FREE-KNOT SPLINES APPROXIMATION OF s-MONOTONE FUNCTIONS

V. N. Konovalov and D. Leviatan[1]

Abstract. Let I be a finite interval and $r, s \in \mathbb{N}$. Given a set M, of functions defined on I, denote by $\Delta_+^s M$ the subset of all functions $y \in M$ such that the s-difference $\Delta_\tau^s y(\cdot)$ is nonnegative on I, $\forall \tau > 0$. Further, denote by $\Delta_+^s W_p^r$, the class of functions x on I with the seminorm $\|x^{(r)}\|_{L_p} \le 1$, such that $\Delta_\tau^s x \ge 0$, $\tau > 0$. Let $M_n(h_k) := \sum_{i=1}^n c_i h_k(w_i t - \theta_i) \mid c_i, w_i, \theta_i \in \mathbb{R}$, be a single hidden layer perceptron univariate model with n units in the hidden layer, and activation functions $h_k(t) = t_+^k$, $t \in \mathbb{R}$, $k \in \mathbb{N}_0$. We give two-sided estimates both of the best unconstrained approximation $E\ \Delta_+^s W_p^r, M_n(h_k)\ _{L_q}$, $k = r-1, r$, $s = 0, 1, \dots, r+1$, and of the best s-monotonicity preserving approximation $E\ \Delta_+^s W_p^r, \Delta_+^s M_n(h_k)\ _{L_q}$, $k = r-1, r$, $s = 0, 1, \dots, r+1$. The most significant results are contained in Theorem 2.

§1. Introduction

Let X be a real linear space of vectors x with a norm $\|x\|_X$, $W \subset X$, $W \ne \emptyset$, and $M \subset X$, $M \ne \emptyset$. Let

$$E(x, M)_X := \inf_{y \in M} \|x - y\|_X,$$

denote the best approximation of the vector $x \in X$ by M and let

$$E(W, M)_X := \sup_{x \in W} E(x, M)_X,$$

denote the deviation of the set W from M.

Let $s = 0, 1, \dots,$ and for a function x defined on I, let

$$\Delta_\tau^s x(t) := \sum_{k=0}^{s} (-1)^{s-k} \binom{s}{k} x(t + k\tau), \quad \{t, t + s\tau\} \subset I, \quad s = 0, 1, \dots,$$

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be the s-th difference of the function x, with step $\tau > 0$. Denote by $\Delta_+^s M$ the subclass of functions $x \in M$ for which $\Delta_\tau^s x(t) \geq 0$, for all $\tau > 0$ such that $[t, t + s\tau] \subseteq I$. Further denote by

$$E\big(x, \Delta_+^s M\big)_X := \inf_{y \in \Delta_+^s M} \|x - y\|_X,$$

the best approximation of the vector $x \in X$ by $\Delta_+^s M$, and by

$$E\big(W, \Delta_+^s M\big)_X := \sup_{x \in W} E(x, \Delta_+^s M)_X,$$

the deviation of the set W from $\Delta_+^s M$.

For $r \in \mathbb{N}$, denote as usual

$$W_p^r := W_p^r(I) := \{x : I \to \mathbb{R} \mid x^{(r-1)} \in AC_{loc}(a,b), \|x^{(r)}\|_{L_p(I)} \leq 1\}, \quad 1 \leq p \leq \infty,$$

where $I = [a,b]$, and where $AC_{loc}(a,b)$ denotes the set of absolutely continuous functions in every compact subinterval of (a,b).

In this paper we discuss shape preserving free-knot polynomial spline approximation which may be viewed as a single hidden layer perceptron univariate model with n units in the hidden layer, and activation functions $h_k(t) := t_+^k$, $t \in \mathbb{R}$, $k \in \mathbb{N}_0$, where $t_+ := \max\{0, t\}$. Namely, the function

$$y(t) := \sum_{i=1}^n c_i h_k(w_i t - \theta_i), \quad t \in \mathbb{R},$$

where $c_i \in \mathbb{R}$, $w_i \in \mathbb{R}$ and $\theta_i \in \mathbb{R}$, that is called a single hidden layer perceptron model, is viewed as a polynomial spline $\sigma_{k,n}(\cdot)$ of degree k, belonging to $C^{k-1}(\mathbb{R})$, with knots $w_i^{-1}\theta_i$. The reader is referred to the survey [Pi] where various approximation-theoretic problems that arise in the multilayer feedforward perceptron (MLP) model in neural networks are discussed.

Thus, let

$$M_n(h_k) := \left\{ \sum_{i=1}^n c_i h_k(w_i t - \theta_i) \mid c_i, w_i, \theta_i \in \mathbb{R} \right\}, \quad t \in \mathbb{R},$$

be a $3n$ parameter family of polynomial splines. For $k=r-1$ and $k=r$, we are interested in the asymptotic behavior of the best unconstrained approximation $E\left(\Delta_+^s W_p^r, M_n(h_k)\right)_{L_q}$, $s=0,1,\dots,r+1$. Further, we obtain the asymptotic behavior of the best s-monotonicity preserving approximation, $E\left(\Delta_+^s W_p^r, \Delta_+^s M_n(h_k)\right)_{L_q}$, $k=r-1,r,\ s=0,1,\dots,r+1$.

§2. Main results

Our first result is

Theorem 1. *Let $1\le p,q\le\infty$, $r\in\mathbb{N}$, and $0\le s\le r$. Then*

$$E\big(\Delta_+^s W_p^r, M_n(h_{r-1})\big)_{L_q} \asymp E\big(\Delta_+^s W_p^r, \Delta_+^s M_n(h_{r-1})\big)_{L_q} \asymp n^{-r}. \tag{2.1}$$

Furthermore, if $s=r+1$, then

$$E\big(\Delta_+^{r+1} W_p^r, M_n(h_{r-1})\big)_{L_q} \asymp n^{-r}, \tag{2.2}$$

while

$$E\big(\Delta_+^{r+1} W_p^r, \Delta_+^{r+1} M_n(h_{r-1})\big)_{L_q} \asymp 1. \tag{2.3}$$

Remarks. (i) The upper bounds in (2.1) for $s=0,1,\dots,r-1$ are an immediate consequence of [H, Theorem 1.1]. We show that in order to obtain the upper bounds in (2.1) for $s=r$, we can still use the approach of [H]. What (2.1) shows as a special case, is that Hu's upper estimates (which are given only for $0\le s<r$, $p=1$ and $q=\infty$), are best possible. We give a simple proof of this fact. Also, the upper bound in (2.2) follows immediately from well known estimates on the degree of approximation of elements in W_p^r, by free knot splines with n knots.

(ii) The upper bound in (2.1) for $s=1,2$, and to some extent the lower bound in those cases, also follows from the work of Leviatan and Shadrin [LS, Theorems 1 and 2].

(iii) Note that (2.3) is not surprising as the set $\Delta_+^{r+1}M_n(h_{r-1})$ only contains polynomials of degree $\le r-1$.

Next we state our main result. We show that there is no improvement in (2.1), if we replace h_{r-1} with h_r, but that such a replacement improves significantly the orders of approximation in (2.2) and (2.3). Namely, we prove

Theorem 2. *Let $1 \le p, q \le \infty$, $r \in \mathbb{N}$, and $0 \le s \le r$. Then*

$$E\big(\Delta_+^s W_p^r, M_n(h_r)\big)_{L_q} \asymp E\big(\Delta_+^s W_p^r, \Delta_+^s M_n(h_r)\big)_{L_q} \asymp n^{-r}. \tag{2.4}$$

Furthermore, if $1 < r \in \mathbb{N}$, and if $r = 1$ and either $p = \infty$ or $1 \le p, q < \infty$, then

$$E\big(\Delta_+^{r+1} W_p^r, M_n(h_r)\big)_{L_q} \asymp E\big(\Delta_+^{r+1} W_p^r, \Delta_+^{r+1} M_n(h_r)\big)_{L_q} \asymp n^{-r-1}. \tag{2.5}$$

On the other hand, if $r = p = 1$ and $q = \infty$, then

$$E\big(\Delta_+^2 W_1^1, M_n(h_1)\big)_{L_\infty} \asymp E\big(\Delta_+^2 W_1^1, \Delta_+^2 M_n(h_1)\big)_{L_\infty} \asymp n^{-1}, \tag{2.6}$$

and finally, if $r = 1$, $1 < p < \infty$ and $q = \infty$, then there exist absolute constants $c_1 > 0$ and c_2, such that for any $\varepsilon > 0$,

$$c_1 n^{-2} \le E\big(\Delta_+^2 W_p^1, M_n(h_1)\big)_{L_\infty} \le E\big(\Delta_+^2 W_p^1, \Delta_+^2 M_n(h_1)\big)_{L_\infty} \le c_2 \varepsilon^{-1} n^{-2+\varepsilon}. \tag{2.7}$$

Obviously in view of the lefthand inequality in (2.7), we hope that ε may be removed from the righthand side, but we have not succeeded in that.

