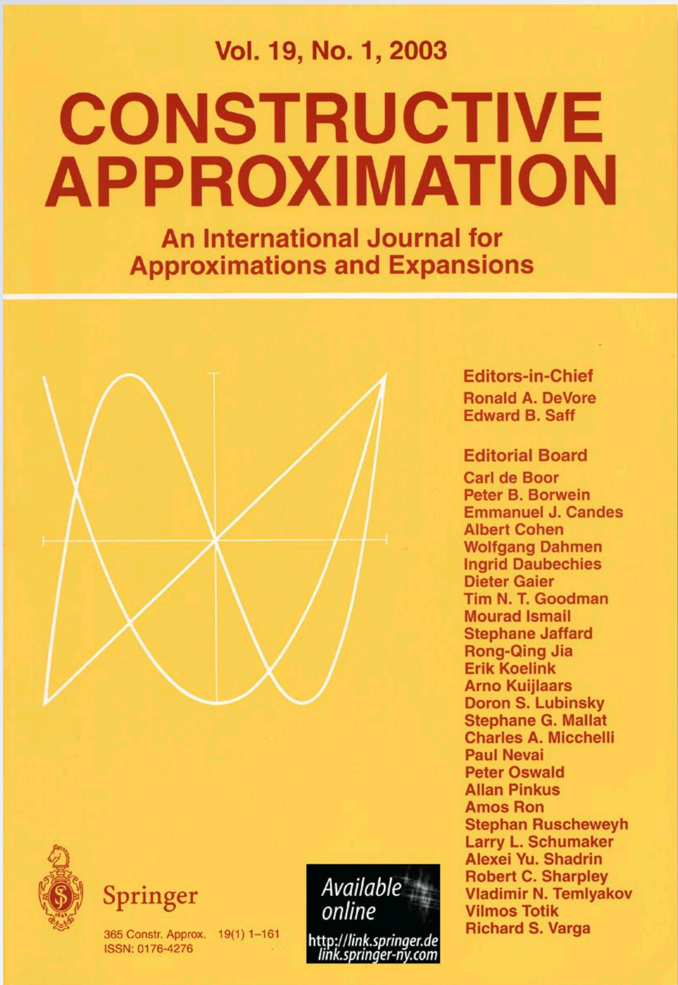

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CONSTRUCTIVE APPROXIMATION

Positive Results and Counterexamples in Comonotone Approximation

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Abstract We estimate the degree of comonotone polynomial approximation of continuous functions f, on $[-1,1]$, that change monotonicity $s\ge 1$ times in the interval, when the degree of unconstrained polynomial approximation $E_n(f)\le n^{-\alpha}$, $n\ge 1$. We ask whether the degree of comonotone approximation is necessarily $\le c(\alpha,s)n^{-\alpha}$, $n\ge 1$, and if not, what can be said. It turns out that for each $s\ge 1$, there is an exceptional set A_s of α's for which the above estimate cannot be achieved.

Keywords Comonotone polynomial approximation · Degree of approximation · Degree of comonotone approximation · Constants in constrained approximation

Mathematics Subject Classification (2000) 41A10 · 41A25 · 41A29

1 Introduction

Let $\mathbf{P}_n$ be the space of algebraic polynomials of degree $<n$. For $f\in C[a,b]$, set

$$\|f\|_{[a,b]} := \max_{x\in[a,b]}\bigl|f(x)\bigr|,$$

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and let

$$E_n(f)_{[a,b]} := \inf_{P_n \in \mathbf{P}_n} \|f - P_n\|_{[a,b]}$$

denote its degree of best approximation. In particular, for $f \in C[-1,1]$, denote $\|f\| := \|f\|_{[-1,1]}$ and $E_n(f) := E_n(f)_{[-1,1]}$.

Denote by Δ^1 the set of monotone, say, nondecreasing functions $f \in C[-1,1]$, and, as usual, for $f \in \Delta^1$, let

$$E_n^{(1)}(f) = \inf_{P_n \in \mathbf{P}_n \cap \Delta^1} \|f - P_n\|$$

be the degree of best monotone approximation of f. Clearly,

$$E_n(f) \le E_n^{(1)}(f). \tag{1.1}$$

The inverse inequality, in general, cannot be had, since Lorentz and Zeller [12] constructed a function $f \in \Delta^1$ such that

$$\limsup_{n\to\infty} \frac{E_n^{(1)}(f)}{E_n(f)} = \infty.$$

The first result we have is that in certain cases one may still achieve a kind of inverse to (1.1), in the sense that one may obtain information on the degree of best monotone approximation from knowledge of the degree of best unconstrained approximation. We have:

Theorem 1 *Let $\alpha > 0$. Then there exists $c(\alpha)$, a constant depending only on α such that, if $f \in C[-1,1]$ is a monotone function and*

$$n^\alpha E_n(f) \le 1, \quad n \ge 1,$$

then

$$n^\alpha E_n^{(1)}(f) \le c(\alpha), \quad n \ge 1.$$

For $\alpha < 2$, Theorem 1 follows from [8]; for $\alpha > 2$, it follows from [5]; and for $\alpha = 2$, it is proved in [9].

We wish to extend Theorem 1 to functions that have a finite number changes of monotonicity in $[-1,1]$. It turns out that if α is not an integer, we can always do that, whereas for integer α's it is not always so.

2 Definitions and Formulation of the Main Results

Given $s \ge 1$, let $\mathbf{Y}_s[a,b]$ denote the set of all collections $Y_s = \{y_i\}_{i=1}^s$, of points y_i such that $a < y_1 < \cdots < y_s < b$. For a collection $Y_s = \{y_i\}_{i=1}^s \in \mathbf{Y}_s[a,b]$, we write $f \in \Delta^1(Y_s;[a,b])$ if $f \in C[a,b]$ is nondecreasing on $[y_s,b]$, nonincreasing on $[y_{s-1}, y_s]$, and so on, and, finally, $(-1)^s f$ is nondecreasing on $[a, y_1]$. Clearly, if $f \in C^1(a,b)$, then $f \in \Delta^1(Y_s;[a,b])$ if and only if

$$f'(x)\prod_{i=1}^{s}(x-y_i) \ge 0, \quad x \in (a,b). \tag{2.1}$$

For $f \in \Delta^1(Y_s;[a,b])$, we define by

$$E_n^{(1)}(f,Y_s)_{[a,b]} := \inf_{P_n \in \mathbb{P}_n \cap \Delta^1(Y_s;[a,b])} \|f-P_n\|_{[a,b]}, \tag{2.2}$$

the degree of best comonotone approximation of f, relative to Y_s. In the formulations of some of the theorems below, we do not wish to specify Y_s; rather, f is such that $f \in \Delta^1(Y_s;[a,b])$ for some $Y_s \in \mathbf{Y}_s$. In this case, we write $f \in \Delta_s^1([a,b])$ and put

$$E_n^{1,s}(f)_{[a,b]} := \sup_{Y_s \in \mathbf{Y}_s[a,b]:\, f \in \Delta^1(Y_s;[a,b])} E_n^{(1)}(f,Y_s)_{[a,b]}. \tag{2.3}$$

Again, in the case $[a,b]=[-1,1]$, we suppress reference to the interval; namely, we write $\Delta^1(Y_s) := \Delta^1(Y_s;[-1,1])$ and $\Delta_s^1 := \Delta_s^1([-1,1])$.

In order to formulate the main negative result, we define exceptional sets of integers A_s, where $s \ge 1$ is going to be the number of changes of monotonicity of the function f.

Definition Set $A_1 := \{2\}$, and for each $s \ge 2$, let

$$A_s := \{j \mid 1 \le j \le s-1, \text{ or } j = 2i, 1 \le i \le s\}.$$

E.g.,

$$A_2 = \{1,2,4\}, \qquad A_3 = \{1,2,4,6\},$$
$$A_4 = \{1,2,3,4,6,8\}, \qquad A_5 = \{1,2,3,4,6,8,10\}, \qquad \text{etc.}$$

Theorem 2 *Given $s \in \mathbf{N}$, let $\alpha \in A_s$. Then there is a constant $c(s) > 0$, which depends only on s, such that for each $m \in \mathbf{N}$, there exists a function $f \in C^1(-1,1) \cap \Delta_s^1$ satisfying*

$$n^\alpha E_n(f) \le 1, \quad n \ge 1, \tag{2.4}$$

while

$$m^\alpha E_m^{1,s}(f) \ge c(s)\ln m.$$

We now formulate the positive results. In particular, we show that all exceptional cases are covered by Theorem 2.

