On the relation between piecewise polynomial and rational approximation in $L_p\left(\mathbb{R}^2\right)$

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Abstract: In the univariate case there are certain equivalences between the nonlinear approximation methods that use piecewise polynomial and those that use rational functions. It is known that for certain parameters the respective approximation spaces are identical and may be described as Besov spaces. The characterization of the approximation spaces of the multivariate nonlinear approximation by piecewise polynomials and by rational functions is not known. In this work we compare between the two methods in the bivariate case. We show some relations between the approximation spaces of piecewise polynomials defined on n triangles and those of bivariate rational functions of total degree n which are described by n parameters. Thus we compare two classes of approximants with the same number Cn of parameters. We consider this the proper comparison between the two methods.

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1 Introduction

Let $\Pi_n(\mathbb{R}^2)$ be the space of polynomials of total degree n and $\mathcal{R}_n$, the set of all rational functions of degree n, i.e.,

$$\mathcal{R}_n := \left\{ R = P_1 / P_2 : P_1, P_2 \in \Pi_n(\mathbb{R}^2),\ P_2 > 0 \right\}.$$

We wish to restrict ourselves to elements of $\mathcal{R}_n$ that are taken from a collection of functions that can be described by m parameters. We will denote this collection by $\widetilde{\mathcal{R}}_{m,n}$ (see description in the proof of Lemma 3.3). Thus, for $f \in L_p(\mathbb{R}^2)$, $0 < p \le \infty$ we denote

$$\tilde{r}_{m,n}(f)_p := \inf_{R \in \widetilde{\mathcal{R}}_{m,n}} \|f - R\|_p .$$

Whenever $m = n$, we will use the notations $\widetilde{\mathcal{R}}_n := \widetilde{\mathcal{R}}_{n,n}$ and $\tilde{r}_n(f)_p := \tilde{r}_{n,n}(f)_p$.

Let $S_m^r(\mathbb{R}^2)$ denote the space of all piecewise polynomial functions of degree r over triangles,

$$\sum_{j=1}^{m} \mathbf{1}_{\Delta_j}(x)\, P_j(x), \tag{1.1}$$

where $\operatorname{int}(\Delta_j) \cap \operatorname{int}(\Delta_k) = \emptyset$ for $j \ne k$ and $P_j \in \Pi_r(\mathbb{R}^2)$. For $b > 0$ we denote by $S_m^{r,b}(\mathbb{R}^2)$ the subspace of piecewise polynomial functions over triangles whose minimal angle is $\ge b$. For $f \in L_p(\mathbb{R}^2)$, $0 < p \le \infty$ we denote

$$s_{m,r}(f)_p := \inf_{j \in S_m^r} \|f - j\|_p, \qquad s_{m,r,b}(f)_p := \inf_{j \in S_m^{r,b}} \|f - j\|_p .$$

Observe that any piecewise polynomial function of degree r of the form

$$\sum_{k=1}^{\tilde{m}} \mathbf{1}_{\Omega_k}(x)\, P_k(x), \tag{1.2}$$

where Ω_k are compact polyhedral domains (not necessarily disjoint) can be represented in the form (1.1) by triangulation. If the domains are pairwise disjoint then the number of triangles m in (1.1) is proportional to $n_e \tilde{m}$, where $\tilde{m}$ is the number of domains in (1.2) and n_e is an upper bound on the complexity of the domains' boundaries. However, we note that whenever the approximation takes place in $L_p(\mathbb{R}^2)$, $1 < p < \infty$, we restrict ourselves to triangles that are not too "skinny". For example, if the domains Ω_k are rectangles, then this implies that for $1 < p < \infty$ we restrict ourselves to rectangles whose dimensions' ratio is bounded away from 0 and ∞.

We have been motivated in this work by a manuscript of Pencho Petrushev [P]. We follow the approach in [P] that uses nested sets of rectangles and define nested sets of triangles.

Definition 1.1 We call a sequence of sets of triangles $\Lambda = \{\Lambda_n\}_{n \ge 0}$ an **almost nested sequence** if it has the following properties

(i) $|\Lambda_n| \le C_\Lambda 2^n$, for some $C_\Lambda > 0$,

(ii) The triangles of Λ_n have disjoint interiors,

(iii) For each $\Delta \in \Lambda_{n+1}$ and $\tilde{\Delta} \in \Lambda_n$ either $\operatorname{int}(\Delta) \cap \operatorname{int}(\tilde{\Delta}) = \emptyset$ or $\Delta \subseteq \tilde{\Delta}$,

(iv) $\bigcup_{\Delta \in \Lambda_n} \Delta \subseteq \bigcup_{\Delta \in \Lambda_{n+1}} \Delta$.

We denote by $S^r(\Lambda_n)$ the set of piecewise polynomial functions associated with the level n

$$\sum_{\substack{\Delta \in \Lambda_n \\ P_\Delta \in \Pi_r}} \mathbf{1}_\Delta(x) P_\Delta(x).$$

Observe that conditions (ii)-(iv) above imply the two-scale relation $S^r(\Lambda_n) \subset S^r(\Lambda_{n+1})$, $n \ge 0$. Therefore, whenever $\inf_{\Delta \in \Lambda} (minangle(\Delta)) = b > 0$, for any $j_n \in S^r(\Lambda_n)$ and $j_{n+1} \in S^r(\Lambda_{n+1})$, the sum $j_n + j_{n+1}$, is in $S^r(\Lambda_{n+1})$ and thus in $S^{r,b}_{C_\Lambda 2^{n+1}}(\mathbb{R}^2)$. The degree of approximation by the level n is

$$E_{2^n,r}(f,\Lambda)_p := \min_{j \in S^r(\Lambda_n)} \| f - j \|_p. \tag{1.3}$$

The following two theorems are bivariate analogues of Theorem 10.6.2 in [LGM]. Note that by restricting ourselves to approximation from $\widetilde{\mathcal{R}}_{2^n}$, **we compare two nonlinear methods of approximation with** $C2^n$ **free parameters**. We thank Pencho Petrushev for pointing out to us the importance of comparing the two methods with comparable number of free parameters. We consider this the proper comparison.

Theorem 1.2 Let $f \in L_p(\mathbb{R}^2)$, $a > 0$, $r \ge 0$ and let $\Lambda = \{\Lambda_n\}_{n \ge 0}$ be any almost nested sequence.

(i) For $0 < p \le 1$ and any $n \ge 0$ we have

$$\tilde{r}_{2^n}(f)_p \le C 2^{-an} \left(\sum_{k=0}^{n} \left[2^{ak} E_{2^k,r}(f,\Lambda)_p \right]^p + \|f\|_p^p \right)^{1/p}, \tag{1.4}$$

with $C = C(p, r, a, C_\Lambda)$.

(ii) Let $1 < p < \infty$ and assume $\inf_{\Delta \in \Lambda} (minangle(\Delta)) = b > 0$. Then for any $n \ge 0$ we have

$$\tilde{r}_{2^n}(f)_p \le C 2^{-an} \left(\sum_{k=0}^{n} 2^{ak} E_{2^k,r}(f,\Lambda)_p + \|f\|_p \right), \tag{1.5}$$

with $C = C(p, r, a, b, C_\Lambda)$.

Remark Observe that when the approximation takes place in $L_p(\mathbb{R}^2)$, $1 < p < \infty$, we are forced to restrict ourselves to triangles that are not too "skinny". This assumption is needed in the proof of Lemma 2.7 where we use a variant of the Hardy-Littlewood maximal function (see Section 2.2).