We prove the upper bounds in (2.5) through (2.7) by applying in a more delicate way the basic idea of 'balanced partition' of Hu [H].

It is interesting to compare the above asymptotic relations with our earlier estimates of the Kolmogorov and shape preserving widths of the sets $\Delta_+^s W_p^r$, $0 \le s \le r+1$, which in most cases are bigger (see [KL1] and [KL2]). In the theorems we quote below $d_n(\Delta_+^s W_p^r)_{L_q}$ denotes the usual Kolmogorov n width, and

$$d_n(\Delta_+^s W_p^r, \Delta_+^s L_q)_{L_q} = \inf_{M^n} E(\Delta_+^s W_p^r, M^n \cap \Delta_+^s L_q),$$

where the infimum is taken over all linear manifolds M^n, of dimension n.

Theorem KL1. *Let $r \in \mathbb{N}$ and $1 \le p, q \le \infty$, be so that $r - \frac{1}{p} + \frac{1}{q} > 0$. If $(r,p) \ne (1,1)$, and if $(r,p) = (1,1)$ and $1 \le q \le 2$, then for each $s = 0, 1, \dots, r$,*

$$d_n(\Delta_+^s W_p^r)_{L_q} \asymp n^{-r+(\max\{\frac{1}{p},\frac{1}{2}\}-\max\{\frac{1}{q},\frac{1}{2}\})_+}, \quad n \ge r.$$

If, on the other hand, $(r,p)=(1,1)$ *and* $2<q<\infty$*, then for* $s=0,1$*,*

$$c_1 n^{-\frac{1}{2}} \le d_n(\Delta_+^s W_1^1)_{L_q} \le c_2 n^{-\frac{1}{2}}\big(\log(n+1)\big)^{\frac{3}{2}}, \quad n\ge 1,$$

where $c_1>0$ *and* c_2 *do not depend on* n*. Furthermore,*

$$d_n(\Delta_+^{r+1} W_p^r)_{L_q} \asymp n^{-r-\max\{\frac{1}{q},\frac{1}{2}\}}. \quad n>r.$$

And

Theorem KL2. *Let* $s=1,2$*,* $s\le r\in\mathbb{N}$*, and* $1\le p,q\le\infty$*, be so that* $r-\frac{1}{p}+\frac{1}{q}>0$*. Then*

$$d_n(\Delta_+^s W_p^r, \Delta_+^s L_q)_{L_q} \asymp n^{-r+(\frac{1}{p}-\frac{1}{q})_+}, \quad n\ge r.$$

If, on the other hand, $s=r+1=2$*, then*

$$d_n(\Delta_+^2 W_p^1, \Delta_+^2 L_q)_{L_q} \asymp n^{-1-\frac{1}{q}}, \quad n\ge 1. \tag{2.8}$$

For $3\le s\le r+1$ the shape preserving widths were obtained in [KL3]. Namely,

Theorem KL3. *Let* $r\in\mathbb{N}$*,* $s\in\mathbb{N}$ *and* $1\le p,q\le\infty$*. For* $3\le s\le r$*, we have*

$$d_n\left(\Delta_+^s W_p^r, \Delta_+^s L_q\right)_{L_q} \asymp n^{-r+s+\frac{1}{p}-3}, \quad n\ge r.$$

Also, if $s=r+1$*,* $r\ge 2$*, then*

$$d_n\left(\Delta_+^{r+1} W_p^r, \Delta_+^{r+1} L_q\right)_{L_q} \asymp n^{-2}, \quad n\ge r.$$

§3. Approximation by free-knot splines of degree $r-1$

This section contains the proof of Theorem 1. We begin with a lemma.

Lemma 1. *For all* $n\ge 1$*,*

$$E\big(h_r, M_n(h_{r-1})\big)_{L_1} \ge 2^{-2r}(n+1)^{-r}. \tag{3.1}$$

Proof. Let $P_{r-1}(J)$ denote the space of all algebraic polynomials of degree $\le r-1$ on the interval $J \subset \mathbb{R}$, and set $P_{r-1} := P_{r-1}(I)$. Denote by

$$y(t) := \sum_{i=1}^{n} c_i h_{r-1}(w_i t - b_i), \quad w_i \in \mathbb{R}, \quad b_i \in \mathbb{R}, \quad i = 1, \dots, n,$$

an arbitrary function from $M_n(h_{r-1})$. If $T_m := \{t_i\}_{i=1}^m \subset \mathbb{R}$, is the collection of $0 \le m \le n$ distinct knots $t_i := w_i^{-1} b_i$, $t_1 < \cdots < t_m$, then $T_m \cap (0,1) = \emptyset$, implies $y \in P_{r-1}([0,1])$. Thus in this case

$$\begin{aligned} \|h_r(\cdot) - y(\cdot)\|_{L_1} &\ge \inf_{\pi_{r-1} \in P_{r-1}} \int_{-1}^{1} \big| t_+^r - \pi_{r-1}(t) \big| dt \\ &\ge 2^{-2r}, \end{aligned} \tag{3.2}$$

where we have applied the well known formula (see, e.g., [T, p. 96])

$$\inf_{\pi_{r-1} \in P_{r-1}} \int_{-1}^{1} |t^r - \pi_{r-1}(t)| dt = 2^{-r+1}. \tag{3.3}$$

Otherwise, denote $\Theta_\mu := \{\theta_i\}_{i=0}^{\mu+1}$, where $0 < \theta_1 < \cdots < \theta_\mu < 1$ are the knots $T_m \cap (0,1)$, and let $S_{r-1}^0(\Theta_\mu) := S_{r-1}^0(\Theta_\mu; [0,1])$, denote the space of all polynomial splines on $[0,1]$ of degree $\le r-1$, with knots θ_i, $i = 1, \dots, \mu$. If $I_i := [\theta_i, \theta_{i+1}]$, $i = 0, \dots, \mu$, where $\theta_0 := 0$, and $\theta_{\mu+1} := 1$, then

$$\begin{aligned} \|h_r - y\|_{L_1} &\ge \|h_r - y\|_{L_1[0,1]} \\ &\ge E\big(h_r, S_{r-1}^0(\Theta_\mu)\big)_{L_1[0,1]} \\ &= \sum_{i=0}^{\mu} E\big(h_r, P_{r-1}(I_i)\big)_{L_1(I_i)}. \end{aligned} \tag{3.4}$$

By virtue of (3.3), it is easily seen that

$$\begin{aligned} E\big(h_r, P_{r-1}(I_i)\big)_{L_1(I_i)} &= \inf_{\pi_{r-1} \in P_{r-1}} \int_{I_i} \big| (t - \bar{\theta}_i)^r - \pi_{r-1}(t) \big| dt \\ &= 2^{-2r} |I_i|^{r+1}, \quad i = 1, \dots, \mu + 1, \end{aligned}$$

where $\bar{\theta}_i := \frac{1}{2}(\theta_i + \theta_{i+1})$. Hence if we write $x_i := |I_i|$, $i = 0, \dots, \mu$, then $\sum_{i=0}^{\mu} x_i = 1$, and if we wish to have a lower bound to the righthand side of (3.4), then we have to consider the extremal problem

$$f(x) := \sum_{i=0}^{\mu} x_i^{r+1} \to \inf; \quad x_i \ge 0, \quad i = 0, \dots, \mu, \quad \sum_{i=0}^{\mu} x_i = 1, \tag{3.5}$$

We use Lagrange multipliers, namely, we let

$$L_\lambda(x;f) := \sum_{i=0}^{\mu} x_i^{r+1} - \lambda\left(\sum_{i=0}^{\mu} x_i - 1\right)$$

and we impose that the partial derivatives vanish, i.e.,

$$\frac{\partial}{\partial x_i} L_\lambda(x;f) = (r+1)x_i^r - \lambda = 0, \quad i = 0,\dots,\mu.$$

The solution is $x_i = (r+1)^{-\frac{1}{r}}\lambda^{\frac{1}{r}}$, $i = 0,\dots,\mu$, so that

$$1 = \sum_{i=1}^{\mu+1} x_i = (\mu+1)(r+1)^{-\frac{1}{r}}\lambda^{\frac{1}{r}}.$$

Hence, $\lambda = (r+1)(\mu+1)^{-r}$ and $x_i = (\mu+1)^{-1}$. The minimal value of $f(x)$ in (3.5) is obtained at $x_* := \big((\mu+1)^{-1},\dots,(\mu+1)^{-1}\big)$ and

$$f(x_*) = \sum_{i=1}^{\mu+1}\big((\mu+1)^{-1}\big)^{r+1} = (\mu+1)^{-r}.$$

Thus we conclude by (3.4) that

$$\|h_r(\cdot) - y(\cdot)\|_{L_1} \geq 2^{-r+1}(\mu+1)^{-r} \geq 2^{-2r}(n+1)^{-r}.$$

Combining this with (3.2) yields (3.1) and completes the proof. $\square$

We are ready to prove Theorem 1.