Theorem 3 *Given $s \in \mathbf{N}$, let $\alpha > 0$ be such that $\alpha \notin A_s$. Then there exists $c(\alpha,s)$, a constant which depends only on α and s such that, if $f \in \Delta_s^1$ and satisfies*

$$n^\alpha E_n(f) \le 1, \quad n \ge 1, \tag{2.5}$$

then

$$n^\alpha E_n^{1,s}(f) \le c(\alpha, s), \quad n \ge 1. \tag{2.6}$$

Note that if $\alpha > 1$, then (2.5) implies $f \in C^1(-1,1)$. If $\alpha < 1$, then Theorem 3 is [9, Theorem 1]. Therefore, in the proofs, we will concentrate on the case $\alpha > 1$; thus the definition of $\Delta^1(Y_s)$ given by (2.1) will apply.

For the sake of completeness, we emphasize that, Theorem 2 notwithstanding, all is not lost for $\alpha \in A_s$, since we still have:

Theorem 4 *Let $s \in \mathbf{N}$ and $\alpha \in A_s$. Then there exist constants $c(s)$ and $N(Y_s)$, depending only on s and Y_s, respectively, such that for each function $f \in \Delta^1(Y_s)$ satisfying*

$$n^\alpha E_n(f) \le 1, \quad n \ge 1,$$

we have

$$n^\alpha E_n^{(1)}(f, Y_s) \le c(s), \quad n \ge N(Y_s).$$

For $\alpha \neq 2$, Theorem 4 follows from [10, Theorem 4], and for $\alpha = 2$, it follows from [11, Corollary 2].

It is worth mentioning that similar investigation for coconvex approximation is done in [7]. However, in coconvex approximation, there are no results analogous to those of Theorem 3 for $s \ge 2$.

In Sect. 3, we give some auxiliary notation and known results. Then we prove Theorem 2 in Sect. 4, and in Sect. 5 we prove Theorem 3.

Above and subsequently, we have positive constants c, depending on certain parameters and only on those parameters. We indicate this dependence by $c(\cdot, \dots, \cdot)$. The constants may differ from one another even when they look exactly the same and appear on the same line. Sometimes we will need to single out a constant which we will need to return to. Such a constant will have a subscript, i.e., $c_k(\cdot, \dots, \cdot)$. Finally, it is obvious that some of the constants below depend on the function Ψ, defined below, which we keep fixed throughout the paper. Thus, we suppress reference to this dependence.

3 Auxiliary Results

Let $g \in C[a,b]$, and recall that

$$\begin{aligned} \Delta_h(g,x) &:= \begin{cases} g(x+\frac{h}{2}) - g(x-\frac{h}{2}), & x \pm \frac{h}{2} \in [a,b], \\ 0, & \text{otherwise}, \end{cases} \\ \Delta_h^k(g,x) &:= \Delta_h\big(\Delta_h^{k-1}(g,x)\big), \quad k > 1, \end{aligned} \tag{3.1}$$

and denote by

$$\omega\big(g,t,[a,b]\big) := \sup_{0<h\le t} \big\|\Delta_h(g,\cdot)\big\|_{[a,b]}$$

and

$$\omega_k\big(g,t,[a,b]\big) := \sup_{0<h\le t} \big\| \Delta_h^k(g,\cdot)\big\|_{[a,b]}, \quad k\ge 1,$$

respectively, its modulus of continuity and its kth modulus of smoothness. (Note that $\omega(g,t,[a,b]) \equiv \omega_1(g,t,[a,b])$ and that $\omega_k(g,t,[a,b]) \equiv \omega_k(g,(b-a)/k,[a,b])$ for $t\ge (b-a)/k$.)

Again, when $[a,b]=[-1,1]$, we suppress the interval in all moduli; i.e., we write $\omega_k(g,t) := \omega_k(g,t,[-1,1])$.

We write $f\in Z[a,b]$, the Zygmund class, if $f\in C[a,b]$ and

$$\omega_2\big(f,t,[a,b]\big)\le t, \quad t\ge 0.$$

It is well known that if $f\in C^r[a,b]$, $r\ge 0$, and $f^{(r)}\in Z[a,b]$, then

$$E_n(f)_{[a,b]} \le \frac{c(r)(b-a)^{r+1}}{n^{r+1}}, \quad n\ge r+1. \tag{3.2}$$

We write $f\in B^r[a,b]$, $r\ge 1$, the Babenko class (first introduced by Babenko [1]), if $f\in C[a,b]$ has a locally absolutely continuous $(r-1)$st derivative in (a,b) and

$$\Big|\big((x-a)(b-x)\big)^{r/2} f^{(r)}(x)\Big| \le 1, \quad \text{a.e., in } [a,b].$$

It is well known (see, e.g., [2, Theorems 2.2.1 and 7.2.1]) that, if $f\in B^r[a,b]$, then $\omega_r^{\varphi}(f,t,[a,b])\le ct^r$, so that

$$E_n(f)_{[a,b]} \le \frac{c(r)}{n^r}, \quad n\ge r. \tag{3.3}$$

Next we state Dzyadyk's inequality for the derivatives of polynomials (see, e.g., [3, Chap. 7, Lemma 2.1 (p. 384)], see also [6, Lemma 5.2] for a short proof).

Lemma 1 *Given $x_0\in[a,b]$, assume that a polynomial $P_n\in\mathbf{P}_n$ satisfies*

$$\big|P_n(x)\big| \le 1+\left(\frac{|x-x_0|}{\rho_n(x_0)}\right)^m, \quad x\in[a,b],$$

for some $m\in\mathbf{N}$, where

$$\rho_n(x) := \frac{b-a}{2n^2} + \frac{\sqrt{(x-a)(b-x)}}{n}. \tag{3.4}$$

Then for each $j\in\mathbf{N}$,

$$\big|P_n^{(j)}(x_0)\big| \le \frac{c(j,m)}{\rho_n^j(x_0)}.$$

Recall that the $(k-1)$st divided difference of g, at the distinct points $\{u_1,\dots,u_k\}$, is defined by

$$[u_1,u_2,\dots,u_k;g] := \sum_{i=1}^k \frac{g(u_i)}{\varpi'(u_i)},$$

where

$$\varpi(x) := \prod_{j=1}^{k}(x-u_j),$$

and that if $[a,b] := [\min u_j, \max u_j]$, and $g \in C[a,b]$ possesses a $(k-1)$st derivative in (a,b), then

$$[u_1,u_2,\dots,u_k;g] = \frac{g^{(k-1)}(\zeta)}{(k-1)!} \tag{3.5}$$

for some $\zeta \in (a,b)$.

The Lagrange polynomial interpolating g at $\{u_1,\dots,u_k\}$ is defined by

$$L_k(x) = \sum_{j=1}^{k} g(u_j)l_j(x), \tag{3.6}$$

where

$$l_j(x) := \frac{\prod_{i\neq j}(x-u_i)}{\varpi'(u_j)}.$$

The following Newton representation of the Lagrange polynomial is well known:

$$\begin{aligned} L_k(x) &:= L_k(g;u_1,\dots,u_k;x) \\ &:= g(u_1) + [u_1,u_2;g](x-u_1) + \cdots \\ &\quad + [u_1,\dots,u_k;g](x-u_1)\cdots(x-u_{k-1}). \end{aligned} \tag{3.7}$$

It is also well known that

$$g(x) - L_k(x) = [x,u_1,\dots,u_k;g](x-u_1)\cdots(x-u_k). \tag{3.8}$$

Hence,

$$\begin{aligned} g(x) - L_k(x) &= g(x) - L_{k-1}(g;u_1,\dots,u_{k-1};x) \\ &\quad - [u_1,\dots,u_k;g](x-u_1)\cdots(x-u_{k-1}) \\ &= (x-u_1)\cdots(x-u_{k-1})\big([x,u_1,\dots,u_{k-1};g] - [u_1,\dots,u_k;g]\big) \\ &= \frac{g^{(k-1)}(\zeta_1) - g^{(k-1)}(\zeta_2)}{(k-1)!}(x-u_1)\cdots(x-u_{k-1}) \\ &=: \frac{\omega}{(k-1)!}(x-u_1)\cdots(x-u_{k-1}), \end{aligned} \tag{3.9}$$

where $\zeta_1, \zeta_2 \in (a.b)$, and we note that

$$|\omega| = \big|g^{(k-1)}(\zeta_1) - g^{(k-1)}(\zeta_2)\big| \le \omega\big(g^{(k-1)}, b-a, [a,b]\big) \tag{3.10}$$

if $g \in C^{k-1}[a,b]$.