Theorem 1.3 Let $f \in L_p(\mathbb{R}^2)$, $0 < p \le 1$, $a > 0$ and $r \ge 0$. Then for any $n \ge 0$ we have

$$\tilde{r}_{4^n}(f)_p \le C 2^{-an} \left(\sum_{k=0}^{n} \left[2^{ak} s_{2^k,r}(f)_p \right]^p + \|f\|_p^p \right)^{1/p}, \tag{1.6}$$

with $C = C(p,r,a)$.

Definition 1.4 (Piecewise polynomial approximation spaces) Let $0 < p \le \infty$ and $r \ge 0$. For $0 < q \le \infty$ and $a > 0$ we define the approximation space $A_q^a(L_p, \Sigma_r)$ as the set of functions $f \in L_p(\mathbb{R}^2)$ for which

$$|f|_{A_q^a(L_p,\Sigma_r)} := \begin{cases} \left(\sum_{m=0}^{\infty} \left(2^{ma} s_{2^m,r}(f)_p \right)^q \right)^{1/q} & 0 < q < \infty, \\ \sup_{m \ge 0} 2^{ma} s_{2^m,r}(f)_p & q = \infty, \end{cases} \tag{1.7}$$

is finite. In a similar manner we use the sequence $\left\{ s_{2^m,r,b}(f)_p \right\}_{m \ge 0}$ to define $A_q^a(L_p, \Sigma_r, b)$ for some minimal angle b. Observe that if $b_1 \le b_2$ then $A_q^a(L_p, \Sigma_r, b_2) \subseteq A_q^a(L_p, \Sigma_r, b_1)$. For an almost nested sequence $\Lambda = \{\Lambda_n\}_{n \ge 0}$ we define $A_q^a(L_p, \Sigma_r, \Lambda)$ by replacing in (1.7) the terms $s_{2^m,r}(f)_p$ with $E_{2^m,r}(f,\Lambda)_p$ (see (1.3)). Finally, for $f \in L_p(\mathbb{R}^2)$ we let

$$|f|_{A_q^a(L_p,\Sigma_r,nest)} := \inf_{\Lambda} |f|_{A_q^a(L_p,\Sigma_r,\Lambda)},$$

$$|f|_{A_q^a(L_p,\Sigma_r,nest,b)} := \inf_{\Lambda} \left\{ |f|_{A_q^a(L_p,\Sigma_r,\Lambda)} : \inf_{\Delta \in \Lambda} \left(minangle(\Delta) \right) \ge b \right\}.$$

Definition 1.5 (Rational approximation spaces) Let $0 < p \le \infty$. For $0 < q \le \infty$ and $a > 0$ we define the **rational logarithmic approximation** space $A_{q,\log}^a(L_p, \widetilde{\mathcal{R}})$ as the set of functions $f \in L_p(\mathbb{R}^2)$ for which

$$|f|_{A_{q,\log}^a(L_p,\widetilde{\mathcal{R}})} := \begin{cases} \left(\sum_{n=0}^{\infty} \left(2^{na} \tilde{r}_{2^n, \left\lceil 2^n \log^2\left(2^{n(a+e)}\right) \right\rceil}(f)_p \right)^q \right)^{1/q} & 0 < q < \infty, \\ \sup_{n \ge 0} 2^{na} \tilde{r}_{2^n, \left\lceil 2^n \log^2\left(2^{na}\right) \right\rceil}(f)_p & q = \infty, \end{cases} \tag{1.8}$$

is finite for some $e > 0$. We denote by $A_q^a(L_p, \widetilde{\mathcal{R}})$ the space corresponding to rational approximation where the terms $\tilde{r}_{2^n}(f)_p$ are used in (1.8).

The following is our main result.

Theorem 1.6 Let $0 < q \le \infty$ and $a > 0$. For $0 < p \le 1$ we have

$$A_q^a(L_p, \Sigma_r, nest) \subseteq A_q^a(L_p, \widetilde{\mathcal{R}}), \tag{1.9}$$

$$A_q^a(L_p, \Sigma_r) \subseteq A_{q,\log}^a(L_p, \widetilde{\mathcal{R}}) \subseteq A_q^g(L_p, \widetilde{\mathcal{R}}), \qquad \forall g < a. \tag{1.10}$$

For $1 < p < \infty$ and any $b > 0$ we have

$$A_q^a\left(L_p, \Sigma_r, nest, b\right) \subseteq A_q^a\left(L_p, \widetilde{\mathcal{R}}\right), \tag{1.11}$$

$$A_q^a\left(L_p, \Sigma_r, b\right) \subseteq A_{q,\log}^a\left(L_p, \widetilde{\mathcal{R}}\right) \subseteq A_q^g\left(L_p, \widetilde{\mathcal{R}}\right), \qquad \forall g < a \; . \tag{1.12}$$

Example 1.7 Let $f(x) = \mathbf{1}_\Omega(x)$ with $\Omega = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\right\}$. By using the rational functions

$$R_{2^n}(x_1, x_2) = Q_{2^n,u}\left(x_1^2 + x_2^2\right), \quad n \ge 3,$$

where $Q_{2^n,u}$ is the rational function of Lemma 3.1 below with parameters $k = 2^n, u = \max(1, 1/p)$, one can verify that $f \in A_q^a\left(L_p, \widetilde{\mathcal{R}}\right)$ for all $0 < p, q < \infty$, $a > 0$. On the other, it is plausible that a near-best approximation of the disk using 2^n triangles is obtained by triangulating the regular $2^n + 2-$gon inscribed in the unit circle. Actually, as one of the referees pointed out, a slightly better approximation is obtained by triangulating a larger $2^n + 2-$gon with a side length of $4\Big/\sqrt{1 + 4\cot^2\left(p / \left(2^n + 2\right)\right)}$. In any case, we believe that there exists a constant $C_1 > 0$ such that

$$s_{2^n,r}(f)_p \ge C_1\left(p - \left(2^{n-1} + 1\right)\sin\left(p \Big/ \left(2^{n-1} + 1\right)\right)\right)^{1/p}, \qquad n \ge 2, \; r \ge 0.$$

Since there also exists $C_2 > 0$ such that

$$p - \left(2^{n-1} + 1\right)\sin\left(p \Big/ \left(2^{n-1} + 1\right)\right) \ge C_2 2^{-2n},$$

we have that $f \notin A_q^a\left(L_p, \Sigma_r\right)$ whenever $a > 2/p$ for any $0 < q \le \infty$. In the case $p = 2$, we have the Curvelets [CD] that are designed to achieve the same performance as piecewise polynomials, and indeed we have $f \notin A_q^a\left(L_2, Curvelets\right)$ whenever $a > 1$. For Wavelets that, as we know, do not perform as well in the multivariate case, we have (see Section 3 in [CD] or Section 7.7 in [C]) $f \notin A_q^a\left(L_2, Wavelets\right)$ whenever $a > 1/2$. The spaces $A_q^a\left(L_2, Curvelets\right)$ and $A_q^a\left(L_2, Wavelets\right)$ are the analogues of the above approximation spaces for nonlinear Curvelet approximation and nonlinear Wavelet approximation, respectively.

Example 1.7 is typical for indicator functions of domains whose boundary is a simple (i.e. not self-intersecting) algebraic curve. While rational approximation can exploit the implicit representation of the curve, piecewise polynomial approximation over triangles requires many triangles near the domain's boundary.