Proof of Theorem 1. As we have remarked above (Remark (i)), the upper bounds in (2.1) for $s = 0,1,\dots,r-1$, follows from [H, Theorem 1.1], since $\|x\|_p \leq 2\|x\|_\infty$ and $W_p^r \subseteq 2^{1/p'}W_1^r$, for all $1 \leq p \leq \infty$ ($\frac{1}{p} + \frac{1}{p'} = 1$). Also, since $\Delta_+^{r+1}W_p^r \subseteq W_p^r$, the upper bound in (2.2) follows immediately from well known estimates on the degree of approximation of elements in W_p^r, by free knot splines with n knots (see, e.g., [H, Theorem 2.1]). In order to complete the proof of the upper bound in (2.1) for $s = r$, we may without loss of generality, again assume that $x \in W_1^r$, and apply Hu's construction [H], to obtain a balanced partition of $[-1,1]$, $T : -1 = t_0 < t_1 < \cdots < t_n = 1$, so that

$$(t_{i+1} - t_i)^{r-1}\|x^{(r)}\|_{L_1[t_i,t_{i+1}]} \leq n^{-r}. \tag{3.6}$$

Due to the fact that $x^{(r-1)}$ is nondecreasing, for a fixed $0 \le i < n$ and $r_0 := \left[\frac{r+1}{2}\right]$, there is a quadrature

$$\int_{t_i}^{t_{i+1}} g(\tau) dx^{(r-1)}(\tau) = \sum_{k=1}^{r_0} A_k g(u_k) + R(g),$$

where $A_k > 0$ and $t_i =: u_0 < u_1 < \cdot < u_{r_0} < u_{r_0+1} := t_{i+1}$, which is exact for $P_{r-1}([t_i, t_{i+1}])$. (See Petrov [Pe] for a similar idea.) Thus we have

$$\int_{t_i}^{t_{i+1}} (t_{i+1} - \tau)^j dx^{(r-1)}(\tau) = \sum_{k=1}^{r_0} A_k (t_{i+1} - u_k)^j, \quad j = 0, \dots, r-1, \tag{3.7}$$

and, in particular, for $j = 0$ we have

$$\sum_{k=1}^{r_0} A_k = x^{(r-1)}(t_{i+1}) - x^{(r-1)}(t_i).$$

Hence, if S_0 is defined on $[t_i, t_{i+1})$, $0 \le i \le n$, by

$$S_0(\tau) = x^{(r-1)}(t_i) + \sum_{k=1}^{m} A_k =: s_m, \quad u_m \le \tau < u_{m+1}, \quad m = 0, \dots, r_0,$$

then evidently, $x^{(r-1)}(t_i) \le s_m \le x^{(r-1)}(t_{i+1})$, $m = 0, \dots, r_0 - 1$. Furthermore, it follows by (3.7) that

$$\int_{t_i}^{t_{i+1}} (t_{i+1} - \tau)^j dx^{(r-1)}(\tau) = \int_{t_i}^{t_{i+1}} (t_{i+1} - \tau)^j dS_0(\tau), \quad 0 \le j \le r-1. \tag{3.8}$$

Now set

$$S_r(t) := \sum_{k=0}^{r-2} \frac{x^{(k)}(-1)}{k!}(t+1)^k + \frac{1}{(r-2)!} \int_{-1}^{t} (t-\tau)^{r-2} S_0(\tau) d\tau, \quad -1 \le t \le 1.$$

Then we have

$$S_r^{(j)}(t_i) = x^{(j)}(t_i), \quad j = 0, \dots, r-1, \quad 0 \le i \le n,$$

which by virtue of (3.6) and (3.8), yields for $t_i \le t < t_{i+1}$,

$$\begin{aligned}
|x(t) - S_r(t)| &\le \frac{1}{(r-2)!}\left|\int_{t_i}^{t}(t-\tau)^{r-2}\big(x^{(r-1)}(\tau) - S_0(\tau)\big)d\tau\right| \\
&\le \frac{1}{(r-2)!}(t_{i+1}-t_i)^{r-2}\int_{t_i}^{t_{i+1}}\big|x^{(r-1)}(\tau) - S_0(\tau)\big|d\tau \\
&\le \frac{1}{(r-2)!}(t_{i+1}-t_i)^{r-2}\int_{t_i}^{t_{i+1}}\big(x^{(r-1)}(t_{i+1}) - x^{(r-1)}(t_i)\big)d\tau \\
&= \frac{1}{(r-2)!}(t_{i+1}-t_i)^{r-2}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} x^{(r)}(\tau)d\tau \\
&\le (t_{i+1}-t_i)^{r-1}\|x^{(r)}\|_{L_1[t_i,t_{i+1}]} \le n^{-r},
\end{aligned}$$

where in the third inequality we used the fact that $x^{(r-1)}$ is nondecreasing and that $x^{(r-1)}(t_i) \le S_0(\tau) \le x^{(r-1)}(t_{i+1})$. This completes the proof of the upper bound in (2.1).

The lower bounds in (2.1) and (2.2) readily follow from Lemma 1. Indeed, $\frac{1}{r!}h_r \in \Delta_+^s W_p^r$, for all $s = 0, 1, \dots, r+1$, $1 \le p \le \infty$, and for $1 \le q \le \infty$ we have

$$\begin{aligned}
E\big(\Delta_+^s W_p^r, \Delta_+^s M_n(h_{r-1})\big)_{L_q} &\ge E\big(\frac{1}{r!}h_r, M_n(h_{r-1})\big)_{L_q} \\
&\ge 2^{-\frac{1}{q}} E\big(\frac{1}{r!}h_r, M_n(h_{r-1})\big)_{L_1} \\
&\ge 2^{-\frac{1}{q}}\frac{1}{r!}2^{-2r}(n+1)^{-r} \\
&\ge \frac{1}{r!}2^{-3r-1}n^{-r}.
\end{aligned}$$

The upper bound in (2.3) readily follows by observing that for every $x \in W_1^r$, Taylor's formula yields,

$$\left|x(t) - \sum_{s=0}^{r-1} x^{(s)}(0)t^s\right| = \frac{1}{(r-1)!}\left|\int_0^t x^{(r)}(\tau)(t-\tau)^{r-1}d\tau\right| \le \frac{1}{r!},$$

and that $P_{r-1} \subset \Delta_+^{r+1} M_n(h_{r-1})$.

For the lower bound in (2.3) we observe that $x \in \Delta_+^{r+1} M_n(h_{r-1})$ if and only if $x^{(r)}(t) \equiv 0$, $t \in I$, i.e., if and only if $x \in P_{r-1}$. Thus, we take $x_{r+1} := \frac{1}{(r+1)!}h_{r+1} \in \Delta_+^{r+1} W_p^r$,

$1 \le p \le \infty$, and for all $1 \le q \le \infty$, we obtain (see (3.2))

$$\begin{aligned} E\big(\Delta_+^{r+1} W_p^r, \Delta_+^{r+1} M_n(h_{r-1})\big)_{L_q} &\ge E\big(\Delta_+^{r+1} W_p^r, P_{r-1}\big)_{L_q} \\ &\ge E\big(\Delta_+^{r+1} W_p^r, P_r\big)_{L_q} \\ &\ge \frac{1}{(r+1)!} E\big(h_{r+1}, P_r\big)_{L_q} \\ &\ge 2^{-\frac{1}{q}} \frac{1}{(r+1)!} E\big(h_{r+1}, P_r\big)_{L_1} \\ &\ge 2^{-\frac{1}{q}} \frac{1}{(r+1)!} E\big(h_{r+1}, P_r\big)_{L_1[0,1]} \\ &\ge \frac{1}{(r+1)!} 2^{-2r-3}. \end{aligned}$$

This completes the proof of the lower bound in (2.3) and thus the proof of Theorem 1. $\square$