Finally, for the proof of Theorem 3, we need two propositions and a lemma. Both propositions are immediate consequences of [4, Corollary 3.1]; however, since this paper is not easily accessible, we provide short proofs.

Proposition 1 *Given a collection* $\{y_i\}_{i=1}^s$ *of distinct points* $y_i \in (a,b)$, *if a function* $g \in C^l[a,b]$ *satisfies*

$$g(x)\prod_{i=1}^s (x-y_i) \ge 0, \quad x \in (a,b),$$

then there is a polynomial $P_{l+1} \in \mathbf{P}_{l+1}$ *such that*

$$\|g - P_{l+1}\|_{C[a,b]} \le (b-a)^l \omega\bigl(g^{(l)}, b-a, [a,b]\bigr) \tag{3.11}$$

and

$$P_{l+1}(x)\prod_{i=1}^s (x-y_i) \ge 0, \quad x \in (a,b).$$

Proof For $l < s$, we take $P_{l+1}(x) := L_{l+1}(g; y_1, \dots, y_{l+1}; x) \equiv 0$, and Proposition 1 follows by (3.9) and (3.10). Similarly, for $l = s$, we take

$$P_{l+1}(x) := L_{l+1}(g; y_1, \dots, y_s, b; x) = g(b)\prod_{i=1}^s \frac{x-y_i}{b-y_i}.$$

Otherwise $l > s$. We take u_j, $j = s+1, \dots, l+1$, to be arbitrary distinct points in (a,b), different from y_i, $1 \le i \le s$. Finally, we set

$$\begin{aligned} P_{l+1}(x) &:= L_{l+1}(g; y_1, \dots, y_s, u_{s+1}, \dots, u_{l+1}; x) \\ &\quad + \frac{(b-a)^{l-s}}{l!}\omega\bigl(g^{(l)}, b-a, [a,b]\bigr)\prod_{i=1}^s (x-y_i). \end{aligned}$$

By virtue of (3.9) and (3.10), it readily follows that

$$P_{l+1}(x)\prod_{i=1}^s (x-y_i) \ge g(x)\prod_{i=1}^s (x-y_i) \ge 0$$

and that (3.11) holds. □

An immediate consequence is:

Corollary 1 *If* $g \in \Delta_s^1[a,b] \cap C^r[a,b]$, *then*

$$E_{r+1}^{1,s}(g)_{[a,b]} \le (b-a)^r \omega\bigl(g^{(r)}, b-a, [a,b]\bigr). \tag{3.12}$$

Proof Note that g' satisfies the conditions of Proposition 1 with $l = r - 1$. Hence, there exists P_r such that $P_r(x)g'(x) \ge 0$, $x \in (a, b)$, and

$$\left\| g' - P_r \right\|_{C[a,b]} \le (b-a)^{r-1} \omega\bigl(g^{(r)}, b-a, [a,b]\bigr).$$

The polynomial $P_{r+1}(x) := \int_a^x P_r(t)\,dt + g(a)$ readily yields (3.12). □

Proposition 2 *Given a collection* $\{y_i\}_{i=1}^s$ *of distinct points* $y_i \in (a, b)$ *and* $l \ge s - 1$, *if a function* $g \in C^l[a, b]$ *satisfies*

$$g(y_i) = 0, \quad i = 1, \dots, s,$$

then there is a polynomial $P_{l+2} \in \mathbf{P}_{l+2}$ *such that*

$$\|g - P_{l+2}\|_{C[a,b]} \le c(l)(b-a)^l \omega_2\bigl(g^{(l)}, b-a, [a,b]\bigr)$$

and

$$P_{l+2}(y_i) = 0, \quad i = 1, \dots, s.$$

Furthermore, if

$$g(x) \prod_{i=1}^{s} (x - y_i) \ge 0, \quad x \in (a, b), \tag{3.13}$$

then we may take the polynomial to satisfy

$$P_{l+2}(x) \prod_{i=1}^{s} (x - y_i) \ge 0, \quad x \in (a, b). \tag{3.14}$$

Proof We begin by noting that it follows by [3, p. 239, Theorem 3.6.4] that if $a \le v_0 < v_1 < \cdots < v_{l+2} \le b$, then

$$\bigl|[v_0, \dots, v_{l+2}; g]\bigr| \le c(l) \frac{\omega_2(g^{(l)}, b-a, [a,b])}{(v_{l+2} - v_1)(v_{l+1} - v_0)}. \tag{3.15}$$

For $l = s - 1$, we take

$$P_{l+2}(x) := L_{l+2}(g; y_0, y_1, \dots, y_s; x) = g(y_0) \prod_{i=1}^{s} \frac{x - y_i}{y_0 - y_i},$$

where $y_0 = a$, if $y_1 - a > b - y_1$, and $y_0 = b$ otherwise. Then substituting (3.15) in (3.8) and applying simple calculations according to the various possibilities for the locations of y_0 and x yields the desired estimate. If $l \ge s$, then we take u_j, $j = s+1, \dots, l$, to be arbitrary distinct points in (a, b), different from y_i, $1 \le i \le s$, and set

$$L_{l+2}(x) := L_{l+2}(g; y_1, \dots, y_s, u_{s+1}, \dots, u_l, a, b; x).$$

Again substituting (3.15) in (3.8) and applying simple calculations yields

$$\left|g(x) - L_{l+2}(x)\right| \le c_*(l)(b-a)^{l-s}\omega_2\big(g^{(l)}, b-a, [a,b]\big)\prod_{i=1}^{s}|x-y_i|, \quad x \in [a,b].$$

Hence, we take

$$P_{l+2}(x) := L_{l+2}(x) + c_*(l)(b-a)^{l-s}\omega_2\big(g^{(l)}, b-a, [a,b]\big)\prod_{i=1}^{s}(x-y_i)$$

and obtain the desired estimate with $c(l) = 2c_*(l)$. Furthermore, if g satisfies (3.13), then we also obtain (3.14). □

An immediate consequence is:

Corollary 2 *If $g \in \Delta_s^1[a,b] \cap C^r[a,b]$, $r \ge s$, then*

$$E_{r+2}^{1,s}(g)_{[a,b]} \le c(r)(b-a)^r\omega_2\big(g^{(r)}, b-a, [a,b]\big). \tag{3.16}$$

Proof Again, g' satisfies the conditions of Proposition 2 with $l = r-1$. Hence we proceed as in the proof of Corollary 1. □

We also need a similar result for $r = s-1$.