2 Preliminaries

2.1 Some triangle geometry

For a triangle $\Delta \subset \mathbb{R}^2$ and $m \ge 1$, we construct a similar triangle Δ^m that contains Δ as follows. Without loss of generality the center of the inscribed circle of Δ is at the origin. Denote by v_i, $1 \le i \le 3$ the vectors of the vertices of Δ. Then Δ^m is defined as the triangle with vertices mv_i. The following properties of Δ^m can be easily verified using basic vector calculus:

(i) The sides of Δ^m are parallel to the sides of Δ.

(ii) The lengths of the heights of Δ^m are m times the lengths of the heights of Δ.

(iii) We have $\left|\Delta^m\right| = m^2 \left|\Delta\right|$, where for any domain in $\Omega \subset \mathbb{R}^2$, $\left|\Omega\right|$ denotes the area of the domain.

We denote by $\Delta(b)$ the set of triangles in $\mathbb{R}^2$ whose minimal angle is $\geq b$. Let $\Delta \in \Delta(b)$ and let $Cir(\Delta)$ be the circumscribed disk of Δ with radius R. Then the following equivalence of areas holds

$$\left|\Delta\right| < \left|Cir(\Delta)\right| \leq C(b)\left|\Delta\right|. \qquad (2.1)$$

The left hand side inequality is obvious since $\Delta \subset Cir(\Delta)$. Let e_i, b_i, $1 \leq i \leq 3$ be the sides and angles of Δ with the angle b_i opposite to the side e_i. Then,

$$R = \frac{e_i}{2\sin(b_i)} \leq \frac{e_i}{2\sin(b)}.$$

Therefore

$$\left|Cir(\Delta)\right| = p R^2 \leq \frac{p e_1 e_2 e_3}{8\sin^3(b) R} = C(b)\frac{e_1 e_2 e_3}{4R} = C(b)\left|\Delta\right|,$$

which is the right hand side of (2.1).
The following is a Vitali-type covering lemma for constrained triangles.

Lemma 2.1 Let Ω be an arbitrary measurable subset of $\mathbb{R}^2$ of finite measure. Let $\mathcal{F}$ be a subset of $\Delta(b)$ for some $b > 0$ that covers Ω. Then there exist finitely many disjoint triangles $\{\Delta_i\}_{i=1}^N$ from $\mathcal{F}$ such that

$$\sum_{i=1}^N \left|\Delta_i\right| \geq C(b)\left|\Omega\right|. \qquad (2.2)$$

Proof The proof essentially follows the proof of Lemma 3.3.2 in [BS]. As in [BS] we may assume without loss of generality that Ω is compact and therefore that $\mathcal{F}$ is finite. We select Δ_1 as the triangle in $\mathcal{F}$ with the largest circumscribed disk. We continue and select inductively the triangle Δ_i as the triangle in $\mathcal{F}$ with the largest circumscribed disk that is disjoint from all the previously selected triangles Δ_j, $1 \leq j < i$. Since $\mathcal{F}$ is finite this process ends after finitely many steps yielding disjoint triangles $\{\Delta_i\}_{i=1}^N$. For $1 \leq i \leq N$, let $\widetilde{C}_i$ be the disk concentric with $Cir(\Delta_i)$ with a radius that is 3 times bigger than the radius of $Cir(\Delta_i)$. We claim that the disks $\{\widetilde{C}_i\}_{i=1}^N$ cover Ω. If not, there exist $x \in \Omega \setminus \bigcup_{i=1}^N \widetilde{C}_i$ and $\Delta \in \mathcal{F}$ such that $x \in \Delta$. From our construction $Cir(\Delta)$ is not bigger than $Cir(\Delta_1)$. Therefore, $x \notin \widetilde{C}_1$ implies that $Cir(\Delta)$ and $Cir(\Delta_1)$ are disjoint and thus so are Δ, Δ_1. Continuing this way we see that Δ is disjoint from all $\{\Delta_i\}_{i=1}^N$ which is impossible because then the process would not have ended after N steps. Therefore $\{\widetilde{C}_i\}_{i=1}^N$ cover Ω and by (2.1)

$$\left|\Omega\right| \leq \left|\bigcup_{i=1}^N \widetilde{C}_i\right| \leq \sum_{i=1}^N \left|\widetilde{C}_i\right| = 9\sum_{i=1}^N \left|Cir(\Delta_i)\right| \leq C(b)\sum_{i=1}^N \left|\Delta_i\right|.$$

♦

2.2 A Maximal function over constrained triangles

Definition 2.2 Let $0 < b < p/4$. For a locally integrable function f we define the Maximal function $M_b f$ by

$$M_b f(x) = \sup_{x \in \Delta \in \Delta(b)} \frac{1}{|\Delta|} \int_\Delta |f(y)| dy .$$

Lemma 2.3 Let $\Delta_0, \Delta_1 \in \Delta(b)$ for some $b > 0$ with $\Delta_0 \subseteq \Delta_1$. Then for any $f \in L_1(\Delta_1)$

$$\frac{1}{|\Delta_1|} \int_{\Delta_1} |f(x)| dx \le \frac{1}{|\Delta_0|} \int_{\Delta_0} M_b f(x) dx . \tag{2.3}$$

Proof Let $x \in \Delta_0$. Then, since $\Delta_0 \subseteq \Delta_1$

$$\frac{1}{|\Delta_1|} \int_{\Delta_1} |f(y)| dy \le M_b f(x) . \tag{2.4}$$

Now, take the average over Δ_0 of both sides of (2.4) to obtain (2.3).

♦

The minimal angle constraint comes into play in the proof of the following variant of the Hardy-Littlewood maximal inequality.

Theorem 2.4 Let $f \in L_p(\mathbb{R}^2)$, $1 < p \le \infty$. Then

$$\|M_b f\|_p \le C(p, b) \|f\|_p . \tag{2.5}$$

Sketch of proof For $p = \infty$ the proof is obvious. For $1 < p < \infty$, the proof follows Section 3.3 in [BS]. We recall that for a locally integrable f the Hardy-Littlewood maximal function Mf is defined over cubes

$$Mf(x) := \sup_{\substack{x \in Q \\ Q\,cube}} \frac{1}{|Q|} \int_Q |f(y)| dy .$$

The multivariate Hardy-Littlewood inequality establishes the L_p boundedness of the maximal operator for $1 < p < \infty$

$$\|Mf\|_{L_p(\mathbb{R}^d)} \le C(p, d) \|f\|_{L_p(\mathbb{R}^d)} .$$

The only difference in the proof of our inequality is that we use Lemma 2.1, that is, a variant of the Vitali covering lemma for cubes (Lemma 3.3.2 in [BS]). We note that the constraint we place on the triangles is important just as the choice of cubes (equal side lengths!) is crucial for the original Hardy-Littlewood maximal inequality. Indeed, the choice of constrained triangles leads to the weak type inequality

$$\left| x \in \mathbb{R}^2 : M_b f(x) > l \right| \le C \frac{\|f\|_1}{l} ,$$

which implies that the maximal function is a bounded operator from L_1 to weak-L_1.

♦

Remark If the cubes are replaced by rectangles, one then obtains the so-called strong maximal inequality

$$\left|x \in \mathbb{R}^2 : Mf(x) > l\right| \le C \int_{\mathbb{R}^2} \frac{|f(x)|}{l} \left(1 + \log^+\left(\frac{|f(x)|}{l}\right)\right) dx,$$

which implies that the maximal function over rectangles is a bounded operator from the smaller Orlicz space $L(1+\log^+ L)$ to weak L_1 (see e.g. [CF]).