§4. Improved approximation by free-knot splines of degree r

This section contains the proof of Theorem 2. First we need

Lemma 2. *For every $n > 1$ and each $\alpha > 1$,*

$$\sup_{\substack{x_i>0, i=1,\dots,n \\ x_1+\cdots+x_n=1}} \left(\sum_{i=1}^n x_i^{-\alpha} \left(\sum_{j=i}^n x_j \right) \right)^{-1} \le \left(1 - \frac{1}{\alpha}\right)^{-\alpha} n^{-\alpha-1}. \tag{4.1}$$

Proof. We begin with the well known inequality (see [HLP])

$$\left(\frac{1}{n} \sum_{i=1}^n y_i^{-\alpha} \right)^{-\frac{1}{\alpha}} \le \left(\prod_{i=1}^n y_i \right)^{\frac{1}{n}} \le \left(\frac{1}{n} \sum_{i=1}^n y_i^{\beta} \right)^{\frac{1}{\beta}}$$

where $y_i > 0$, $1 \le i \le n$, and $\alpha, \beta > 0$. Taking $\beta = 1$ and setting

$$y_i := x_i \left(\sum_{j=i}^n x_j \right)^{-\frac{1}{\alpha}}, \quad i = 1, \dots, n,$$

we see that

$$\left(\sum_{i=1}^n x_i^{-\alpha} \left(\sum_{j=i}^n x_j \right) \right)^{-1} \le n^{-\alpha-1} \left(\sum_{i=1}^n x_i \left(\sum_{j=i}^n x_j \right)^{-\frac{1}{\alpha}} \right)^{\alpha}.$$

Thus it remains to prove that

$$\sum_{i=1}^{n} x_i \left(\sum_{j=i}^{n} x_j\right)^{-\frac{1}{\alpha}} \le \left(1-\frac{1}{\alpha}\right)^{-1}, \tag{4.2}$$

for all $x_i > 0$, $1 \le i \le n$, such that $x_1 + \cdots + x_n = 1$. To this end, set

$$z_i := \sum_{j=i}^{n} x_j, \quad i = 1, \ldots, n, \quad \text{and} \quad z_{n+1} = 0.$$

Then $x_i = z_i - z_{i+1}$, $i = 1, \ldots, n$, hence

$$\begin{aligned}
\sum_{i=1}^{n} x_i \left(\sum_{j=i}^{n} x_j\right)^{-\frac{1}{\alpha}} &= \sum_{i=1}^{n} (z_i - z_{i+1}) z_i^{-\frac{1}{\alpha}} \\
&\le \sum_{i=1}^{n} \int_{z_{i+1}}^{z_i} t^{-\frac{1}{\alpha}}\,dt \\
&= \int_0^1 t^{-\frac{1}{\alpha}}\,dt \\
&= \left(1-\frac{1}{\alpha}\right)^{-1}.
\end{aligned}$$

So (4.2) is valid and Lemma 2 is proved. $\square$

We are ready to prove Theorem 2.

Proof of Theorem 2. We begin with the proof of (2.4). To this end, we first observe that

$$E(\Delta_+^s W_\infty^r, M_n(h_r))_{L_1} \le E(\Delta_+^s W_p^r, \Delta_+^s M_n(h_r))_{L_q} \le E(\Delta_+^s W_1^r, \Delta_+^s M_n(h_r))_{L_\infty},$$

for all $1 \le p, q \le \infty$. Hence in order to prove (2.4), it suffices to prove the following two inequalities.

$$E(\Delta_+^s W_1^r, \Delta_+^s M_n(h_r))_{L_\infty} \le c n^{-r}, \quad s = 0, 1, \ldots, r, \tag{4.3}$$

and

$$E(\Delta_+^s W_\infty^r, M_n(h_r))_{L_1} \ge c n^{-r}, \quad s = 0, 1, \ldots, r. \tag{4.4}$$

We begin with (4.3), and take $x \in \Delta_+^s W_1^r$.

If $r = 1$, then $0 \le s \le 1$, and $\int_{-1}^1 |x'(\tau)|\,d\tau \le 1$. Let $-1 =: \tau_0 < \tau_1 < \cdots < \tau_{n+1} := 1$, be such that

$$\int_{\tau_j}^{\tau_{j+1}} |x'(\tau)|\,d\tau \le \frac{1}{n}.$$

Then the piecewise linear function $\sigma_{1,n}(t;x)$ which interpolates x at $t = \tau_j$, $j = 0, \dots, n+1$, has n knots, and it is in $\Delta_+^s M_n(h_1)$. It is readily seen that

$$\|x - \sigma_{1,n}(\cdot;x)\|_{L_\infty} \le cn^{-1},$$

and (4.3) follows.

For $r > 1$, by Theorem 1 there is a $\sigma_{r-1,n}(\cdot;x) \in \Delta_+^s M_n(h_{r-1})$ such that

$$\|x - \sigma_{r-1,n}(\cdot;x)\|_{L_\infty} \le cn^{-r}. \tag{4.5}$$

The $(r-1)$st derivative $\sigma_{r-1,n}^{(r-1)}(\cdot;x)$ is piecewise constant so that for an arbitrary $\epsilon > 0$ to be prescribed, it is easy to see that there exists a piecewise linear function $\sigma_{1,2n,\epsilon}(\cdot;x)$, with $2n$ knots such that

$$\|\sigma_{r-1,n}^{(r-1)}(\cdot;x) - \sigma_{1,2n,\epsilon}(\cdot;x)\|_{L_1} \le \epsilon. \tag{4.6}$$

Moreover if $r-1 \le s \le r$, then $\sigma_{1,2n,\epsilon}(\cdot;x) \in \Delta_+^{s-r+1} M_{2n}(h_r)$. Let

$$\sigma_{r,2n,\epsilon}(t;x) := \sum_{k=0}^{r-2} \sigma_{r-1,n}^{(k)}(0;x)\frac{1}{k!}t^k + \int_0^t \int_0^{\tau_{r-2}} \cdots \int_0^{\tau_1} \sigma_{1,2n,\epsilon}(\tau;x)\,d\tau d\tau_1 \cdots d\tau_{r-2}.$$

If $r-1 \le s \le r$, then clearly $\sigma_{r,2n,\epsilon}(\cdot;x) \in \Delta_+^s M_{2n}(h_r)$. However, if $0 \le s \le r-2$, then we cannot guarantee this. Still, from (4.6), for every $k = 0, \dots, r-2$

$$\|\sigma_{r-1,n}^{(k)}(\cdot;x) - \sigma_{r,2n,\epsilon}^{(k)}(\cdot;x)\|_{L_\infty} \le \epsilon, \tag{4.7}$$

so that the function

$$\tilde{\sigma}_{r,2n,\epsilon}(t;x) := \sigma_{r,2n,\epsilon}(t;x) + \frac{\epsilon}{s!}t^s \in \Delta_+^s M_{2n}(h_r), \quad 0 \le s \le r.$$

Indeed, we have to show this only for $s \le r-2$, but then by (4.7) with $k = s$, we have

$$\begin{aligned}\tilde{\sigma}^{(s)}_{r,2n,\epsilon}(t;x) &= \sigma^{(s)}_{r,2n,\epsilon}(t;x) + \epsilon \\ &\ge \sigma^{(s)}_{r-1,n}(t;x) - \epsilon + \epsilon \\ &\ge 0,\end{aligned}$$

where we used the fact that $\sigma^{(s)}_{r-1,n}(t;x) \ge 0$ for $t \in I$. Also, by virtue of (4.7) with $k = 0$, we obtain

$$\|\sigma_{r-1,n}(\cdot;x) - \tilde{\sigma}_{r,2n,\epsilon}(\cdot;x)\|_{L_\infty} \le 2\epsilon,$$

and combining with (4.5), we conclude that

$$\begin{aligned}\|x(\cdot) - \tilde{\sigma}_{r,2n,\epsilon}(\cdot;x)\|_{L_\infty} &\le \|x(\cdot) - \sigma_{r-1,n}(\cdot;x)\|_{L_\infty} + \|\sigma_{r-1,n}(\cdot;x) - \tilde{\sigma}_{r,2n,\epsilon}(\cdot;x)\|_{L_\infty} \\ &\le cn^{-r} + 2\epsilon.\end{aligned}$$

Taking $\epsilon := n^{-r}$, yields (4.3).