Lemma 2 *Let $r = \sigma - 1$ and $a < y_1 < \cdots < y_\sigma < b$, and take*

$$\frac{1}{9}(b-a) < a - a_1 < 9(b-a) \quad \textit{and} \quad \frac{1}{9}(b-a) < b_1 - b < 9(b-a). \tag{3.17}$$

If $g \in \Delta^1(Y_\sigma; [a_1,b_1]) \cap C^r[a_1,b_1]$, then

$$E_{r+2}^{1,\sigma}(g)_{[a,b]} \le E_1^{1,\sigma}(g)_{[a,b]} \le c(r)(b-a)^r\omega_2\big(g^{(r)}, b-a, [a_1,b_1]\big). \tag{3.18}$$

Proof First assume that $r > 0$. Since $g'(y_i) = 0$, $1 \le i \le \sigma-1$, and $g' \in C^{r-1}[a_1,b_1]$, it follows by Proposition 2 with $l = r-1$ and $s = \sigma - 1$ that there exists a polynomial $P_{r+1} \in \mathbb{P}_{r+1} = \mathbb{P}_\sigma$ such that

$$\left\|g' - P_{r+1}\right\|_{[a_1,b_1]} \le c(r)(b_1-a_1)^{r-1}\omega_2\big(g^{(r)}, b_1-a_1, [a_1,b_1]\big) \tag{3.19}$$

and

$$P_{r+1}(y_i) = 0, \quad 1 \le i \le \sigma - 1.$$

As P_{r+1} is of degree $\sigma - 1$, it is

$$P_{r+1}(x) = A\prod_{i=1}^{\sigma-1}(x-y_i).$$

Now, g' changes its sign once more in $[a_1, b_1]$ (at y_σ), whereas P_{r+1} cannot change its sign again. Hence, with either $J := [a_1, a]$ or $J := [b, b_1]$, we have

$$g'(x)P_{r+1}(x) \le 0, \quad x \in J.$$

This, in turn, implies (applying (3.19))

$$\begin{aligned}\|P_{r+1}\|_J &\le \|g' - P_{r+1}\|_J \le \|g' - P_{r+1}\|_{[a_1,b_1]} \\ &\le c(r)(b_1 - a_1)^{r-1}\omega_2\big(g^{(r)}, b_1 - a_1, [a_1, b_1]\big).\end{aligned} \tag{3.20}$$

Now, by (3.19),

$$\begin{aligned}\|g'\|_{[a_1,b_1]} &\le \|g' - P_{r+1}\|_{[a_1,b_1]} + \|P_{r+1}\|_{[a_1,b_1]} \\ &\le c(r)(b_1 - a_1)^{r-1}\omega_2\big(g^{(r)}, b_1 - a_1, [a_1, b_1]\big) + \frac{(b_1 - a_1)^{\sigma-1}}{|J|^{\sigma-1}}\|P_{r+1}\|_J \\ &\le c(r)(b - a)^{r-1}\omega_2\big(g^{(r)}, b - a, [a_1, b_1]\big),\end{aligned}$$

where we used the fact that $|J| \sim b_1 - a_1 \sim b - a$ (see (3.17)), and applied (3.20).

Thus,

$$\|g - g(a)\|_{[a,b]} \le \int_a^b |g'(t)|\,dt \le c(r)(b - a)^r\omega_2\big(g^{(r)}, b - a, [a_1, b_1]\big),$$

whence the required polynomial may be taken to be $P_1(x) \equiv g(a)$. This completes the proof for $r > 0$.

If $r = 0$, then $\sigma = 1$. Without loss of generality, we may assume that $g(a_1) \le g(b_1)$. Then the linear $P_1(x) \equiv g(a_1)$ (which clearly is comonotone with g) interpolates g at two points the distance between which is $\ge y_1 - a_1 > a - a_1 \sim b_1 - a_1$ (where y_1 is the point of monotonicity change of g, and see (3.17)). Hence, by Whitney's theorem (see, e.g., [3, Chap. 3, (6.2) (p. 230)]),

$$\|g - P_1\|_{[a,b]} \le c\omega_2\big(g, b - a, [a_1, b_1]\big),$$

as $b - a \sim b_1 - a_1$, again by (3.17). This concludes the proof of Lemma 1. □

4 Proof of Theorem 2

Our strategy is to construct, for each $\alpha \in A_s$, a function which is well approximated by algebraic polynomials when no constraints are imposed on the polynomials, but if certain derivatives of these polynomials have to vanish, then they yield weaker approximation rate. Then by adding an oscillating polynomial to the function, we will guarantee that we have an element with s changes of monotonicity without destroying the above two properties. We will have to deal separately with even α's and with odd ones, because even though the ideas are similar, the functions for even α are defined on $[0, 2]$, while those for odd α are defined on $[-2, 2]$.

Let Ψ be an infinitely differentiable function on $\mathbb{R}$, decreasing on $[1,2]$, and satisfying

$$\Psi(x) = \begin{cases} 1, & \text{if} \quad x \le 1, \\ -1, & \text{if} \quad x \ge 2, \end{cases}$$

and

$$\int_1^2 \Psi(u) \frac{du}{u} = 0.$$

For each $a \in (0, \frac{1}{2}]$, set

$$g_a(x) := \int_{a^2}^x \Psi\left(\frac{u}{a}\right) \frac{du}{u}, \quad x > 0,$$

and note that

$$g_a(x) = \begin{cases} \ln \frac{x}{a^2}, & \text{if } x \in (0, a], \\ \ln \frac{2}{x}, & \text{if } x \ge 2a. \end{cases}$$

It is readily seen that

$$\left(x - a^2\right) g_a(x) \ge 0, \quad x \in (0, 2],$$

and that

$$\max_{x \in (0,2]} x\left|g_a(x)\right| < 1. \tag{4.1}$$

Also, evidently, g_a is infinitely differentiable for $x > 0$, and for each $j \in \mathbf{N}$,

$$\left|g_a^{(j)}(x)\right| \le \frac{c(j)}{x^j}, \quad x > 0. \tag{4.2}$$

Fix $m \ge 2$ throughout this section.

Given an even $r > 0$, let $f_r \in C[0,2]$ be the function defined by

$$f_r(x) = \int_{1/m^4}^x (x-u)^{\frac{r}{2}} \Psi\left(um^2\right) \frac{du}{u}, \quad x \in [0,2]. \tag{4.3}$$

If $r \ge 1$ is odd, then let $f_r \in C[-2,2]$ be the function defined by

$$f_r(x) = \int_0^x u^{r-1} g_{1/m}\left(|u|\right) du, \quad x \in [-2,2]. \tag{4.4}$$

Note that $g_a(|x|)$ is integrable in $[-2,2]$, so that f_1 is well defined, it is continuous at $x = 0$, and $f_1(0) = 0$.

For the sake of simplifying notation, we set

$$a := \begin{cases} \frac{1}{m} & \text{if } r \text{ is odd}, \\ \frac{1}{m^2} & \text{if } r \text{ is even}. \end{cases}$$

We begin with two lemmas.

Lemma 3 *If $r>0$ is even, then*

$$E_n(f_r)_{[0,1]} \le \frac{c(r)}{n^r}, \quad n \ge 1. \tag{4.5}$$

Proof Set $p = \frac{r}{2}$. By (4.3) and (4.2),

$$\left|f_r^{(r)}(x)\right| = p!\left|g_a^{(p)}(x)\right| \le \frac{p!c(p)}{x^p} =: \frac{c(r)}{x^{r/2}}, \quad x>0.$$

Therefore, $\frac{1}{c(r)} f_r \in B^r[0,1]$, where B^r is the Babenko class. Hence, (3.3) implies (4.5) for $n \ge r$. For $n<r$, (4.5) follows from the estimate

$$\|f_r\|_{[0,1]} \le p+1.$$

Indeed, we get for $x \in (0,1]$,

$$\begin{aligned}
\left|f_r(x)\right| &= \left|x^p g_a(x) + \int_{a^2}^x \left((x-u)^p - x^p\right)\Psi\left(\frac{u}{a}\right)\frac{du}{u}\right| \\
&\le x^{p-1} + \left|\int_{a^2}^x \left(x^p - (x-u)^p\right)\frac{du}{u}\right| \\
&\le x^{p-1} + p \le p+1,
\end{aligned}$$

where for the first inequality we have applied (4.1). □

Lemma 4 *If r is odd, then*

$$E_n(f_r)_{[-1,1]} \le \frac{c(r)}{n^r}, \quad n \ge 1. \tag{4.6}$$

Proof It follows by (4.4) that $f_r^{(r-1)}$ is continuous in $[0,1]$, and (4.2) yields

$$\left|f_r^{(r+1)}(x)\right| \le \frac{c(r)}{x}, \quad x>0.$$

This implies that $\frac{1}{c(r)} f_r^{(r-1)} \in B^2[0,1]$, which, in turn, yields $\frac{1}{c(r)} f_r^{(r-1)} \in Z[0,1]$. Indeed, it is well known (see, e.g., [3, p. 272 (9.9)]) that $g \in B^2[0,1]$ implies $\omega_2(g,t) \le \omega_2^{\varphi}(g,\sqrt{t}) \le t$. (Here $\varphi(x) := \sqrt{x(1-x)}$.)