2.3 On integration of distance gauges

Let $\Omega \subset \mathbb{R}^2$ be a convex compact polyhedral domain with edges e_i, $1 \le i \le n_e$. We denote by $\square_i$ the infinite strip defined by e_i and a parallel edge going through the vertex of Ω which is the farthest from e_i. Denoting $l_i := width(\square_i)$ and $d_i(x) := d(x, \square_i)$ we define

$$q(x,\Omega) := \prod_{i=1}^{n_e} q_i(x,\Omega) := \prod_{i=1}^{n_e} \frac{l_i}{d_i(x) + l_i}. \tag{2.6}$$

Lemma 2.5 Let Δ be a triangle and $q > 1$. Then

$$\int_{\mathbb{R}^2} q(x,\Delta)^q \, dx \le C(q)|\Delta|. \tag{2.7}$$

Proof The three strips associated with a given triangle subdivide the plane into sub-domains, four of which are Δ itself and three identical copies.

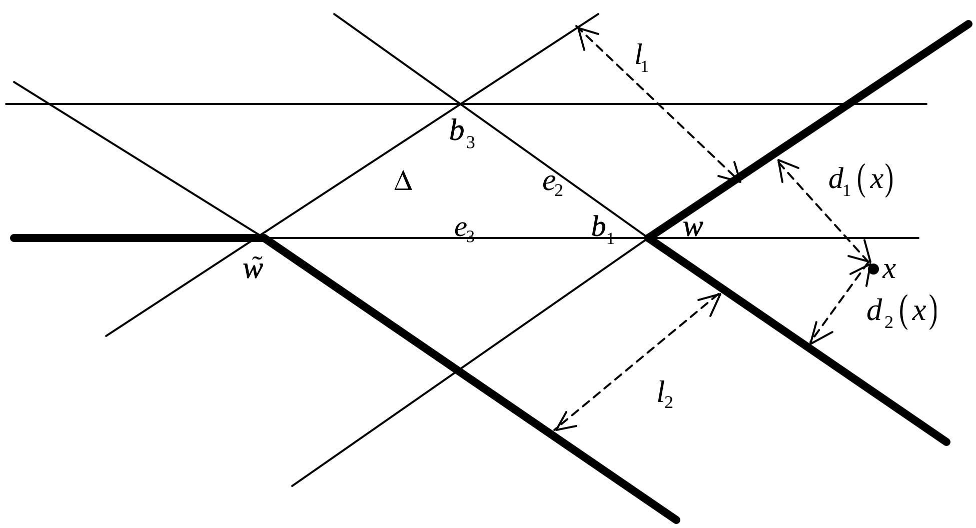

Figure 2-1 **Two types of infinite rays**

Since $q(x,\Delta) \le 1$, we need to prove (2.7) only outside of these four triangles. The rest of the plane is covered by six (overlapping) sectors each defined by the boundaries of two different strips and enclosing a half-strip of the third one. Hence, it suffices to prove (2.7) separately for each of these six sectors. Figure 2-1 illustrates two typical sectors (bounded by the bold lines). The one on the right has a head-angle

$0 < w = p - b_3 \le p/2$, while the one on the left has a head-angle $\tilde{w} = p - b_1 > p/2$. We now elaborate, using the right sector for illustration, on how we estimate the integral in a sector with a head-angle $\le p/2$, and then explain how the other case can be treated similarly.

Thus, assume that the sector's head-angle is $0 < w \le p/2$. By shifting the head of the sector to the origin and rotating so that one of the boundaries coincides with the x-axis we get,

$$\int_{\text{sector}} q(x,\Delta)^q \, dx \le (l_1 l_2)^q \int_{\text{sector}} \frac{dx}{\left(d_1(x)+l_1\right)^q \left(d_2(x)+l_2\right)^q}$$

$$= (l_1 l_2)^q \int_0^\infty dx_1 \int_0^{x_1 \, tg(w)} \frac{dx_2}{\left(x_1 \sin(w) - x_2 \cos(w) + l_1\right)^q \left(x_2 + l_2\right)^q}.$$

A change of variables

$$u = x_1 \sin(w) - x_2 \cos(w), \qquad v = x_2,$$

yields the Jacobian

$$J = \left| \frac{\partial(x_1, x_2)}{\partial(u,v)} \right| = \begin{vmatrix} 1/\sin(w) & \cot(w) \\ 0 & 1 \end{vmatrix} = \frac{1}{\sin(w)},$$

and we have

$$\int_{\text{sector}} q(x,\Delta)^q \, dx \le \frac{(l_1 l_2)^q}{\sin(w)} \int_0^\infty \int_0^\infty \frac{du\,dv}{(u+l_1)^q (v+l_2)^q}$$

$$= \frac{(l_1 l_2)^q}{\sin(w)} \int_0^\infty \frac{du}{(u+l_1)^q} \int_0^\infty \frac{dv}{(v+l_2)^q}$$

$$\le C(q) \frac{l_1 l_2}{\sin(w)}$$

$$= C(q) \frac{e_2 l_2}{2} = C(q)|\Delta|.$$

As for the case $p/2 < \tilde{w} < p$ we again illustrate the proof, this time on the left sector. We first place the sector at the origin and then divide it into two smaller sectors. One is the upper left quadrant of the plane and the other the remainder with a head-angle of $\tilde{w} - p/2 < p/2$. We then proceed with computation similar to the above to estimate the integral over each of the two smaller sectors.

♦

Lemma 2.6 Let $\Delta \in \Delta(b)$ and let $q > 2$. If $j \ge 0$ is locally integrable then

$$\int_{\mathbb{R}^2} j(x) q(x,\Delta)^q \, dx \le C(q,b) \int_\Delta M_b j(x) \, dx. \tag{2.8}$$

Proof We define $\Delta^{(0)} = \Delta$ and construct similar triangles $\Delta^{(k)} := \Delta^{2^k}$, $k \ge 1$ (see Section 2.1). These triangles are associated with the scale 2^k and satisfy

(i) $\Delta^{(k)} \subset \Delta^{(k+1)}$,

(ii) $\left|\Delta^{(k)}\right| = 2^{2k} |\Delta|$,

(iii) The heights of $\Delta^{(k)}$ are 2^k times the heights of Δ.

For $x \in \Delta^{(0)}$ we clearly have $q(x,\Delta)=1$. Let $x \in \Delta^{(k)} \setminus \Delta^{(k-1)}$ for $k \geq 1$. By a geometric argument, there exists $i(x)$, $1 \leq i(x) \leq 3$ such that $d_{i(x)}(x)/l_{i(x)} \geq \frac{1}{3}\left(2^{k-1}-1\right)\sin(b)$. Therefore

$$\begin{aligned} q_{i(x)}(x,\Delta) &= \frac{l_{i(x)}}{d_{i(x)}(x)+l_{i(x)}} \\ &\leq C(b)2^{-k}. \end{aligned}$$

so that for $x \in \Delta^{(k)} \setminus \Delta^{(k-1)}$

$$q(x,\Delta)^q \leq q_{i(x)}(x,\Delta)^q \leq C\,2^{-qk} \leq C\sum_{j=k}^{\infty} 2^{-qj} = C\sum_{j=0}^{\infty} 2^{-qj}\mathbf{1}_{\Delta^{(j)}}(x).$$

Observe that $\Delta^{(j)} \in \Delta(b)$. Now (2.3) yields

$$\begin{aligned} \int_{\mathbb{R}^2} j(x) q(x,\Delta)^q\,dx &\leq C\sum_{j=0}^{\infty} 2^{-qj}\int_{\Delta^{(j)}} j(x)\,dx \\ &= C|\Delta|\sum_{j=0}^{\infty} 2^{-(q-2)j}\frac{1}{\left|\Delta^{(j)}\right|}\int_{\Delta^{(j)}} j(x)\,dx \\ &\leq C|\Delta|\sum_{j=0}^{\infty} 2^{-(q-2)j}\frac{1}{|\Delta|}\int_{\Delta} M_b j(x)\,dx \\ &\leq C(q,b)\int_{\Delta} M_b j(x)\,dx, \end{aligned}$$

where we used that $q>2$.