In order to prove (4.4), we take $t_{n,i} := -1 + \frac{i}{n}$, and denote $I_{ni} := [t_{n,i-1}, t_{n,i}]$, $i = 1, \ldots, 2n$, and their midpoints $\bar{t}_{n,i} := \frac{1}{2}(t_{n,i-1} + t_{n,i}) \quad i = 1, \ldots, 2n$. Now set

$$x_{0,n}(t) := \begin{cases} 0, & t \in (t_{n,i-1}, \bar{t}_{n,i}), \quad i = 1, \ldots, 2n, \\ 1, & t \in (\bar{t}_{n,i}, t_{n,i}), \quad i = 1, \ldots, 2n, \end{cases}$$

and let

$$x_{r,n}(t) := \int_{-1}^{t} \int_{-1}^{\tau_{r-1}} \cdots \int_{-1}^{\tau_1} x_{0,n}(\tau)\, d\tau d\tau_1 \cdots d\tau_{r-1} \quad t \in I.$$

Evidently, $x_{r,n} \in \Delta^s_+ W^r_\infty$ for all $s = 0, 1, \ldots, r$. Also it easy to verify that

$$x_{r,n}(t) = \frac{1}{2r!}|t - \bar{t}_{n,i}|^r + \pi_{r,i}(t), \quad t \in I_{n,i}, \quad i = 1, \ldots, 2n,$$

where $\pi_{r,i} \in P_r$.

If $y \in M_n(h_r)$, then there exist n subintervals I_{n,i_k}, $k = 1, \ldots, n$ (depending, of course, on y), such that $y_{|I_{n,i_k}}$ is a polynomial of degree $\le r$. Hence,

$$
\begin{aligned}
\|x_{n,r} - y\|_{L_1} &\ge \sum_{k=1}^{n} \int_{I_{n,i_k}} |x_{r,n}(t) - y(t)|dt \\
&= \sum_{k=1}^{n} \int_{I_{n,i_k}} \left| \frac{1}{2r!}|t - \bar{t}_{n,i_k}|^r + \pi_{r,i_k}(t) - y(t) \right| dt \qquad (4.8) \\
&\ge \frac{1}{2r!} \sum_{k=1}^{n} \inf_{\pi_r \in P_r} \int_{I_{n,i_k}} ||t - \bar{t}_{n,i}|^r - \pi_r(t)|dt.
\end{aligned}
$$

Denote by

$$
c_r := \inf_{\pi_r \in P_r} \int_{-1}^{1} ||t|^r - \pi_r(t)|dt > 0.
$$

Then it follows that

$$
\inf_{\pi_r \in P_r} \int_{I_{n,i_k}} ||t - \bar{t}_{n,i_k}|^r - \pi_r(t)|dt = c_r(2n)^{-r-1}, \quad k = 1, \dots, n,
$$

which by (4.8) yields

$$
\begin{aligned}
\|x_{n,r} - y\|_{L_1} &\ge \frac{1}{2r!} \sum_{k=1}^{n} c_r(2n)^{-r-1} \\
&= \frac{1}{2r!} c_r 2^{-r-1} n^{-r}.
\end{aligned}
$$

since $y \in M_n(h_r)$ was arbitrary, (4.4) follows.

We proceed to prove the upper bounds in (2.5) through (2.7). Let $n > 1$ and $1 \le q \le \infty$, be fixed and assume that $x \in \Delta_+^{r+1} W_p^r \cap C^r$, satisfies the additional assumption $x^{(r)}(0) = 0$. We remove this assumption in the second part of the proof. Then we have that $x^{(r)}$ is nonnegative on $[0, 1]$ and nonpositive on $[-1, 0]$. We will concentrate on the interval $[0, 1]$, the other interval being a symmetric case, and assume that $\|x^{(r)}\|_{L_p[0,1]} > 0$, otherwise there is nothing to prove.

We begin with some preparatory construction. Let $\mathbf{t} := (t_1, \dots, t_{n-1}) \in S^{n-1} := \{0 \le t_1 \le \cdots \le t_{n-1} \le 1\}$, and set $I_i := [t_{i-1}, t_i]$, $i = 1, \dots, n$. Set

$$
F_{p,q}\big(x^{(r)}; I_i\big) := \begin{cases} |I_i|^{r-1+\frac{1}{q}} \int_{I_i} \big(x^{(r)}(t) - x^{(r)}(t_{i-1})\big)dt, & 1 \le p < \infty, \\ |I_i|^{r+\frac{1}{q}} \big(x^{(r)}(t_i) - x^{(r)}(t_{i-1})\big), & p = \infty. \end{cases}
$$

Denote

$$F_{p,q}^{\max}\big(\mathbf{t};x^{(r)}\big) := \max_{1\le i\le n} F_{p,q}\big(x^{(r)};I_i\big),$$
$$F_{p,q}^{\min}\big(\mathbf{t};x^{(r)}\big) := \min_{1\le i\le n} F_{p,q}\big(x^{(r)};I_i\big),$$

and set

$$\Delta_{p,q}\big(\mathbf{t};x^{(r)}\big) := F_{p,q}^{\max}\big(\mathbf{t};x^{(r)}\big)-F_{p,q}^{\min}\big(\mathbf{t};x^{(r)}\big),$$

which is evidently continuous on the compact set S^{n-1}. Let

$$\Delta_{p,q}^{\min}\big(x^{(r)}\big) := \min_{\mathbf{t}\in S^{n-1}} \Delta_{p,q}\big(\mathbf{t};x^{(r)}\big).$$

We will prove that $\Delta_{p,q}^{\min}\big(x^{(r)}\big)=0$. To this end, assume to the contrary that $\Delta_{p,q}^{\min}\big(x^{(r)}\big)>0$. Denote by T_* the collection of all points $\mathbf{t}_*=(t_{*,1},\dots,t_{*,n-1})$, where the minimum is attained, and let $J(\mathbf{t}_*):=\{j(\mathbf{t}_*)\}$, be the set of all indices such that

$$F_{p,q}\big(x^{(r)};[t_{*,j(\mathbf{t}_*)-1},t_{*,j(\mathbf{t}_*)}]\big) = F_{p,q}^{\max}\big(\mathbf{t}_*;x^{(r)}\big).$$

Then clearly $\operatorname{card}J(\mathbf{t}_*)\ge 1$. Our assumption that $\Delta_{p,q}^{\min}\big(x^{(r)}\big)>0$, implies that there is an index $j(\mathbf{t}_*)\in J(\mathbf{t}_*)$ such that either $j(\mathbf{t}_*)\pm 1$ is not in $J(\mathbf{t}_*)$. Without loss assume it is $j(\mathbf{t}_*)-1$. Then we have

$$F_{p,q}\big(x^{(r)};[t_{*,j(\mathbf{t}_*)-2},t_{*,j(\mathbf{t}_*)-1}]\big) < F_{p,q}\big(x^{(r)};[t_{*,j(\mathbf{t}_*)-1},t_{*,j(\mathbf{t}_*)}]\big). \tag{4.9}$$

Observing that $F_{p,q}\big(x^{(r)};[t_{i-1},t_i]\big)$ increases continuously when t_{i-1} decrease and when t_i increase, we increase $t_{*,j(\mathbf{t}_*)-1}$ a little to, say, $t'_{*,j(\mathbf{t}_*)-1}$, so that we still preserve the inequality

$$F_{p,q}\big(x^{(r)};[t_{*,j(\mathbf{t}_*)-2},t'_{*,j(\mathbf{t}_*)-1}]\big) \le F_{p,q}\big(x^{(r)};[t'_{*,j(\mathbf{t}_*)-1},t_{*,j(\mathbf{t}_*)}]\big),$$

while the lefthand side is bigger and the righthand side is smaller than their counterparts in (4.9). This provides a new point $\mathbf{t}'_*:=\big(t_{*,1},\dots,t_{*,j(\mathbf{t}_*)-2},t'_{*,j(\mathbf{t}_*)-1},t_{*,j(\mathbf{t}_*)},\dots,t_{*,n-1}\big)$

in $\mathbf{T}_*$, such that $\mathrm{card}J(\mathbf{t}'_*) = \mathrm{card}J(\mathbf{t}'_*) - 1$. Repeating this process $\mathrm{card}J(\mathbf{t}_*)$ times, we end up with $\hat{\mathbf{t}}_* \in \mathbf{T}_*$ such that

$$F_{p,q}\big(x^{(r)};[\hat{t}_{*,i-1},\hat{t}_{*,i}]\big) - F^{\min}_{p,q}\big(\hat{\mathbf{t}}_*;x^{(r)}\big)< \Delta^{\min}_{p,q}\big(x^{(r)}\big)$$

for all $i = 1,\dots,n$, a contradiction. Thus we have shown that for every fixed $1 \le p,q \le \infty$, there exists a partition $0 =: t^*_0 < t^*_1 < \cdots < t^*_{n-1} < t^*_n := 1\big\}$ of $[0,1]$ (for the sake of simplicity in the notation we suppress the indices p and q) such that

$$F_{p,q}\big(x^{(r)};[t^*_{i-1},t^*_i]\big) = F^{\max}_{p,q}\big(\mathbf{t}_*;x^{(r)}\big)= F^{\min}_{p,q}\big(\mathbf{t}_*;x^{(r)}\big)> 0, \quad i = 1,\dots,n. \tag{4.10}$$

Doing the same in $[-1,0]$, we end up with a partition

$$T_n : -1 =: t^*_{-n} < \cdots < t^*_{-1} < 0 = t^*_0 < t^*_1 \cdots < t^*_n,$$

of I, which in addition to (4.10), satisfies

$$F_{p,q}\big(x^{(r)};[t^*_{-n},t^*_{-n+1}]\big) = \cdots = F_{p,q}\big(x^{(r)};[t^*_{-1},t^*_0]\big) > 0. \tag{4.11}$$

Set $I^*_i := [t^*_{i-1},t^*_i]$, $1 \le i \le n$, and $I^*_i := [t^*_i,t^*_{i+1}]$, $-n \le i \le -1$.