Since $f_r^{(r-1)}$ is odd, we have $\frac{1}{c(r)} f_r^{(r-1)} \in Z[-1,1]$. Hence (3.2) implies (4.6) for $n \ge r+1$. For $n \le r$, (4.6) follows from the estimate

$$\|f_r\|_{[-1,1]} < 1.$$

This completes the proof. □

Next we show that some constrained polynomials do not approximate f_r so well. First for even r, we have:

Lemma 5 *If $r > 0$ is even, then there exists a constant $c(r) > 0$ such that for any polynomial $P_m \in \mathbf{P}_m$ satisfying $P_m^{(\frac{r}{2})}(\theta) = 0$ for some point $\theta \in [0, \frac{1}{m^2}]$, we have*

$$\|f_r - P_m\|_{[0,1]} \ge c(r)\frac{\ln m}{m^r}.$$

Proof Let $p = \frac{r}{2}$, and set

$$L(x) := \int_{a^2}^{a} (x-u)^p \frac{du}{u},$$

where we recall that $a = \frac{1}{m^2}$ and, by (4.3),

$$G(x) := f_r(x) - L(x) = \int_a^x (x-u)^p \Psi\left(\frac{u}{a}\right)\frac{du}{u}.$$

For $x \ge a$, we have

$$\begin{aligned}
\left|G(x)\right| &\le x^p \int_a^x \frac{du}{u} = x^p \ln\left(m^2 x\right) \\
&< x^{p+1} m^2 = \frac{1}{m^{2p}}\left(xm^2\right)^{p+1} \\
&\le \frac{c(p)}{m^{2p}}\left(\theta^{p+1} + (x-\theta)^{p+1}\right)m^{2(p+1)} \\
&\le \frac{c(p)}{m^{2p}}\left(1 + \left(\frac{x-\theta}{\rho_m(\theta)}\right)^{p+1}\right),
\end{aligned}$$

since by (3.4), for the interval [0, 1], $\frac{1}{m^2} \le \rho_m(\theta) = \frac{1}{m^2} + \frac{1}{m}\sqrt{\theta(1-\theta)} < \frac{2}{m^2}$.

If, on the other hand, $x \in [0, a]$, then

$$\left|G(x)\right| \le \left|G(0)\right| = \frac{1}{pm^{2p}}.$$

Now let

$$A := m^{2p}\|f_r - P_m\|_{[0,1]}.$$

It follows that for all $x \in [0, 1]$,

$$\begin{aligned}
\left|L(x) - P_m(x)\right| &\le \left|f_r(x) - P_m(x)\right| + \left|L(x) - f_r(x)\right| \le \frac{A}{m^{2p}} + \left|G(x)\right| \\
&\le \frac{c(p)\max\{A, 1\}}{m^{2p}}\left(1 + \left(\frac{|x-\theta|}{\rho_m(\theta)}\right)^{p+1}\right).
\end{aligned}$$

By virtue of Lemma 1, we obtain

$$2(p!)\ln m = \left|L^{(p)}(\theta)\right| = \left|L^{(p)}(\theta) - P_m^{(p)}(\theta)\right|$$

$$\begin{aligned}&\le \frac{c(p)\max\{A,1\}}{m^{2p}}\rho_m^{-p}(\theta)\\&\le c(p)\max\{A,1\},\end{aligned}$$

whence

$$A > c(r)\ln m.$$

This concludes our proof. □

Next, for odd r, we have:

Lemma 6 *If r is odd, then there exists a constant $c(r)>0$ such that for any polynomial $P_m\in\mathbf{P}_m$ satisfying $P_m^{(r)}(\theta)=0$ for some point $\theta\in[-\frac{1}{m},\frac{1}{m}]$, we have*

$$\|f_r-P_m\|_{[-1,1]}\ge c(r)\frac{\ln m}{m^r}.$$

Proof Let

$$L(x):=\int_0^x u^{r-1}\,du\int_{a^2}^a\frac{dt}{t}=\frac{1}{r}x^r\ln 1/a,$$

where we recall that $a=\frac{1}{m}$ and, by (4.4),

$$G(x):=f_r(x)-L(x)=\int_0^x u^{r-1}\left(\int_{1/m}^{|u|}\Psi\left(\frac{t}{a}\right)\frac{dt}{t}\right)du.$$

For $|x|\ge a$, we have

$$\begin{aligned}\bigl|G(x)\bigr|&\le |x|^r\ln\bigl(m|x|\bigr)\\&< x^{r+1}m=\frac{1}{m^r}(xm)^{r+1}\\&\le\frac{c(r)}{m^r}\left(1+\left(\frac{x-\theta}{\rho_m(\theta)}\right)^{r+1}\right),\end{aligned}$$

since by (3.4), for the interval $[-1,1]$, $\frac{1}{m}<\rho_m(\theta)=\frac{1}{m^2}+\frac{1}{m}\sqrt{1-\theta^2}<\frac{2}{m}$.

If, on the other hand, $|x|<\frac{1}{m}$, then

$$\bigl|G(x)\bigr|\le\bigl|G(a)\bigr|=\int_0^a u^{r-1}\ln\frac{a}{u}\,du=\frac{a^r}{r^2}=\frac{1}{r^2m^r}.$$

We now proceed just as in the proof of Lemma 6, to obtain

$$\begin{aligned}(r-1)!\ln m=\bigl|L^{(r)}(\theta)\bigr|&=\bigl|L^{(r)}(\theta)-P_m^{(r)}(\theta)\bigr|\\&\le\frac{c(r)\max\{A,1\}}{m^r}\rho_m^{-r}(\theta)\\&\le c(r)\max\{A,1\},\end{aligned}$$

whence

$$A \geq c(r) \ln m.$$

This completes the proof. □

We set

$$\mu_r := \min_{x \in [a,1]} f_r'(x),$$

and it is readily seen that $\mu_r > 0$.

We proceed with two lemmas for an even $r > 0$.

Lemma 7 *Let $r > 0$ be even. Then there exists a function $f \in \Delta^1_{\frac{r}{2}}[0,1]$, satisfying*

$$n^r E_n(f)_{[0,1]} \leq 1, \quad n \geq 1, \tag{4.7}$$

and

$$m^r E_m^{1,\frac{r}{2}}(f)_{[0,1]} \geq c(r) \ln m, \tag{4.8}$$

where $c(r) > 0$.

Proof Let $p := \frac{r}{2}$. Since $f_r^{(j)}(a^2) = 0$ for all $0 \leq j \leq p$, it follows that there is $\delta > 0$ such that $a^2 + \delta < a$ and

$$\|f_r^{(j)}\|_{[a^2,a^2+\delta]} < \frac{1}{e} \min\left\{\mu_r, \frac{1}{m^r}\right\}, \quad j = 1, \dots, p. \tag{4.9}$$

Take a collection of p distinct points

$$a^2 < y_1 < \cdots < y_p < a^2 + \delta,$$

and denote by $L_p(x) := L(f_r'; y_1, \dots, y_p; x)$ the Lagrange polynomial interpolating f_r' at these points. By virtue of (3.7) and (3.5), (4.9) yields

$$\begin{aligned}\|L_p\|_{[0,1]} &\leq \|f_r'\|_{[a^2,a^2+\delta]} + \frac{\|f_r''\|_{[a^2,a^2+\delta]}}{1!} + \cdots + \frac{\|f_r^{(p)}\|_{[a^2,a^2+\delta]}}{(p-1)!} \\ &< \min\left\{\mu_r, \frac{1}{m^r}\right\}.\end{aligned}$$