♦

Lemma 2.7 Let Δ_j be triangles and $a_j \geq 0$ for $j=1,\ldots,m$.

(i) For $0<p\leq 1$ and $u>1/p$ we have

$$\int_{\mathbb{R}^2}\left(\sum_{j=1}^{m} a_j q(x,\Delta_j)^u\right)^p dx \leq C(p,u)\sum_{j=1}^{m} a_j^p\left|\Delta_j\right|. \tag{2.9}$$

(ii) Let $1<p<\infty$ and $u>2$. Assume that Δ_j are disjoint and $\Delta_j \in \Delta(b)$, $j=1,\ldots,m$, for some $b>0$. Then

$$\int_{\mathbb{R}^2}\left(\sum_{j=1}^{m} a_j q(x,\Delta_j)^u\right)^p dx \leq C(p,u,b)\sum_{j=1}^{m} a_j^p\left|\Delta_j\right|. \tag{2.10}$$

Proof

(i) If $0<p\leq 1$, then

$$\int_{\mathbb{R}^2}\left(\sum_{j=1}^{m} a_j q(x,\Delta_j)^u\right)^p dx \leq \sum_{j=1}^{m} a_j^p\int_{\mathbb{R}^2} q(x,\Delta_j)^{up}\,dx,$$

and since $up>1$, (2.9) follows from (2.7).

(ii) Assume $1<p<\infty$. Then there exists $j \in L_{p'}(\mathbb{R}^2)$, with p' being the dual index of p such that $j \geq 0$, $\|j\|_{L_{p'}(\mathbb{R}^2)}=1$ and

$$\left\| \sum_{j=1}^{m} a_j q\left(x,\Delta_j\right)^u \right\|_{L_p\left(\mathbb{R}^2\right)} = \int_{\mathbb{R}^2} j\,(x) \sum_{j=1}^{m} a_j q\left(x,\Delta_j\right)^u dx .$$

By (2.5) we have $\left\| M_b j \right\|_{p'} \le C\left(p', b\right) := C\left(p, b\right)$. Lemma 2.6 and Holder's continuous and discrete inequalities then yield

$$\begin{aligned} \left\| \sum_{j=1}^{m} a_j q\left(x,\Delta_j\right)^u \right\|_p &\le C \sum_{j=1}^{m} a_j \int_{\Delta_j} M_b j\,(x) dx \\ &\le C \sum_{j=1}^{m} a_j \left|\Delta_j\right|^{1/p} \left\| M_b j \right\|_{L_{p'}\left(\Delta_j\right)} \\ &\le C \left(\sum_{j=1}^{m} a_j^p \left|\Delta_j\right| \right)^{1/p} \left(\sum_{j=1}^{m} \left\| M_b j \right\|_{L_{p'}\left(\Delta_j\right)}^{p'} \right)^{1/p'} \\ &\le C \left(\sum_{j=1}^{m} a_j^p \left|\Delta_j\right| \right)^{1/p} . \end{aligned}$$

♦

Remark We do not know whether the dependence of the constant in (2.10) on the minimal angle is essential. If not, then clearly the results of this paper become independent of the "thinness" of triangles and (1.9) and (1.10) hold for all $0 < p < \infty$.

2.4 Polynomial approximation over triangles

The following two results follow from [KP] Lemmas 2.6 and 2.7.

Lemma 2.8 Let $P \in \Pi_r\left(\mathbb{R}^2\right)$ and $\Delta_0 \subseteq \Delta_1$ such that $\left|\Delta_1\right| \le m\left|\Delta_0\right|$. Then for $0 < p \le \infty$

$$\left\| P \right\|_{L_p(\Delta_1)} \le C(p, r)\, m^{r+1/p} \left\| P \right\|_{L_p(\Delta_0)} .$$

Proof The method of proof is similar to the method used to prove [KP] Lemma 2.6, except that here the dependence on m is made explicit.

♦

Lemma 2.9 Let $P \in \Pi_r\left(\mathbb{R}^2\right)$ and $0 < p, q \le \infty$. Then, for any triangle $\Delta \subset \mathbb{R}^2$

$$\left\| P \right\|_{L_q(\Delta)} \le C\left(p, q, r\right) \left|\Delta\right|^{1/q-1/p} \left\| P \right\|_{L_p(\Delta)} \tag{2.11}$$

We need an estimate on the rate of increase of a polynomial as we move away from a given triangle. We prove

Lemma 2.10 Let $x \in \mathbb{R}^2$, $\Delta_0 \subset \mathbb{R}^2$ and $P \in \Pi_r\left(\mathbb{R}^2\right)$. Then,

$$\left| P\left(x\right) \right| \le C(r)\, q\left(x, \Delta_0\right)^{-2r} \left\| P \right\|_{L_\infty(\Delta_0)} . \tag{2.12}$$

Proof If $x \in \Delta_0$ then clearly $q(x,\Delta_0) = 1$ and $|P(x)| \le \|P\|_{L_\infty(\Delta_0)}$ so that (2.12) holds. Thus, we may assume that $x \notin \Delta_0$. There are two cases:

Case I: The point x is inside a sector defined by two edges, say e_1, e_2. The triangle Δ_1 that is defined by the intersection of e_1, e_2 with a line going through x and parallel to e_3 as in Figure 2-2, contains Δ_0 and is similar to it.

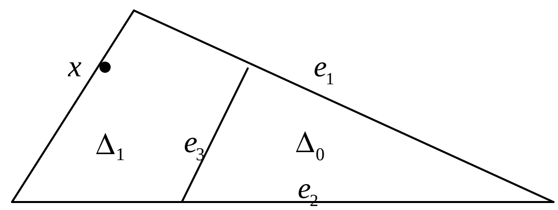

Figure 2-2 Construction of an including triangle, case I

Hence

$$|\Delta_1| = \left(\frac{d_3(x) + l_3}{l_3} \right)^2 |\Delta_0| = q_3(x,\Delta_0)^{-2} |\Delta_0|$$
$$\le q(x,\Delta_0)^{-2} |\Delta_0|,$$

and by Lemma 2.8

$$|P(x)| \le \|P\|_{L_\infty(\Delta_1)} \le C(r) q(x,\Delta_0)^{-2r} \|P\|_{L_\infty(\Delta_0)}.$$

Case II: The point x is in a wedge defined by two edges, say e_1, e_2. In this case let Δ_1 be defined, as in Figure 2-3, by the vertices v_1, v_2, x, where v_1, v_2 are the vertices opposite to e_1, e_2, respectively.