We are ready to proceed with the proof. Put $r_0 := \big[\frac{r+2}{2}\big]$. Then following the proof of Theorem 1 with r replaced by $r+1$, we conclude that for the above partition there exists a spline $\sigma_r(\cdot;x)$, of degree r (with r_0 additional knots in every interval I^*_i) the rth derivative of which is nondecreasing in I, and such that

$$\sigma^{(s)}_r\big(t^*_{i-1};x\big) = x^{(s)}\big(t^*_{i-1}\big), \quad \sigma^{(s)}_r\big(t^*_i;x\big) = x^{(s)}\big(t^*_i\big), \quad s = 0,\dots,r-1, \tag{4.12}$$

and

$$x^{(r)}\big(t^*_{i-1}\big) \le \sigma^{(r)}_r\big(t;x\big) \le x^{(r)}\big(t^*_i\big), \quad t^*_{i-1} \le t \le t^*_i, \quad -n < i \le n. \tag{4.13}$$

Finally,

$$\big|x(t) - \sigma_r\big(t;x\big)\big| \le \frac{1}{(r-1)!}\int_{t^*_{i-1}}^{t}\big|x^{(r)}(\tau) - \sigma^{(r)}_r\big(\tau;x\big)\big|(t-\tau)^{r-1}d\tau,$$
$$t^*_{i-1} \le t \le t^*_i, \quad -n < i \le n,$$

whence

$$\big\|x(\cdot)-\sigma_r\big(\cdot;x\big)\big\|_{L_q(I_i^*)} \le \frac{\big|I_i^*\big|^{r-1+\frac{1}{q}}}{(r-1)!}\int_{I_i^*}\big|x^{(r)}(t)-\sigma_r^{(r)}\big(t;x\big)\big|dt. \tag{4.14}$$

We will restrict our discussion to $[0,1]$, the other case is symmetric. It follows by virtue of (4.12) that

$$\begin{aligned}\int_{I_i^*}\big|x^{(r)}(t)-\sigma_r^{(r)}(t;x)\big|dt &= 2\int_{I_i^*}\big(x^{(r)}(t)-\sigma_r^{(r)}(t;x)\big)_+dt\\ &\le 2\int_{I_i^*}\big(x^{(r)}(t)-x^{(r)}(t_{i-1}^*)\big)dt\\ &\le 2|I_i^*|\big(x^{(r)}(t_i^*)-x^{(r)}(t_{i-1}^*)\big), \quad 1\le i\le n,\end{aligned}$$

where for the first inequality we used the monotonicity of $\sigma_r^{(r)}(\cdot;x)$ and in the last inequality we applied the monotonicity of $x^{(r)}$. This combined with (4.14) implies

$$\big\|x(\cdot)-\sigma_r(\cdot;x)\big\|_{L_q(I_i^*)} \le \frac{2}{(r-1)!}F_{p,q}^{\max}\big(\mathbf{t}_*;x^{(r)}\big), \quad 1\le p\le\infty,$$

which in turn yields

$$\big\|x(\cdot)-\sigma_r(\cdot;x)\big\|_{L_q[0,1]} \le \frac{2n^{\frac{1}{q}}}{(r-1)!}F_{p,q}^{\max}\big(\mathbf{t}_*;x^{(r)}\big), \quad 1\le p\le\infty. \tag{4.15}$$

Next we wish to estimate $\big\|x^{(r)}\big\|_{L_p[0,1]}$ from below and we first assume that $1\le p<\infty$. Then by Hölder's inequality we obtain,

$$\begin{aligned}&\big\|x^{(r)}\big\|_{L_p[0,1]}^p = \sum_{i=1}^n\int_{I_i^*}\big(x^{(r)}(t)\big)^p dt\\ &= \int_{I_1^*}\big(x^{(r)}(t)-x^{(r)}(t_0^*)\big)^p dt\\ &\quad+\sum_{i=2}^n\int_{I_i^*}\Big[\big(x^{(r)}(t)-x^{(r)}(t_{i-1}^*)\big)+\sum_{j=1}^{i-1}\big(x^{(r)}(t_j^*)-x^{(r)}(t_{j-1}^*)\big)\Big]^p dt\\ &\ge |I_1^*|^{-p+1}\left(\int_{I_1^*}\big(x^{(r)}(t)-x^{(r)}(t_0^*)\big)dt\right)^p\\ &\quad+\sum_{i=2}^n|I_i^*|^{-p+1}\left(\int_{I_i^*}\big(x^{(r)}(t)-x^{(r)}(t_{i-1}^*)\big)dt\right.\\ &\quad\left.+\sum_{j=1}^{i-1}\int_{I_i^*}\big(x^{(r)}(t_j^*)-x^{(r)}(t_{j-1}^*)\big)dt\right)^p,\end{aligned}$$

since by assumption $x^{(r)}(t_0^*) = x^{(r)}(0) = 0$. Hence,

$$
\begin{aligned}
1 &\geq \left\| x^{(r)} \right\|_{L_p([0,1])} \\
&\geq \left(\sum_{i=1}^{n} |I_i^*| \left(\sum_{j=1}^{i} |I_j^*|^{-1} \int_{I_j^*} \big(x^{(r)}(t) - x^{(r)}(t_{j-1}^*) \big) dt \right)^p \right)^{\frac{1}{p}} \\
&\geq \left(\sum_{i=1}^{n} |I_i^*| \left(\sum_{j=1}^{i} |I_j^*|^{-r-\frac{1}{q}} \right)^p \right)^{\frac{1}{p}} \\
&\quad \times \min_{1 \leq i \leq n} \left(|I_i^*|^{r-1+\frac{1}{q}} \int_{I_i^*} \big(x^{(r)}(t) - x^{(r)}(t_{i-1}^*) \big) dt \right) \\
&= \left(\sum_{i=1}^{n} |I_i^*| \left(\sum_{j=1}^{i} |I_j^*|^{-r-\frac{1}{q}} \right)^p \right)^{\frac{1}{p}} F_{p,q}^{\min}\big(\mathbf{t}_*; x^{(r)} \big),
\end{aligned} \tag{4.16}
$$

where we applied that for all i and j,

$$
\int_{I_i^*} \big(x^{(r)}(t_j^*) - x^{(r)}(t_{j-1}^*) \big) dt \geq \frac{|I_i^*|}{|I_j^*|} \int_{I_j^*} \big(x^{(r)}(t) - x^{(r)}(t_{j-1}^*) \big) dt,
$$

since $x^{(r)}$ is nondecreasing. In view of (4.10), we may combine (4.15) and (4.16) to obtain

$$
\begin{aligned}
&\left\| x(\cdot) - \sigma_r(\cdot; x) \right\|_{L_q[0,1]} \\
&\leq \frac{2n^{\frac{1}{q}}}{(r-1)!} \left(\sum_{i=1}^{n} |I_i^*| \left(\sum_{j=1}^{i} |I_j^*|^{-r-\frac{1}{q}} \right)^p \right)^{-\frac{1}{p}}.
\end{aligned} \tag{4.17}
$$

For $p = \infty$ we have

$$
\begin{aligned}
1 &\geq \left\| x^{(r)} \right\|_{L_\infty[0,1]} \\
&= \sum_{i=1}^{n} \big(x^{(r)}(t_i^*) - x^{(r)}(t_{i-1}^*) \big) \\
&\geq \left(\sum_{i=1}^{n} \left| I_i^* \right|^{-r-\frac{1}{q}} \right) \left(\min_{1 \leq i \leq n} \left(\left| I_i^* \right|^{r+\frac{1}{q}} \big(x^{(r)}(t_i^*) - x^{(r)}(t_{i-1}^*) \big) \right) \right) \\
&= \left(\sum_{i=1}^{n} \left| I_i^* \right|^{-r-\frac{1}{q}} \right) F_{\infty,q}^{\min}\big(\mathbf{t}_*; x^{(r)} \big).
\end{aligned} \tag{4.18}
$$