Now, set

$$F' := f_r' - L_p,$$

and define

$$F(x) := \int_{a^2}^{x} F'(t)\,dt, \quad x \in [0,1].$$

By virtue of (3.8) and (3.5), we have for $x \in (0, a]$,

$$\begin{aligned} F'(x) = f_r'(x) - L_p(x) &= \big[x, y_1, \dots, y_p; f_r'\big](x - y_1) \cdots (x - y_p) \\ &= \frac{f_r^{(p+1)}(\zeta)}{p!}(x - y_1) \cdots (x - y_p) \\ &= \frac{1}{\zeta}(x - y_1) \cdots (x - y_p), \end{aligned}$$

where $\zeta = \zeta(x) \in (0, a)$, and we used the fact that $f_r^{(p+1)}(x) = p! g_a'(x) = \frac{p!}{x}$, $x \in (0, a)$. Applying the estimate $\|L_p\|_{[0,1]} < \mu_r$, we conclude that $F \in \Delta_p^1[0, 1]$ and, by virtue of Lemma 3, that

$$n^r E_n(F)_{[0,1]} \le c_1(r). \tag{4.10}$$

Finally, if $P_m'(x) F'(x) \ge 0$, $x \in (0, a)$, then $P_m^{(p)}(\theta) = 0$ for some point $\theta \in (0, a)$. Hence, by Lemma 5,

$$\|f_r - P_m\|_{[0,1]} \ge c(r) \frac{\ln m}{m^r}.$$

This implies

$$\begin{aligned} \|F - P_m\|_{[0,1]} &\ge \|f_r - P_m\|_{[0,1]} - \frac{1}{m^r} \\ &\ge c(r) \frac{\ln m}{m^r} - \frac{1}{m^r}, \end{aligned} \tag{4.11}$$

where for the first inequality we have applied the estimate $\|L_p\|_{[0,1]} < \frac{1}{m^r}$.

Clearly, by (4.10), $f := \frac{F}{c_1(r)}$ satisfies (4.7), and (4.8) follows from (4.11). □

Lemma 8 *Let $r > 0$ be even and $s > \frac{r}{2}$, and let $b := \frac{\pi}{2} + (s - \frac{r}{2})\pi$. Then there exists a function $\tilde f \in \Delta_s^1[0, b]$ satisfying*

$$n^r E_n(\tilde f)_{[0,b]} \le 1, \quad n \ge 1, \tag{4.12}$$

and

$$m^r E_m^{1,s}(\tilde f)_{[0,b]} \ge c(r, s) \ln m, \tag{4.13}$$

where $c(r, s) > 0$.

Proof Let

$$h_r'(x) := \frac{1}{2}\big(1 + \Psi(x)\big) f_r'(x) + \frac{1}{2}\big(1 - \Psi(x)\big) \sin x, \quad x \in [0, b],$$

and set

$$h_r(x) := \int_{a^2}^x h_r'(u)\, du.$$

Note that

$$h_r'(x) = f_r'(x), \quad x \in [0,1], \tag{4.14}$$

that $h_r'(x) = \sin x$ for $x \geq 2$, and that

$$\min_{x\in[1,2]} h_r'(x) =: \nu_r > 0. \tag{4.15}$$

By virtue of Lemma 3, it follows that $c(r,s)h_r \in B^r[0,b]$, whence

$$n^r E_n(h_r)_{[0,b]} \leq c_2(r,s). \tag{4.16}$$

Let L_p be defined as in the proof of Lemma 7, but with δ so small that, in addition to (4.9), it guarantees that the polynomial L_p on $[0,b]$ satisfies

$$\|L_p\|_{[1,2]} < \nu_r, \quad \|L_p\|_{[1,b]} < \sin 1, \quad \text{and} \quad \left\|L_p'\right\|_{[1,b]} < \cos 1.$$

This together with (4.15) guarantees that L_p does not intersect h_r' in $[1,2]$ and intersects it exactly $s-\frac{r}{2}$ times in $[2,b]$ (exactly once in each interval $(\pi k - \frac{\pi}{2}, \pi k + \frac{\pi}{2}]$, $k = 1,\dots,s-\frac{r}{2}$, where $|\sin x| < \sin 1$). If we set $\tilde{F}' := h_r' - L_p$ and define

$$\tilde{F}(x) := (-1)^{s-r/2} \int_{a^2}^{x} \tilde{F}'(u)\,du,$$

then we conclude that $\tilde{F} \in \Delta_s^1[0,b]$. Finally, since by (4.14), $\tilde{F}(x) = (-1)^{s-r/2}F(x)$, for $x \in [0,1]$, where F is the function from the Proof of Lemma 7, we obtain

$$\begin{aligned} m^r E_m^{1,s}(\tilde{F})_{[0,b]} &\geq m^r E_m^{1,\frac{r}{2}}(F)_{[0,1]} \\ &\geq c(r)\ln m. \end{aligned} \tag{4.17}$$

Again, taking $\tilde{f} := \frac{\tilde{F}}{c_2(r,s)}$, (4.12) follows from (4.16), and (4.13) follows from (4.17). $\square$

We now proceed to discuss the odd r's.

Lemma 9 *Let $r \geq 3$ be odd. Then there exists a function $f \in \Delta_{r+1}^1[-1,1]$ satisfying*

$$n^r E_n(f) \leq 1, \quad n \geq 1, \tag{4.18}$$

and

$$m^r E_m^{1,r+1}(f) \geq c(r)\ln m, \tag{4.19}$$

where $c(r) > 0$.

Proof Recall that $a = \frac{1}{m}$, and take $p := \frac{r-1}{2}$. Define

$$F_r(t) := f_r'(\sqrt{t}) = t^p g_a(\sqrt{t}), \quad t \in [0,1],$$

and observe that,

$$F_r(t) := \frac{1}{2} t^p \ln \frac{t}{a^4}, \quad t \in [0, a^2].$$

For $\delta \in (0, \frac{a^4}{2p})$, write $z_j := j\delta$, $1 \le j \le p$, and $z_{p+1} := a^4$, and let (see (3.6))

$$L_{p+1}(t) = L_{p+1}(F_r; z_1, \dots, z_{p+1}; t) = \sum_{i=1}^{p+1} F_r(z_i) l_i(t)$$

be the Lagrange polynomial interpolating F_r at the z_j's. Since $F_r(z_{p+1}) = 0$, and for all $1 \le i \le p$,

$$\begin{aligned}\left|F_r(z_i) l_i(t)\right| &< c(p) \frac{\delta^p}{\delta^{p-1}(a^4 - p\delta)} \ln \frac{a^4}{\delta} \\ &\le c(p) \frac{\delta}{a^4} \ln \frac{a^4}{\delta}, \quad t \in [0, 1],\end{aligned}$$

we may take δ so small that

$$\|L_{p+1}\|_{[0,1]} < \min\left\{\frac{1}{m^r}, \min_{t \in [a^2, 1]} F_r(t)\right\} < 1. \tag{4.20}$$

Again, by virtue of (3.8) and (3.5), we have for $t \in (0, a^2)$,

$$\begin{aligned}F_r(t) - L_{p+1}(t) &= [t, z_1, \dots, z_{p+1}; F_r](t - z_1) \cdots (t - z_{p+1}) \\ &= \frac{F_r^{(p+1)}(\zeta)}{(p+1)!}(t - z_1) \cdots (t - z_{p+1}) \\ &= \frac{1}{2(p+1)\zeta}(t - z_1) \cdots (t - z_{p+1}),\end{aligned}$$

where $\zeta = \zeta(t) \in (0, a^2)$. Thus, we set

$$F'(x) := F_r\left(x^2\right) - L_{p+1}\left(x^2\right) = f_r'(x) - L_{p+1}\left(x^2\right), \quad x \in [-1, 1],$$

and define

$$F(x) := \int_0^x F'(u)\, du, \quad x \in [-1, 1].$$

It follows that

$$F'(x)\left(x^2 - z_1\right) \dots \left(x^2 - z_{p+1}\right) \ge 0, \quad x \in [-a, a];$$

that is, $F \in \Delta^1_{2p+2}[-a, a] = \Delta^1_{r+1}[-a, a]$. By (4.20), we obtain

$$\left|L_{p+1}\left(x^2\right)\right| < f_r'(x), \quad |x| \in [a, 1],$$

which, in turn, implies that

$$F \in \Delta^1_{r+1}[-1,1] = \Delta^1_{r+1}.$$

At the same time, Lemma 4 and (4.20) yield

$$n^r E_n(F)_{[-1,1]} \le c_3(r). \tag{4.21}$$

Finally, if $P_m'(x)F'(x) \ge 0$, $x \in (-1,1)$, then there is a point $\theta \in (-a,a)$ such that

$$P_m^{(r)}(\theta) = 0.$$

(In fact, there are at least two such points.) By Lemma 6,

$$\|f_r - P_m\|_{[-1,1]} \ge c(r)\frac{\ln m}{m^r},$$

whence, by (4.20), we get

$$\|F - P_m\|_{[-1,1]} \ge \|f_r - P_m\|_{[-1,1]} - \frac{1}{m^r} \ge c(r)\frac{\ln m}{m^r} - \frac{1}{m^r}. \tag{4.22}$$

Once again, taking $f := \frac{F}{c_3(r)}$, (4.18) follows from (4.21), and (4.19) follows from (4.22). □

We need an analogous result for $r = 1$.