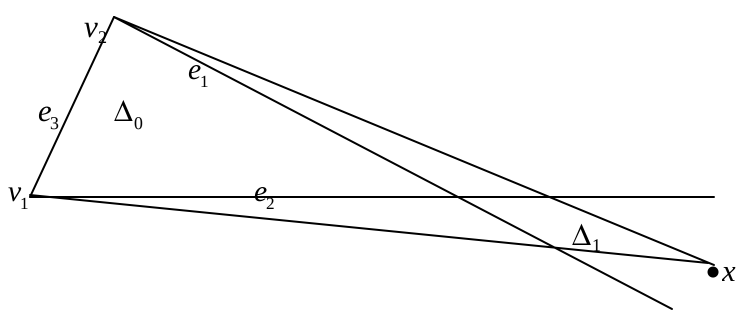

Figure 2-3 Construction of an including triangle, case II

It follows that

$$|\Delta_0| = \frac{|e_3| l_3}{2}, \quad |\Delta_1| = \frac{|e_3|(d_3(x) + l_3)}{2}.$$

Therefore

$$|\Delta_1| = \frac{d_3(x) + l_3}{l_3} |\Delta_0| = q_3(x,\Delta_0)^{-1} |\Delta_0|.$$

Again Lemma 2.8 yields

$$\left|P(x)\right| \le \|P\|_{L_\infty(\Delta_1)} \le C(r) q_3(x,\Delta_0)^{-r} \|P\|_{L_\infty(\Delta_0)}$$
$$\le C(r) q(x,\Delta_0)^{-2r} \|P\|_{L_\infty(\Delta_0)},$$

and (2.12) is proved.

♦

3 Main results

At the core of our proof lies the ability of rational functions of low degree to approximate well characteristic functions. We begin by quoting [LGM], Lemma 10.6.5 (or [PP] Lemma 8.3).

Lemma 3.1 (Rational approximation of a characteristic function) For $k \ge 5$ and $u \ge 1$ there exists a univariate rational function $Q \in \mathcal{R}_{4k+2u}$ such that

$$0 \le Q(t) \le 1, \qquad \forall t \in \mathbb{R},$$

$$1 - Q(t) \le e^{-2\sqrt{k}}, \qquad |t| \le 1 - 2e^{-\sqrt{k}},$$

$$Q(t) \le |t|^{-2u} e^{-2\sqrt{k}}, \qquad |t| \ge 1.$$

Let $\Omega \subset \mathbb{R}^2$ be a convex compact polyhedral domain with edges e_i, $1 \le i \le n_e$. We would like to construct a rational function that approximates well the indicator function of Ω away from the domain's boundary. To this end we construct for each strip $\square_i$ associated with e_i (see Section 2.3), using a rotation and a shift, an operator $T_i : \mathbb{R}^2 \to \mathbb{R}$, $T_i(x_1,x_2) = a_i x_1 + b_i x_2 + c_i$, such that $T_i(\square_i) = [0, l_i]$ and $d_i(x) := d(x,\square_i) = d(T_i(x),[0,l_i])$.

Lemma 3.2 Let $\Omega \subset \mathbb{R}^2$ be a convex compact polyhedral domain with n_e edges and let $q(x,\Omega)$ be defined in (2.6). For $k \ge 5$ we denote by $\Omega_k \subset \Omega$ the "inner" boundary

$$\Omega_k := \left\{ y \in \Omega : \exists i, 1 \le i \le n_e,\ 0 \le T_i(y) \le e^{-\sqrt{k}} l_i \text{ or } 0 \le l_i - T_i(y) \le e^{-\sqrt{k}} l_i \right\}. \tag{3.1}$$

Then, for any $u \ge 1$ there exists a bivariate rational function $R \in \widetilde{\mathcal{R}}_{4n_e n_e(4k+2u)}$ such that $0 \le R(x) \le 1$ and for $x \notin \Omega_k$

$$\left| R(x) - \mathbf{1}_\Omega(x) \right| \le n_e e^{-2\sqrt{k}} q(x,\Omega)^{2u}.$$

Proof We set

$$R(x) = \prod_{i=1}^{n_e} Q\left(\frac{2T_i(x) - l_i}{l_i} \right), \tag{3.2}$$

where Q is from Lemma 3.1.

(i) Let $x \in \Omega \setminus \Omega_k$. Then $q(x,\Omega) = 1$ and by Lemma 3.1

$$\mathbf{1}_\Omega(x) - R(x) \le 1 - \left(1 - e^{-2\sqrt{k}}\right)^{n_e}$$
$$\le n_e e^{-2\sqrt{k}} q(x,\Omega)^{2u}.$$

(ii) Let $x \notin \Omega$. Then

$$\begin{aligned}\left|\mathbf{1}_\Omega(x)-R(x)\right| = \left|R(x)\right| &= \prod_{i=1}^{n_e} Q\left(\frac{2T_i(x)-l_i}{l_i}\right) \\ &= \prod_{0\le T_i(x)\le l_i} Q\left(\frac{2T_i(x)-l_i}{l_i}\right) \prod_{T_i(x)\notin[0,l_i]} Q\left(\frac{2T_i(x)-l_i}{l_i}\right) \\ &\le e^{-2\sqrt{k}} \prod_{T_i(x)\notin[0,l_i]} \left|\frac{l_i}{2T_i(x)-l_i}\right|^{2u} \\ &\le e^{-2\sqrt{k}} \prod_{T_i(x)\notin[0,l_i]} \left|\frac{l_i}{d_i(x)+l_i}\right|^{2u} \\ &= e^{-2\sqrt{k}} \prod_{i=1}^{n_e} \left|\frac{l_i}{d_i(x)+l_i}\right|^{2u} \\ &= e^{-2\sqrt{k}} q(x,\Omega)^{2u}.\end{aligned}$$

♦

The following is a bivariate analogue of Lemma 10.6.4 in [LGM].

Lemma 3.3 Let $\varphi \in S_m^r(\mathbb{R}^2)$ for $m \ge 1$ and $r \ge 0$.

(i) If $0 < p \le 1$, then for each $n \ge 1$ there exists $R \in \widetilde{\mathcal{R}}_{Cm,n}$ such that

$$\|\varphi - R\|_{L_p(\mathbb{R}^2)} \le C_1 \exp\left(-C_2\sqrt{n/m}\right)\|\varphi\|_{L_p(\mathbb{R}^2)}, \tag{3.3}$$

where the constants C_1, C_2 depend on p, r.

(ii) If $1 < p < \infty$ and $\varphi \in S_m^{r,b}(\mathbb{R}^2)$, $b > 0$, then for each $n \ge 1$ there exists $R \in \widetilde{\mathcal{R}}_{Cm,n}$ such that

$$\|\varphi - R\|_{L_p(\mathbb{R}^2)} \le C_1 \exp\left(-C_2\sqrt{n/m}\right)\|\varphi\|_{L_p(\mathbb{R}^2)}, \tag{3.4}$$

where the constants C_1, C_2 depend on p, r, b.

Proof Let $\varphi = \sum_{j=1}^m \mathbf{1}_{\Delta_j} P_j$, where $P_j \in \Pi_r(\mathbb{R}^2)$. Let $l_{j,i}$ be the $i-$th height of the triangle Δ_j and let $T_{j,i}$ be the linear transformation corresponding to the $i-$th edge of Δ_j. We construct

$$R(x) = \sum_{j=1}^m R_j(x) P_j(x),$$

with

$$R_j(x) := \prod_{i=1}^3 Q\left(\frac{2T_{j,i}(x)-l_{j,i}}{l_{j,i}}\right).$$

We take

$$u > r + \max(1, 1/2p), \quad n > \left(6(4k+2u)+r^2\right)m, \quad k \ge 5, \tag{3.5}$$

where we first fix u and then take k to be the biggest possible integer satisfying (3.5). This is possible if $n \ge Cm$ for a sufficiently large $C(p,r)$. Observe that the number of free parameters in $R(x)$ is $\le (r^2+12)m$ so that $R \in \widetilde{\mathcal{R}}_{Cm,n}$. We shall assume that $n \ge Cm$, otherwise there is nothing to prove. For each $1 \le j \le m$ let $\Omega_{j,k}$ be the "inner" boundary of Δ_j defined by (3.1). Note that $|\Omega_{j,k}| \le 6e^{-\sqrt{k}} |\Delta_j|$. Denote $E_k := \bigcup_{j=1}^m \Omega_{j,k}$. First we estimate the error away from the inner boundaries of the triangles. Let $x \in \mathbb{R}^2 \setminus E_k$. By Lemma 3.2 for all $1 \le j \le m$

$$\left| R_j(x) - \mathbf{1}_{\Delta_j}(x) \right| \le 3e^{-2\sqrt{k}} q(x,\Delta_j)^{2u}. \tag{3.6}$$

Therefore, by Lemma 2.10

$$\begin{aligned} |j(x) - R(x)| &\le \sum_{j=1}^m \left| R_j(x) - \mathbf{1}_{\Delta_j}(x) \right| |P_j(x)| \\ &\le 3e^{-2\sqrt{k}} \sum_{j=1}^m q(x,\Delta_j)^{2u} |P_j(x)| \\ &\le 3e^{-2\sqrt{k}} \sum_{j=1}^m q(x,\Delta_j)^{2(u-r)} \|P_j\|_{L_\infty(\Delta_j)}. \end{aligned}$$

Since by (3.5) $u > r + \max(1, 1/2p)$ we may apply Lemma 2.7 so that

$$\begin{aligned} \|j - R\|^p_{L_p(\mathbb{R}^2 \setminus E_k)} &\le Ce^{-2p\sqrt{k}} \int_{\mathbb{R}^2} \left(\sum_{j=1}^m \|P_j\|_{L_\infty(\Delta_j)} q(x,\Delta_j)^{2(u-r)} \right)^p dx \\ &\le Ce^{-2p\sqrt{k}} \sum_{j=1}^m \|P_j\|^p_{L_\infty(\Delta_j)} |\Delta_j|, \end{aligned}$$

where for $0 < p \le 1$ we applied (2.9) and $C = C(p,r)$ and for $1 < p < \infty$ we applied (2.10) and $C = C(p,r,b)$. By Lemma 2.9 we conclude that

$$\begin{aligned} e^{-2p\sqrt{k}} \sum_{j=1}^m \|P_j\|^p_{L_\infty(\Delta_j)} |\Delta_j| &\le Ce^{-2p\sqrt{k}} \sum_{j=1}^m \|P_j\|^p_{L_p(\Delta_j)} \\ &= Ce^{-2p\sqrt{k}} \|j\|_p^p. \end{aligned}$$

We now estimate on the "inner" boundaries of the triangles. Let $x \in E_k$. Then there exists $1 \le s \le m$ such that $x \in \Omega_{s,k}$ and $x \notin \Omega_{j,k}$, $j \ne s$. Again by (3.6) and Lemma 2.10 we have

$$\begin{aligned} |j(x) - R(x)| &\le \sum_{j=1}^m \left| \mathbf{1}_{\Delta_j}(x) - R_j(x) \right| |P_j(x)| \\ &\le |P_s(x)| + \sum_{j \ne s} |R_j(x)| |P_j(x)| \end{aligned}$$

$$\leq \|P_s\|_{L_\infty(\Delta_s)} + 3e^{-2\sqrt{k}} \sum_{j\neq s} q\left(x,\Delta_j\right)^{2(u-r)} \|P_j\|_{L_\infty(\Delta_j)}$$

$$\leq \sum_{j=1}^{m} \|P_j\|_{L_\infty(\Delta_j)} \mathbf{1}_{\Omega_{j,k}}(x) + 3e^{-2\sqrt{k}} \sum_{j=1}^{m} q\left(x,\Delta_j\right)^{2(u-r)} \|P_j\|_{L_\infty(\Delta_j)}.$$

We can now estimate $\|\varphi - R\|_{L_p(E_k)}$ using the same techniques we applied in the first part of the proof for the estimate over the domain $\mathbb{R}^2 \setminus E_k$. For $1 < p < \infty$ we get

$$\begin{aligned}\|\varphi(x) - R(x)\|_{L_p(E_k)} &\leq \left\| \sum_{j=1}^{m} \|P_j\|_{L_\infty(\Delta_j)} \mathbf{1}_{\Omega_{j,k}}(x) \right\|_{L_p(E_k)} + 3e^{-2\sqrt{k}} \left\| \sum_{j=1}^{m} q\left(x,\Delta_j\right)^{2(u-r)} \|P_j\|_{L_\infty(\Delta_j)} \right\|_{L_p(E_k)} \\ &\leq \left(\sum_{j=1}^{m} \|P_j\|_{L_\infty(\Delta_j)}^p |\Omega_{j,k}| \right)^{1/p} + 3e^{-2\sqrt{k}} \left\| \sum_{j=1}^{m} q\left(x,\Delta_j\right)^{2(u-r)} \|P_j\|_{L_\infty(\Delta_j)} \right\|_{L_p(\mathbb{R}^2)} \\ &\leq C \left(e^{-\sqrt{k}} \sum_{j=1}^{m} \|P_j\|_{L_\infty(\Delta_j)}^p |\Delta_j| \right)^{1/p} + Ce^{-2\sqrt{k}} \|\varphi\|_p \\ &\leq Ce^{-C\sqrt{k}} \|\varphi\|_p,\end{aligned}$$

where $C = C(p,r,b)$. If on the other hand $0 < p \leq 1$, then

$$\begin{aligned}\int_{E_k} |\varphi(x) - R(x)|^p dx &\leq \sum_{j=1}^{m} \|P_j\|_{L_\infty(\Delta_j)}^p |\Omega_{j,k}| + Ce^{-2p\sqrt{k}} \int_{E_k} \left(\sum_{j=1}^{m} q\left(x,\Delta_j\right)^{2(u-r)} \|P_j\|_{L_\infty(\Delta_j)} \right)^p dx \\ &\leq Ce^{-\sqrt{k}} \sum_{j=1}^{m} \|P_j\|_{L_\infty(\Delta_j)}^p |\Delta_j| + Ce^{-2p\sqrt{k}} \sum_{j=1}^{m} \|P_j\|_{L_\infty(\Delta_j)}^p |\Delta_j| \\ &\leq Ce^{-p\sqrt{k}} \sum_{j=1}^{m} \|P_j\|_{L_\infty(\Delta_j)}^p |\Delta_j| \\ &\leq Ce^{-p\sqrt{k}} \|\varphi\|_p^p,\end{aligned}$$

with $C = C(p,r)$. Since k was chosen as the biggest possible integer satisfying (3.5), we have that $k > Cn/m$ and the proof is completed by adding the bounds over the domains E_k and $\mathbb{R}^2 \setminus E_k$.

♦

Proof of Theorem 1.2 We assume $1 < p < \infty$ and prove (1.5). Let $\Lambda = \{\Lambda_n\}_{n\geq 0}$ be an almost nested sequence (see Definition 1.1) with $\inf_{\Delta\in\Lambda}(minangle(\Delta)) = b > 0$. For each $k \geq 0$ there exists $f_k \in S^r(\Lambda_k)$ such that $E_{2^k,r}(f,\Lambda)_p = \|f - f_k\|_p$. We let $\varphi_0 = f_0$ and for $k \geq 1$ we set $\varphi_k = f_k - f_{k-1}$. Then, we have

$$\|\varphi_0\|_p \leq E_{1,r}(f,\Lambda)_p + \|f\|_p,$$

and

$$\|\varphi_k\|_p \leq \|f - f_k\|_p + \|f - f_{k-1}\|_p \leq E_{2^{k-1},r}(f,\Lambda)_p + E_{2^k,r}(f,\Lambda)_p, \quad k \geq 1.$$

Furthermore, since Λ is an almost nested sequence, j_k is in $S^{r,b}_{C_\Lambda 2^k}(\mathbb{R}^2)$. We now fix j and for $0 \le k \le j$ we approximate j_k by the rational function $R_k \in \widetilde{\mathcal{R}}_{n(k)}$, where $n(k) = \lceil C_2^{-2} C_\Lambda 2^k \log^2 (2^{a(j-k)}) \rceil$ that is guaranteed by Lemma 3.3. By virtue of (3.4)

$$\|j_k - R_k\|_p \le C_1 \exp\left(-C_2 \sqrt{\frac{C_2^{-2} C_\Lambda 2^k \ \log^2 (2^{a(j-k)})}{C_\Lambda 2^k}}\right) \|j_k\|_p$$

$$\le C_1 2^{-a(j-k)} \left(E_{2^{k-1},r}(f,\Lambda)_p + E_{2^k,r}(f,\Lambda)_p\right),$$

where $E_{2^{-1},r}(f,\Lambda)_p := \|f\|_p$. Therefore for $R := \sum_{k=0}^{j} R_k$ we have that $R \in \widetilde{\mathcal{R}}_{C2^j}$ with $C = C(p, r, a, C_\Lambda, b)$ and

$$\|f - R\|_p \le \|f - f_j\|_p + \sum_{k=0}^{j} \|j_k - R_k\|_p$$

$$\le C 2^{-aj} \left(\|f\|_p + \sum_{k=0}^{j} 2^{ak} E_{2^k,r}(f,\Lambda)_p\right).$$

Given $n \ge 0$, (1.5) follows from the above by a suitable choice of j (depending on n). The proof of (1.4) is similar only we use (3.3) instead of (3.4).

♦

Proof of Theorem 1.3 Assume $0 < p \le 1$. For each $k \ge 0$ there exists $f_k \in S^r_{2^k}(\mathbb{R}^2)$ such that $\|f - f_k\|_p^p \le 2 s_{2^k,r}(f)_p^p$. We let $j_0 = f_0$ and for $k \ge 1$ we set $j_k = f_k - f_{k-1}$. Then, we have

$$\|j_0\|_p^p \le 2\left(s_{1,r}(f)_p^p + \|f\|_p^p\right),$$

and

$$\|j_k\|_p^p \le \|f - f_k\|_p^p + \|f - f_{k-1}\|_p^p \le 2\left(s_{2^{k-1},r}(f)_p^p + s_{2^k,r}(f)_p^p\right), \quad k \ge 1.$$

Since j_k is a sum of two piecewise polynomial functions in $S^r_{2^k}(\mathbb{R}^2)$, there exists a constant C_3, independent of k and r, such that $j_k \in S^r_{C_3 4^k}(\mathbb{R}^2)$. We now fix j and for $0 \le k \le j$ we approximate j_k by the rational function $R_k \in \widetilde{\mathcal{R}}_{n(k)}$ where $n(k) = \lceil C_2^{-2} C_3 4^k \log^2 (2^{a(j-k)}) \rceil$ that is guaranteed by Lemma 3.3. By virtue of (3.3)

$$\|j_k - R_k\|_p^p \le C_1^p \exp\left(-C_2 \sqrt{\frac{C_2^{-2} C_3 4^k \ \log^2 (2^{a(j-k)})}{C_3 4^k}}\right)^p \|j_k\|_p^p$$

$$\le C 2^{-a(j-k)p} \left(s_{2^{k-1},r}(f)_p^p + s_{2^k,r}(f)_p^p\right),$$

where $s_{2^{-1},r}(f)_p := \|f\|_p$. Therefore for $R := \sum_{k=0}^{j} R_k$ we have that $R \in \widetilde{\mathcal{R}}_{C4^j}$ with $C = C(p, r, a)$ and

$$\| f - R\|_p^p \le \| f - f_j \|_p^p + \sum_{k=0}^{j} \| j_k - R_k \|_p^p$$
$$\le C 2^{-a jp} \left(\| f \|_p^p + \sum_{k=0}^{j} 2^{akp} s_{2^k, r} (f)_p^p \right).$$

Given $n \ge 0$, (1.6) follows from the above by a suitable choice of j (depending on n).

♦

Proof of Theorem 1.6 A variant of the discrete Hardy inequality (see Lemma 2.3.4 in [DL] and the remark that follows) implies that if for non-negative sequences $a = \{a_n\}_{n\in\mathbb{Z}}$, $b = \{b_n\}_{n\in\mathbb{Z}}$ we have

$$b_n \le C_0 2^{-nl} \left(\sum_{k=-\infty}^{n} \left(2^{kl} a_k\right)^m \right)^{1/m}, \qquad n \in \mathbb{Z},$$

for some $l, m, C_0 > 0$ then for all $q > 0$ and $0 < a < l$ we have that

$$\left(\sum_{k=-\infty}^{\infty} \left(2^{ka} b_k\right)^q \right)^{1/q} \le C(a, q) C_0 \left(\sum_{k=-\infty}^{\infty} \left(2^{ka} a_k\right)^q \right)^{1/q}.$$

Therefore, the proofs for (1.9) and (1.11) follow from (1.4) and (1.5) respectively and a direct application of the discrete Hardy inequality.

We now prove (1.12). The proof of (1.10) is similar. Let $f \in A_q^a(L_p, \Sigma_r, b)$ for $1 < p < \infty$, $0 < q < \infty$ and $b > 0$. For each $n \ge 0$ there exists $j_n \in S_{2^n}^{r,b}(\mathbb{R}^2)$ such that $\| f - j_n \|_p \le 2 s_{2^n, r, b}(f)_p$. Also, there exists a constant $C > 0$ such that $\| j_n \|_p \le C \| f \|_p$. By (3.4) for $m = 2^n$ there exists a rational function $R_n \in \widetilde{\mathcal{R}}_{C2^n, \left[C_2^{-2} 2^n \log^2\left(2^{n(a+e)}\right)\right]}$ such that

$$\| j_n - R_n \|_p \le C_1 \exp\left(-C_2 \sqrt{\frac{C_2^{-2} 2^n \log^2 2^{n(a+e)}}{2^n}} \right) \| j_n \|_p$$
$$\le C_1 2^{-n(a+e)} \| j_n \|_p$$
$$\le C\left(p, r, b, \| f \|_p\right) 2^{-n(a+e)}.$$

Consequently

$$\| f - R_n \|_p \le \| f - j_n \|_p + \| j_n - R_n \|_p$$
$$\le C\left(p, r, b, \| f \|_p\right) \left(s_{2^n, r}(f)_p + 2^{-n(a+e)} \right).$$

Therefore

$$| f |_{A_{q,\log}^a\left(L_p, \widetilde{\mathcal{R}}\right)} \le C\left(p, q, r, b, \| f \|_p\right) \left(| f |_{A_q^a\left(L_p, \Sigma_r, b\right)} + \left\| \left\{ 2^{-ne} \right\} \right\|_{l_q} \right) < \infty.$$

The proof for $q = \infty$ is similar.

♦

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