Again in view of (4.10), we may combine (4.15) and (4.18) to obtain

$$\left\|x(\cdot)-\sigma_{r,n}\left(\cdot;x\right)\right\|_{L_q[0,1]} \le \frac{2n^{\frac{1}{q}}}{(r-1)!}\left(\sum_{i=1}^{n}\left|I_i^*\right|^{-r-\frac{1}{q}}\right)^{-1}. \tag{4.19}$$

Since $W_p^r([0,1]) \subseteq W_1^r([0,1])$, $1 \le p \le \infty$, for $r > 1$ we substitute $p = 1$ and $q = \infty$ in (4.17), and obtain by Lemma 2 with $\alpha = r$,

$$\begin{aligned}\left\|x(\cdot)-\sigma_{r,n}\left(\cdot;x\right)\right\|_{L_\infty[0,1]} &\le \frac{2}{(r-1)!}\left(\sum_{i=1}^{n}|I_i^*|\left(\sum_{j=1}^{i}|I_j^*|^{-r}\right)\right)^{-1}\\ &= \frac{2}{(r-1)!}\left(\sum_{i=1}^{n}|I_i^*|^{-r}\left(\sum_{j=i}^{n}|I_j^*|\right)\right)^{-1}\\ &\le \frac{2}{(r-1)!}\left(1-\frac{1}{r}\right)^{-r} n^{-r-1}.\end{aligned} \tag{4.20}$$

If $r = 1$ and $1 \le q < \infty$, then again we substitute $p = 1$ in (4.17) and obtain by Lemma 2 with $\alpha = 1 + \frac{1}{q}$,

$$\begin{aligned}\left\|x(\cdot)-\sigma_{1,n}\left(\cdot;x\right)\right\|_{L_q[0,1]} &\le 2n^{\frac{1}{q}}\left(\sum_{i=1}^{n}|I_i^*|\left(\sum_{j=1}^{i}|I_j^*|^{-1-\frac{1}{q}}\right)\right)^{-1}\\ &= 2n^{\frac{1}{q}}\left(\sum_{i=1}^{n}|I_i^*|^{-1-\frac{1}{q}}\left(\sum_{j=i}^{n}|I_j^*|\right)\right)^{-1}\\ &\le 2(q+1)^{1+\frac{1}{q}}n^{-2}.\end{aligned} \tag{4.21}$$

If $r = 1$, $p = 1$ and $q = \infty$, then by (4.17)

$$\begin{aligned}\left\|x(\cdot)-\sigma_{1,n}\left(\cdot;x\right)\right\|_{L_\infty[0,1]} &\le 2\left(\sum_{i=1}^{n}|I_i^*|\left(\sum_{j=1}^{i}|I_j^*|^{-1}\right)\right)^{-1}\\ &\le 2\left(\sum_{i=1}^{n}|I_i^*||I_i^*|^{-1}\right)^{-1}\\ &= 2n^{-1}.\end{aligned} \tag{4.22}$$

Finally if $r = 1$ and $p = q = \infty$, then it follows by (4.19) that

$$\begin{aligned} \|x(\cdot) - \sigma_{1,n}(\cdot; x)\|_{L_\infty[0,1]} &\le \left(\sum_{i=1}^{n} |I_i^*|^{-1} \right)^{-1} \\ &\le \sup_{\substack{x_i > 0, i=1,\ldots,n \\ x_1+\cdots+x_n=1}} \left(\sum_{i=1}^{n} x_i^{-1} \right)^{-1} \\ &= n^{-2}, \end{aligned} \tag{4.23}$$

where the last equality readily follows by induction.

So, we are left with the case $r = 1$, $q = \infty$ and $1 < p < \infty$. We take $0 < \varepsilon \le 1 - \frac{1}{p}$ and denote $p^* := (1 - \varepsilon)^{-1}$. Then $1 < p^* \le p$, so that $W_p^1([0,1]) \subseteq W_{p^*}^1([0,1])$. By virtue of (4.17)

$$\|x(\cdot) - \sigma_{1,n}(\cdot; x)\|_{L_\infty[0,1]} \le 2 \left(\sum_{i=1}^{n} |I_i^*| \left(\sum_{j=1}^{i} |I_j^*|^{-1} \right)^{p^*} \right)^{-\frac{1}{p^*}},$$

where here the intervals I_i^* are the ones associated with p^*. Hence we have

$$\begin{aligned} \|x(\cdot) - \sigma_{1,n}(\cdot; x)\|_{L_\infty[0,1]} &\le 2 \sup_{\substack{x_i > 0, i=1,\ldots,n \\ x_1+\cdots+x_n=1}} \left(\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{i} x_j^{-1} \right)^{p^*} \right)^{-\frac{1}{p^*}} \\ &\le \left(\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{i} x_j^{-p_*} \right) \right)^{-\frac{1}{p_*}} \\ &= \left(\sum_{i=1}^{n} x_i^{-p_*} \left(\sum_{j=i}^{n} x_j \right) \right)^{-\frac{1}{p_*}}. \end{aligned}$$

Now we apply Lemma 2 with $\alpha = p_* := (1 - \varepsilon)^{-1}$ to obtain

$$\left(\sum_{i=1}^{n} x_i \left(\sum_{j=1}^{i} x_j^{-1} \right)^{p_*} \right)^{-\frac{1}{p_*}} \le \varepsilon^{-1} n^{-2+\varepsilon}. \tag{4.24}$$

We have estimates similar to (4.20) through (4.24) for the interval $[-1, 0]$ and together they complete the proof of the upper bounds in (2.5), (2.6) and (2.7) under the additional assumptions that $x \in C^r$ and $x^{(r)}(0) = 0$. All that is left is to reduce the general case $x \in \Delta_+^{r+1} W_p^r$, to the above.

First we extend $x \in \Delta_+^{r+1} W_p^r$ to $\mathbb{R}$ by setting

$$\tilde{x}(t) := \begin{cases} \sum_{s=0}^{r-1} \frac{1}{s!} x^{(s)}(-1)(t+1)^s, & t \in (-\infty, -1), \\ x(t), & t \in [-1, 1] \\ \sum_{s=0}^{r-1} \frac{1}{s!} x^{(s)}(1)(t-1)^s, & t \in (1, +\infty), \end{cases}$$

so that all the derivatives $\tilde{x}^{(s)}$, $s = 0, \ldots, r-1$ are locally absolutely continuous on $\mathbb{R}$ and $x^{(r)} \in L_p(\mathbb{R})$ with

$$\|\tilde{x}^{(r)}\|_{L_p(\mathbb{R})} = \|x^{(r)}\|_{L_p(I)}.$$

For $0 < \delta < \frac{1}{2}$, let

$$\tilde{x}_\delta(t) := \frac{1}{\delta} \int_{-\frac{1}{2}\delta}^{\frac{1}{2}\delta} \tilde{x}(t+\tau) d\tau, \quad t \in \mathbb{R},$$

be the Steklov average. Then $\tilde{x}_\delta \in C^r(\mathbb{R})$, $\tilde{x}_\delta^{(r)}(t) = 0$, $t \in \mathbb{R} \setminus [-1 - \frac{1}{2}\delta, 1 + \frac{1}{2}\delta]$, and

$$\lim_{\delta \to 0} \left\| \tilde{x}^{(r)} - \tilde{x}_\delta^{(r)} \right\|_{L_1(\mathbb{R})} = 0.$$

Hence

$$\lim_{\delta \to 0} \left\| x^{(s)} - \tilde{x}_\delta^{(s)} \right\|_{L_\infty(I)} = 0, \quad s = 0, \ldots, r-1.$$

It is obvious that for any $\delta > 0$, $\tilde{x}_\delta^{(r)}$ is nondecreasing on $I_\delta := [-1 + \frac{1}{2}\delta, 1 - \frac{1}{2}\delta]$ and that

$$\left\| \tilde{x}_\delta^{(r)} \right\|_{L_p(I_\delta)} \le \left\| x^{(r)} \right\|_{L_p(I)}.$$