Lemma 10 *There is a function* $f \in \Delta^1_2[-1,1]$ *satisfying*

$$nE_n(f) \le 1, \quad n \ge 1,$$

and

$$mE_m^{1,2}(f) \ge c \ln m,$$

where $c > 0$.

Proof We would have liked to have used f_1 to prove this lemma the way we did in Lemma 9 (using f_r, $r \ge 3$). However, f_1 is not differentiable at $x = 0$. Thus, we modify it a little. Let l_1 be the tangent to f_1' at $x = a^2$, and let l_2 be the tangent to f_1' at $x = -a^2$. Then $l_1(0) = l_2(0) = -1$. Set

$$\bar{f}'(x) := \begin{cases} f_1'(x) & \text{if } a^2 \le |x| \le 1, \\ l_2(x) & \text{if } -a^2 \le x \le 0, \\ l_1(x) & \text{if } 0 \le x \le a^2, \end{cases}$$

and define

$$\bar{f}(x) := \int_0^x \bar{f}'(u)\,du.$$

Now $\bar{f}$ is continuously differentiable in $[-1,1]$, and the proof follows, repeating the arguments of Lemmas 3 and 5 with $\bar{f}$ replacing f_1, and those of Lemma 9 with $\bar{f}$ replacing f_r, $r \geq 3$. □

We are ready for the r odd analogue of Lemma 8.

Lemma 11 *Let $r>0$ be odd and $s>r+1$, and let $b := \frac{\pi}{2} + (s-r-1)\pi$. Then there exists a function $\tilde{f} \in \Delta_s^1[-b,b]$ satisfying*

$$n^r E_n(\tilde{f})_{[-b,b]} \leq 1, \quad n \geq 1,$$

and

$$m^r E_m^{1,s}(\tilde{f})_{[-b,b]} \geq c(r,s) \ln m,$$

where $c(r,s) > 0$.

Proof Let

$$F_r := \begin{cases} f_r & \text{if } r \geq 3, \\ \bar{f} & \text{if } r = 1, \end{cases}$$

where $\bar{f}$ is from Lemma 10. We extend F_r' to $[1,b]$ just as we did in the proof of Lemma 7, and we extend it to $[-b,-1]$ by putting $F_r'(x) \equiv F_r'(-1)$, $x \in [-b,-1]$. Then we repeat the arguments in the proof of Lemma 8. We omit the details. □

We are ready with the completion of the proof of Theorem 2.

Fix $s>0$.

If $\alpha \in A_s$ is odd, then by the definition of A_s, we have to deal with $\alpha+1<s$ if s is odd and $\alpha+1 \leq s$ if s is even. Hence, Theorem 2 follows from Lemmas 9, 10, and 11, taking $r=\alpha$.

If $\alpha \in A_s$ is even, then $\alpha = 2\ell \leq 2s$. Hence, Theorem 2 follows from Lemmas 7 and 8, taking $r=\alpha$, so that $s \geq \frac{r}{2}$. This completes the proof. □

5 Proof of Theorem 3

Let $x_j := \cos(j\pi/n), 0 \leq j \leq n$, be the Chebyshev knots, and denote $J_{j,n} := [x_{j+1}, x_{j-1}]$ and $|J_{j,n}| = x_{j-1} - x_{j+1}$, $1 \leq j \leq n-1$.

The proof of (2.6) follows the lines of the paper [7]. First we observe that (2.6) readily follows from the inequalities

$$E_{r+1}^{1,\sigma}(f)_{J_{j,n}} \leq \frac{c(\alpha,s)}{n^\alpha}, \quad j=1,\dots,n-1, \tag{5.1}$$

for $0 \leq \sigma \leq s$, such that $f \in \Delta_\sigma^1[x_{j+1}, x_{j-1}]$, and

$$r := [\alpha] \geq 1.$$

We provide a detailed proof of this observation in the Appendix.

To prove (5.1), we use the fact that, by virtue of [7, Theorems 2.1 and 2.2], (2.5) together with inequalities [6, (3.4) and (3.5)], imply, for each n:

(a) $f \in C^{r-1}(-1,1)$ and

$$\omega_2\big(f^{(r-1)}, |J_{j,n}|, J_{j,n}\big) \le \frac{c(\alpha)}{|J_{j,n}|^{r-1} n^\alpha}, \quad 2 \le j \le n-2; \tag{5.2}$$

(b) if $\alpha \notin \mathbb{N}$, then $f \in C^r(-1,1)$ and

$$\omega\big(f^{(r)}, |J_{j,n}|, J_{j,n}\big) \le \frac{c(\alpha)}{|J_{j,n}|^{r} n^\alpha}, \quad 2 \le j \le n-2; \tag{5.3}$$

(c) if α is not an even number, then $f \in C^{[\frac{r}{2}]}[-1,1]$,

$$\omega\big(f^{([\frac{r}{2}])}, |J_{j,n}|, J_{j,n}\big) \le \frac{c(\alpha)}{|J_{j,n}|^{[\frac{r}{2}]} n^\alpha}, \quad 1 \le j \le n-1; \tag{5.4}$$

and

(d) if α is an even number, then $f \in C^{\frac{r}{2}-1}[-1,1]$,

$$\omega_2\big(f^{(\frac{r}{2}-1)}, |J_{j,n}|, J_{j,n}\big) \le \frac{c(\alpha)}{|J_{j,n}|^{\frac{r}{2}-1} n^\alpha}, \quad 1 \le j \le n-1. \tag{5.5}$$

We combine (5.2) through (5.5) with the inequalities (3.12), (3.16), and (3.18), and get (5.1) for each $\alpha \notin A_s$. Specifically, for $\alpha \notin \mathbb{N}$, we have (5.1) for all j by virtue of Corollary 1 and inequalities (5.3) and (5.4). For odd $\alpha \in \mathbb{N}$, such that $\alpha > s$, we observe that $\alpha \ge s+1$ so that $r-1 \ge s$. Hence, for $2 \le j \le n-2$, we obtain (5.1) by virtue of Corollary 2 and inequality (5.2), while for $j = 1, n-1$, we apply Corollary 1 and inequality (5.4). Finally, for even $\alpha \ge 2s+2$, we have $r-1 > r/2-1 \ge s$, and (5.1) follows by virtue of Corollary 2 and inequality (5.5). We have one remaining case where $\alpha = s$ is odd. For $j = 1, 2$ and $j = n-2, n-1$, (5.1) follows from Corollary 1 and (5.4), and for $3 \le j \le n-3$ and $\sigma < s$, (5.1) follows from Corollary 2 and (5.2). Thus, we only need to prove (5.1) for $3 \le j \le n-3$ and $\sigma = s$. To this end, the proof follows from Lemma 1, $[a_1, b_1] := J_{j-1,n} \cup J_{j+1,n}$ and $[a, b] := J_{j,n}$. This completes the proof. □

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Appendix

For the sake of completeness, we include the proof of the fact that the inequalities (5.1) imply (2.6).

Let $\varphi(x) := \sqrt{1-x^2}$, and write $C^0_\varphi := C[-1,1]$; for $r \ge 1$, we say that $f \in C^r_\varphi$ if $f \in C^{(r)}(-1,1)$ and $\lim_{x\to\pm1} \varphi^r(x) f^{(r)}(x) = 0$.

Finally, for $f \in C^r_\varphi$, we write

$$\omega^\varphi_{k,r}\bigl(f^{(r)},t\bigr) := \sup_{0\le h\le t}\ \sup_{x:|x|+\frac{kh}{2}\varphi(x)<1} K^r\left(x,\frac{kh}{2}\right)\bigl|\Delta^k_{h\varphi(x)}\bigl(f^{(r)},x\bigr)\bigr|,$$

where $K(x,\mu) := \varphi(|x| + \mu\varphi(x))$ and the symmetric difference Δ^k_u is defined in (3.1).

Note that for $r=0$,

$$\omega^\varphi_{k,0}(f,t) \equiv \omega^\varphi_k(f,t),$$

the kth Ditzian–Totik modulus of smoothness.

Recall the Chebyshev knots $x_j := \cos(j\pi/n), 0 \le j \le n$, and denote $I_j := [x_j, x_{j-1}]$, $1 \le j \le n$. Note that $J_j = I_j \cup I_{j+1}$, $1 \le j \le n-1$. Let $\Sigma_{k,n}$ be the collection of all continuous piecewise polynomials of degree $< k$, on the Chebyshev partition $\{x_j\}_{j=0}^n$. Also, denote by $\Sigma_{k,n}(Y_s) \subset \Sigma_{k,n}$, the set of all piecewise polynomials S with the following property. Let $j(i)$, $1 \le i \le s$, be such that $y_i \in I_{j(i)}$; then $S \in \Sigma_{k,n}(Y_s)$ if for every $1 \le i \le s$, the restriction of S to $(x_{j(i)+1}, x_{j(i)-2}) =: O_i$ is a polynomial, where $x_{n+l} := -1$, $x_{-l} := 1$. Finally, write $O := \bigcup_{i=1}^s O_i$.

Proposition 3 ([10]) *For every $k \ge 1$ and $s \ge 0$, there are constants $c = c(k,s)$ and $c_* = c_*(k,s)$ such that if $Y_s \in \mathbb{Y}_s$ and $S \in \Sigma_{k,n}(Y_s) \cap \Delta^1(Y_s)$, $n \ge 1$, then*

$$E^{(1)}_{c_* n}(S, Y_s) \le c\omega^\varphi_k(S, 1/n).$$

Theorem 5 *Given $s \in \mathbf{N}$, let $\alpha \ge 1$. Suppose that $f \in \Delta^1_s$ is such that (2.4) holds for $n \ge r+1$ and the inequalities (5.1) are satisfied. Then*

$$n^\alpha E^{1,s}_n(f) \le c_1(\alpha,s), \quad n \ge c_2(\alpha,s)N_0. \tag{A.1}$$

Proof Let $f \in \Delta^{(1)}(Y_s)$. Evidently, it suffices to prove (A.1) for $E^{(1)}_n(f, Y_s)$ instead of $E^{1,s}_n(f)$ (recall that $E^{(1)}_n(f,Y_s)$ and $E^{1,s}_n(f)$ are defined in (2.2) and (2.3)). First, let

$$\sigma^{(1)}_{r+1,n}(f) := \inf\bigl\{\|f-S\| : S \in \Sigma_{r+1,n}(Y_s) \cap \Delta^1(Y_s)\bigr\},$$

and assume that (the analog of (A.1)),

$$n^\alpha \sigma^{(1)}_{r+1,n}(f) \le c_3(\alpha,s), \quad n \ge N_1.$$

Take $S \in \Sigma_{r+1,n}(Y_s) \cap \Delta_1(Y_s)$, so that $\|f-S\| \le 2\sigma^{(1)}_{r+1,n}(f,Y_s)$. Then by Proposition 3, we conclude that

$$E^{(1)}_n(S,Y_s) \le c\omega^\varphi_{r+1}\left(S,\frac{1}{n}\right), \quad n \ge c_*,$$

which, in turn, implies

$$E^{(1)}_n(f,Y_s) \le E^{(1)}_n(S,Y_s) + 2\sigma^{(1)}_{r+1,n}(f,Y_s) \le c\omega^\varphi_{r+1}\left(f,\frac{1}{n}\right) + c\sigma^{(1)}_{r+1,n}(f,Y_s)$$

for all $n \ge c_*$. Since (2.4) implies that $\omega_{r+1}^{\varphi}(f, 1/n) \le cn^{-\alpha}$, we only need to verify (A.1) for $N_1 \ge \max\{c_*, c_3(\alpha, s)N_0\}$, $c_3(\alpha, s)$ to be prescribed.

Let $n \ge c_3(\alpha, s)N_0$, and let $O_n = \bigcup_{i=1}^{s'} O_i'$ be a partition of O_n into connected intervals. First we prove that there is an $m \ge n/c_2$ such that:

(i) for each $1 \le i \le s'$, $O_i' \subset J_{j(i),m}$, and
(ii) for $i_1 \ne i_2$, we have that either $j(i_1) = j(i_2)$ or $\{j(i_1), j(i_1)+1\} \cap \{j(i_2), j(i_2)+1\} = \emptyset$.

To this end, we proceed by induction on $s' \le s$. If $s' = 1$, then there is just one block O_1' of length $\le 2s+1$, so that we may take $m = \lceil \frac{n}{2s+1} \rceil$. If $s' > 1$, let $n_1 = \lceil \frac{n}{2s+1} \rceil$, then property (i) holds for $m = n_1$. If property (ii) is also valid, then we are done. Otherwise,

$$O_n = \bigcup_{i=1}^{s'} O_i' \subset \bigcup_{i=1}^{s'} J_{j(i),m} =: \bigcup_{i=1}^{s''} O_i''$$

is a new partition into connected intervals, where $s'' < s'$. We apply the induction step to s'' to get the desired value of m. It is readily seen that we may take $c_3(\alpha, s) \ge (2s+1)^s$.

By virtue of (5.1), for the appropriate m, for every $1 \le i \le s'$, there is a polynomial p_i, of degree r, comonotone with f on $J_{j(i),m}$, satisfying $p_i(x_{j(i)+1,m}) = f(x_{j(i)+1,m})$ and such that

$$\|f - p_i\|_{J_{j(i),m}} \le \frac{c(\alpha, s)}{m^\alpha} \le \frac{c_3(\alpha, s)}{n^\alpha}.$$

For $j \ne j(i), j(i)+1$, $1 \le i \le s'$, f is monotone on $I_{j,m}$, so that, by (5.1), we have a polynomial $\tilde{q}_j$, of degree r, comonotone with it there, such that

$$\|f - \tilde{q}_j\|_{I_{j,m}} \le \frac{c_4(\alpha, s)}{n^\alpha}.$$

Adding the linear

$$\begin{aligned} l(x) := \bigl(f(x_{j,m}) - \tilde{q}_j(x_{j,m})\bigr)\frac{x - x_{j-1,m}}{x_{j,m} - x_{j-1,m}} \\ + \bigl(f(x_{j-1,m}) - \tilde{q}_j(x_{j-1,m})\bigr)\frac{x - x_{j,m}}{x_{j-1,m} - x_{j,m}}, \end{aligned}$$

we obtain q_j that is comonotone with f on $I_{j,m}$, interpolates f at endpoints of $I_{j,m}$, and satisfies

$$\|f - q_j\|_{I_{j,m}} \le \frac{3c_4(\alpha, s)}{n^\alpha}.$$

Now we take piecewise polynomial s_n to be defined p_i on $J_{j(i),m}$ and q_j on $I_{j,m}$ for the other intervals. Then s_n is comonotone with f and satisfies

$$\|f - s_n\| \le \frac{c(\alpha, s)}{n^\alpha}.$$

It may have discontinuities at the points $x_{j(i)-1,m}$, but there are no more than s such points, so if we make it continuous by moving from left to right and making $s_n(x_{j(i)-1,m}+) := s_n(x_{j(i)-1,m}-)$, then our estimate is still valid with $c_1(\alpha, s) := sc(\alpha, s)$. This completes the proof. □

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