Hence $\tilde{x}_\delta \in \Delta_+^{r+1} W_p^r(I_\delta)$ and $\tilde{x}_\delta^{(r-1)}$ is convex on I_δ. Let $\pi_r(x_\delta; \cdot)$ be an rth degree polynomial such that

$$\pi_r^{(r-1)}(\tilde{x}_\delta; 0) = \tilde{x}_\delta^{(r-1)}(0), \quad \pi_r^{(r-1)}(\tilde{x}_\delta; t) \le \tilde{x}_\delta^{(r-1)}(t), \quad t \in I_\delta,$$

and put

$$\breve{x}(t)_\delta := \tilde{x}_\delta(t) - \pi_r(\tilde{x}_\delta; t), \quad t \in I_\delta.$$

Since $\breve{x}_\delta^{(r)}$ is nondecreasing on I_δ and

$$\breve{x}_\delta^{(r)}(0) = 0,$$

it readily follows that

$$\big\|\breve{x}_\delta^{(r)}\big\|_{L_p(I_\delta)} \le 3\big\|\tilde{x}_\delta^{(r)}\big\|_{L_p(I_\delta)} \le 3\big\|x^{(r)}\big\|_{L_p(I)}.$$

Indeed, if $\pi_r^{(r)}(\tilde{x}_\delta;t) = \pi_r^{(r)}(\tilde{x}_\delta;0) \ge 0$, then $0 \le \pi_r^{(r)}(\tilde{x}_\delta;t) \le \tilde{x}_\delta^{(r)}(t)$ in $[0, 1-\frac{1}{2}\delta]$. Hence

$$\begin{aligned}
\big\|\breve{x}_\delta^{(r)}\big\|_{L_p(I_\delta)} &\le \big\|\tilde{x}_\delta^{(r)}\big\|_{L_p(I_\delta)} + \big\|\pi_r^{(r)}(\tilde{x}_\delta;\cdot)\big\|_{L_p(I_\delta)} \\
&\le \big\|\tilde{x}_\delta^{(r)}\big\|_{L_p(I_\delta)} + 2\big\|\pi_r^{(r)}(\tilde{x}_\delta;\cdot)\big\|_{L_p[0,1-\frac{1}{2}\delta]} \\
&\le \big\|\tilde{x}_\delta^{(r)}\big\|_{L_p(I_\delta)} + 2\big\|\tilde{x}_\delta^{(r)}\big\|_{L_p[0,1-\frac{1}{2}\delta]} \\
&\le 3\big\|\tilde{x}_\delta^{(r)}\big\|_{L_p(I_\delta)} \le 3\big\|x^{(r)}(\cdot)\big\|_{L_p(I)}.
\end{aligned}$$

Otherwise, $\pi_r^{(r)}(\tilde{x}_\delta;0) < 0$, so that $0 > \pi_r^{(r)}(\tilde{x}_\delta;t) \ge \tilde{x}_\delta^{(r)}(t)$ in $[-1+\frac{1}{2}\delta, 0]$, and the proof is similar.

Now, by the above proof applied to the function $\breve{x}_\delta \in C^r(I_\delta)$, an $r+1$-monotone spline $\sigma_{r,n}\big(\cdot;\breve{x}_\delta\big)$ exists in I_δ, which satisfies the appropriate righthand inequalities in (2.5) through (2.7), in the interval I_δ, with constants that are independent of δ. We extend it to I by

$$\sigma_{r,n}\big(t;\breve{x}_\delta\big) := \begin{cases} \sum_{s=0}^r \frac{1}{s!}\breve{x}_\delta^{(s)}(-1+\frac{1}{2}\delta)(t+1-\frac{1}{2}\delta)^s, & t \in [-1, -1+\frac{1}{2}\delta], \\ \sum_{s=0}^r \frac{1}{s!}\breve{x}_\delta^{(s)}(1-\frac{1}{2}\delta)(t-1+\frac{1}{2}\delta)^s, & t \in [1-\frac{1}{2}\delta, 1], \end{cases}$$

thus preserving the $r+1$-monotonicity, and set

$$\sigma_{r,n,\delta}\big(t;x\big) := \sigma_{r,n}\big(t;\breve{x}_\delta\big) + \pi_r(x_\delta;t), \quad t \in I.$$

Evidently, $\sigma_{r,n,\delta}\big(\cdot;x\big) \in \Delta_+^{r+1}L_q$, and for sufficiently small δ yields the upper bounds in (2.5) through (2.7). This completes the proof of the upper bounds.

We proceed to prove the lower bounds. Since $x_{r+1} := \frac{1}{(r+1)!}h_{r+1} \in \Delta_+^{r+1}W_p^r$, we apply Lemma 1 and obtain for all $r \in \mathbb{N}$, $1 \le p \le \infty$ and $1 \le q \le \infty$,

$$\begin{aligned} E\big(\Delta_+^{r+1}W_p^r, \Delta_+^{r+1}M_n(h_r)\big)_{L_q} &\ge E\big(x_{r+1}, M_n(h_r)\big)_{L_q} \\ &\ge 2^{-\frac{1}{q}} E\big(x_{r+1}, M_n(h_r)\big)_{L_1} \\ &\ge 2^{-\frac{1}{q}} \frac{1}{(r+1)!} 2^{-2(r+1)}(n+1)^{-r-1} \\ &\ge \frac{1}{(r+1)!} 2^{-3r-4} n^{-r-1}. \end{aligned}$$

Thus the lower bounds in (2.5) and (2.7) are established.

For the remaining case $r = p = 1$ and $q = \infty$, we need another extreme function. Let z_n, be the piecewise linear function with knots $\tau_0 = \tau_{n,0} := -1$, $\tau_i = \tau_{n,i} := 1 - 2^{-i+1}$, $i = 1, \dots, n+1$, and $\tau_{n+2} = \tau_{n,n+2} := 1$, taking the values $z_n(\tau_0) = z_n(\tau_1) := 0$, $z_n(\tau_i) := 2^{i-1}$, $i = 2, \dots, n+2$. Set

$$y_n(t) := \|z_n\|_{L_1}^{-1} \int_{-1}^{t} z_n(\tau) d\tau, \quad t \in I.$$

Straightforward calculations yield $\|z_n\|_{L_1} = \frac{3n+5}{4}$, so that

$$y_n(t) := \frac{4}{3n+5} \int_{-1}^{t} z_n(\tau) d\tau, \quad t \in I.$$

Clearly $\|y_n'\|_{L_1} = 1$ and $y_n \in \Delta_+^2 L_1$, hence $y_n \in \Delta_+^2 W_1^1$. Put $J_i := [\tau_i, \tau_{i+1}]$, $i = 0, 1, \dots, n+1$. Then

$$|J_i| = 2^{-i}, \quad i = 0, \dots, n, \quad \text{and} \quad |J_{n+1}| = 2^{-n}. \tag{4.25}$$

For every $h \in M_n\big(h_1\big)$ there exists an index $0 \le j_0 \le n+1$ for which h is a linear function on J_j. Hence for $1 \le j_0 \le n$,

$$\begin{aligned} \|y_n - h\|_{L_\infty} &\ge \|y_n - h\|_{L_\infty(J_{j_0})} \\ &\ge \inf_{\pi_1 \in P_1} \big\|y_n - \pi_1\big\|_{L_\infty(J_{j_0})} \\ &\ge \frac{4}{3n+5} 2^{2j_0 - 2} \inf_{\pi_1 \in P_1} \max_{t \in J_{j_0}} \big|t^2 - \pi_1(t)\big|, \end{aligned} \tag{4.26}$$

and if $j_0 = n+1$, then

$$\|y_n - h\|_{L_\infty} \geq \frac{4}{3n+5} 2^{2n-1} \inf_{\pi_1 \in P_1} \max_{t \in J_{n+1}} \bigl|t^2 - \pi_1(t)\bigr|, \tag{4.27}$$

The infima on the righthand sides of (4.26) and (4.27) are of course the norm of the Chebyshev polynomial of degree 2 associated with the respective J_{j_0} interval, i.e.,

$$\inf_{\pi_1 \in P_1} \max_{t \in J_{j_0}} \bigl|(t^2 - \pi_1(t)\bigr| = 2^{-3} |J_{j_0}|^2.$$

Thus, by virtue of (4.25), (4.26) and (4.27) we conclude that

$$\begin{aligned} E\bigl(\Delta_+^2 W_1^1, \Delta_+^2 M_n(h_1),\bigr)_{L_\infty} &\geq E\bigl(\Delta_+^2 W_1^1, M_n(h_1)\bigr)_{L_\infty} \\ &\geq \frac{4}{3n+5} 2^{-5} \\ &\geq \frac{1}{64} n^{-1} \end{aligned}$$

and the lower bound in (2.6) follows. This completes the proof of Theorem 2. $\square$

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Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv 01601, Ukraine

School of Mathematical Sciences, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel