Monotone trigonometric approximation *

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Abstract

Let $f \in C[-\omega,\omega]$, $0<\omega<\pi$, be nonlinear and nondecreasing. We wish to estimate the degree of approximation of f by trigonometric polynomials that are nondecreasing in $[-\omega,\omega]$. We obtain estimates involving the second modulus of smoothness of f and show that one in general cannot have estimates with the third modulus of smoothness.

1 Introduction and the main results

The question of estimating the degree of approximation of a nondecreasing continuous function on a finite interval, by nondecreasing algebraic polynomials, called today monotone approximation, has a long history with the first significant result due to Lorentz and Zeller [9]. A few years later Iliev [7] and Newman [12] asked about estimating the degree of, what is called today, comonotone approximation of a continuous function on a finite interval that changes its monotonicity finitely many times in that interval, by algebraic polynomials that change monotonicity exactly at the same points.

Obviously, one could not ask for the analog of monotone approximation by trigonometric polynomials, of continuous 2π periodic functions, as the only nondecreasing such periodic functions are the constants. On the other hand asking the analog question about comonotone approximation of 2π periodic functions is quite natural, and this question has recently been dealt with by Dzyubenko and Pleshakov [3] (see also [13]).

Videnskii [18], Erdelyi [4] and Erdelyi and Szabados [5] dealt with the behavior of trigonometric polynomials on an interval $[-\omega,\omega]$, $0<\omega<\pi$ and obtain, among others, Bernstein and Markov type inequalities (see, *e.g.,* [1, Chapter 5]). Recently, we see renewed interest in the subject, by Kroó [8] and Vianello [17]. Also, questions of trigonometric interpolation and quadratures on $[-\omega,\omega]$ have been discussed in [2]

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and [6], and Nagy and Totik [11], have proved Bernstein type inequalities for algebraic polynomials on the arc $K_\omega := \{e^{i\theta} \mid \theta \in [-\omega, \omega]\}$.

Unlike the interval $[-\pi, \pi]$, one may ask how well can nondecreasing trigonometric polynomials approximate a function $f \in C[-\omega, \omega]$, where $0 < \omega < \pi$ and $C[-\omega, \omega]$ denotes the space of continuous functions equipped with the sup-norm $\|\cdot\|_{[-\omega,\omega]}$. The purpose of this note is to begin to answer this question.

Let $\mathbb{T}_n$ denote the space of trigonometric polynomials of degree $\le n$, and denote by $\Delta^{(1)}[-\omega, \omega] \subset C[-\omega, \omega]$, the set of all nonlinear, nondecreasing, continuous functions on $[-\omega, \omega]$. We are interested in estimating

$$E_n^{(1)}(f, \omega) := \inf \|f - T_n\|_{[-\omega,\omega]},$$

where the infimum is taken over all polynomials $T_n \in \mathbb{T}_n$ that are nondecreasing in $[-\omega, \omega]$.

As usual, define

$$\Delta_t(f, x) := \begin{cases} f(x+t) - f(x), & x, x+t \in [-\omega, \omega], \\ 0, & \text{otherwise}, \end{cases}$$

and let $\Delta_t^k(f, x) := \Delta_t\big(\Delta_t^{k-1}(f, x)\big)$, $k \ge 2$.

Also, define

$$\omega_k(f, t; \omega) := \sup_{0<\tau\le t} \|\Delta_\tau^k(f, \cdot)\|_{[-\omega,\omega]},$$

and denote

$$\Omega_2(t) := \omega_2(f, t; \omega) > 0. \tag{1.1}$$

Our first result is the following estimate.

Theorem 1.1 *Let $0 < \omega < \pi$. If $f \in \Delta^{(1)}[-\omega, \omega]$, then*

$$E^{(1)}(f, \omega) \le C(\omega) \frac{\omega_1(f, \frac{\omega}{3}; \omega)}{\omega_2(f, 2\omega; \omega)} \omega_2(f, \pi/n; \omega), \quad n \ge N(\omega). \tag{1.2}$$

While we would have liked the constant in (1.2) to be independent of f (the dependence on ω, obviously, is necessary), we could not obtain that. Nevertheless, the next result shows that we cannot improve the estimates by involving ω_3.

Theorem 1.2 *Let $0 < \omega < \pi$ be fixed, and let $A > 0$ be given. Then there is a function $f = f_{n,A;\omega} \in \Delta^{(1)}[-\omega, \omega]$, such that*

$$E_n^{(1)}(f, \omega) > A\omega_3(f, \omega; \omega) > 0, \quad n \ge 1. \tag{1.3}$$

A trigonometric polynomial is obviously periodic in $[-\pi, \pi]$, so that forcing it to be monotone nondecreasing in $[-\omega, \omega]$ means that it will have at least two changes of monotonicity in $[\omega, 2\pi - \omega]$. Therefore our strategy in proving Theorem 1.2 will be to construct auxiliary functions that well approximate f on $[-\omega, \omega]$, are monotone nondecreasing there and are sufficiently simple so as to allow us to extend them into periodic functions in the whole interval $[-\pi,\pi]$, with two changes of monotonicity, such that we have control on their second modulus of smoothness in the latter interval. Then we will approximate these functions by their best trigonometric polynomial approximations.

To this end, in Section 2, for selected positive h's ($h=\frac{\pi}{n}$), we will construct a piecewise linear, periodic function g_h in $[-\pi,\pi]$, that is nondecreasing in $[-\omega,\omega]$, changes monotonicity twice in $[-\pi,\pi]\setminus[-\omega,\omega]$, is close enough to f in $[-\omega,\omega]$ and has an appropriate second modulus of smoothness in $[-\pi,\pi]$. Then we will approximate it using results of [3] (see Section 3). Finally in Section 4, we will prove our results. The constants $C(\cdot,\dots,\cdot)$ depend on the parameters inside the parentheses, but may be different in different occurrences even when they appear on the same line.

2 Auxiliary construction

Given $n\ge\frac{2\pi}{\pi-\omega}$, let $h:=\frac{\pi}{n}$ and let p be the natural number satisfying

$$ph\le\omega<(p+1)h\,.$$

We begin by defining g_h in $[-ph,ph]$, as the broken line connecting

$$g_h(ih):=f(ih),\quad -p\le i\le p\,.$$

Denote by $\delta_{p-1}:=\Delta_h(f,(p-1)h)$ and $\delta_{-p}:=\Delta_h(f,-ph)$, and extend g_h to the intervals $[ph,(p+1)h]$ and $[-(p+1)h,-ph]$ by straight lines having the slopes δ_{p-1}/h and δ_{-p}/h, respectively. We note that by Whitney's theorem [19] (see, *e.g.,* [14])

$$\|f-g_h\|_{[-\omega,\omega]}\le c\,\Omega_2(h)\,, \tag{2.1}$$

where c is an absolute constant.

Since we would like to obtain a 2π periodic g_h, in $(-\infty,\infty)$, we define

$$g_h(ih+2\pi k):=g_h(ih),\quad -(p+1)h\le i\le(p+1)h,\quad k\in\mathbb{Z},$$

where $\mathbb{Z}$ denotes the integers, so that, in particular, we have $g_h\big((2n-p-1)h\big)=g_h\big(-(p+1)h\big)$. Note that the choice of n guarantees that $2n-p-1>p+1$.

With this in mind, it remains to define $g_h(ih)$ for $p+1<i<2n-(p+1)$, and extend it periodically, as above.

With parameters $\alpha,\beta>0$, to be prescribed, we define the differences

$$\delta_i:=g_h((i+1)h)-g_h(ih),\quad p\le i<2n-p\,, \tag{2.2}$$

by

$$\begin{aligned}\delta_i&:=\delta_{p-1}-\Omega_2(h)\alpha(i-p),\quad p\le i<n\,,\\ \delta_i&:=\delta_{-p}-\Omega_2(h)\beta(2n-p-i-1),\quad n\le i<2n-p\,.\end{aligned} \tag{2.3}$$

Note that the new definitions of δ_p and δ_{2n-p-1} agree with the construction above. On the other hand, retrieving $g_h(ih)$, $p+1 < i < 2n-p-1$, we observe that we have two definitions for $g(nh)$, namely,

$$g_h(nh) = \sum_{i=p}^{n-1} \delta_i + f(ph)\,,$$

and

$$g_h(nh) = f(-ph) - \sum_{i=n}^{2n-p-1} \delta_i\,.$$

Hence, to guarantee that g_h is well defined as a piecewise linear function in $[-ph,(2n-p)]$, we impose the condition

$$\sum_{i=p}^{n-1} \delta_i + f(ph) = f(-ph) - \sum_{i=n}^{2n-p-1} \delta_i\,,$$

which is,

$$\begin{aligned} 0 &= \sum_{i=p}^{2n-p-1} \delta_i + f(ph) - f(-ph) \\ &= f(ph) - f(-ph) + \sum_{i=p}^{n-1} \big(\delta_{p-1} - \Omega_2(h)\alpha(i-p)\big) \\ &\quad + \sum_{i=n}^{2n-p-1} \big(\delta_{-p} - \Omega_2(h)\beta(2n-p-i-1)\big) \\ &= f(ph) - f(-ph) + (n-p)(\delta_{p-1} + \delta_{-p}) \\ &\quad + \Omega_2(h)(\alpha+\beta)(n-p)(n-p-1)/2\,. \end{aligned}$$

Hence,

$$\alpha + \beta = \frac{f(ph) - f(-ph) + (n-p)(\delta_{p-1} + \delta_{-p})}{\Omega_2(h)(n-p)(n-p-1)/2}\,, \tag{2.4}$$

where we observe that, by the choice of n, we have $n-p-1 \geq 1$.

We are free to impose another condition on the two parameters, so we take $\alpha = \beta$. Then, it follows by (2.4) that

$$\alpha = \beta = \frac{f(ph) - f(-ph)}{\Omega_2(h)(n-p)(n-p-1)} + \frac{\delta_{p-1} + \delta_{-p}}{\Omega_2(h)(n-p-1)} \geq \frac{\delta_{p-1} + \delta_{-p}}{\Omega_2(h)(n-p-1)}\,. \tag{2.5}$$

Remark *Since f is nondecreasing, we know that $f(ph) - f(-ph) \geq 0$. However, it may happen that although f is not a constant, still for some $h > 0$, $f(ph) = f(-ph)$. By virtue of* (2.1)*, in this case, the constant trigonometric polynomial $T_0(x) \equiv f(ph)$ provides the* *required degree of approximation of f in $[-\omega,\omega]$, namely,* $\|f-T_0\|_{[-\omega.\omega]} \le c\,\omega_2(f,h)$, *and we may proceed directly to Section* 4.

Otherwise, we may assume that $f(ph) > f(-ph)$. Then any pair of positive parameters satisfying (2.4) *will produce a well defined function g_h of period 2π. Moreover, since g_h is nondecreasing in $[ph,(p+1)h]$ and in $[(2n-p-1)h,(2n-p)h]$, while already $g_h(ph) > g_h((2n-p)h)$, it readily follows that g_h has a maximum followed by a minimum in the interval $[(p+1)h,(2n-p-1)h]$.*

Now, let i_1 be such that $\delta_{i_1} \ge 0$ and $\delta_{i_1+1} < 0$ and let $i_2 > i_1$ be such that $\delta_{i_2} \ge 0$ and $\delta_{i_1-1} < 0$. In view of the above remark, the existence of i_1 and i_2 is clearly guaranteed. Moreover, straightforward calculations yield,

$$p-1+\frac{\delta_{p-1}}{\alpha\Omega_2(h)} < i_1 \le p+\frac{\delta_{p-1}}{\alpha\Omega_2(h)} \quad \text{and,}$$
$$2n-p-1-\frac{\delta_{-p}}{\beta\Omega_2(h)} \le i_2 < 2n-p-\frac{\delta_{-p}}{\beta\Omega_2(h)}\,,$$

which, in turn, imply that,

$$2(n-p)-1-\frac{\delta_{p-1}}{\alpha\Omega_2(h)}-\frac{\delta_{-p}}{\beta\Omega_2(h)} \le i_2-i_1 < 2(n-p)+1-\frac{\delta_{p-1}}{\alpha\Omega_2(h)}-\frac{\delta_{-p}}{\beta\Omega_2(h)}\,.$$

Hence, by virtue of (2.5), we conclude that

$$i_2-i_1 \ge 2(n-p)-1-\frac{\delta_{p-1}+\delta_{-p}}{\alpha\Omega_2(h)} \ge 2(n-p)-1-(n-p-1) = n-p\,,$$

so that the distance between the maximum of g_h and its minimum is

$$i_2h-i_1h \ge \pi-\omega\,. \tag{2.6}$$

Finally, it readily follows by (2.5) that

$$\alpha=\beta \le \frac{2n\omega_1(f,h;\omega)}{\Omega_2(h)(n-p)(n-p-1)} \le \frac{4n\omega_1(f,h;\omega)}{\Omega_2(h)(n-p)^2} \le \frac{4\pi h\omega_1(f,h;\omega)}{\Omega_2(h)(\pi-\omega)^2}\,, \tag{2.7}$$

where for the second inequality we used the fact that $n-p-1 \ge 1$.

Since g_h is defined in $\mathbb{R}$, we may talk about

$$\Delta_t^2(g_h,x) := g_h(x)-2g_h(x+t)+g_h(x+2t), \quad t>0\,,$$

and denote

$$\omega_2(g_h,h;\mathbb{R}) := \sup_{0<t\le h} \|\Delta_t^2(g_h,\cdot)\|_{\mathbb{R}}\,.$$

Lemma 2.1 *For $h=\frac{\pi}{n}$, $0<t\le h$ and $x\in\mathbb{R}$, we have*

$$|\Delta_t^2(g_h,x)| \le 2\max_{-p\le i<2n-p} |\Delta_h^2(g_h,ih)|.$$

Proof. Since g_h is periodic with period 2π, we may restrict the discussion to $ih \le x < (i+1)h$, $-p \le i < 2n-p$. We will show that

$$|\Delta_t^2(g_h,x)| \le 2\max\{|\Delta_h^2(g_h,ih)|, |\Delta_h^2(g_h,(i+1)h)|\}\,. \tag{2.8}$$

Indeed, we distinguish three cases
1. If $ih \le x < x+t < x+2t \le (i+1)h$, then $\Delta_t^2(g_h,x) = 0$, and there is nothing to prove.

For the other two cases we need the notation,

$$\begin{aligned}\gamma &:= \frac{1}{h}\Delta_h(g_h,ih)\,,\\ \eta &:= \frac{1}{h}\Delta_h(g_h,(i+1)h)\,,\\ \mu &:= \frac{1}{h}\Delta_h(g_h,(i+2)h)\,.\end{aligned}$$

2. We have $ih \le x < x+t \le (i+1)h < x+2t$ or $x < (i+1)h < x+t < x+2t \le (i+2)h$. We deal with the former, the latter being similar.

To this end, we have

$$\begin{aligned} g_h(x) &= \big(x-(i+1)h\big)\gamma + g_h\big((i+1)h\big)\,,\\ g_h(x+t) &= \big(x+t-(i+1)h\big)\gamma + g_h\big((i+1)h\big)\,,\\ g_h(x+2t) &= \big(x+2t-(i+1)h\big)\eta + g_h\big((i+1)h\big)\,. \end{aligned} \tag{2.9}$$

Hence,

$$\begin{aligned}\Delta_t^2(g_h,x) &= \big(x-(i+1)h\big)\gamma - 2\big(x+t-(i+1)h\big)\gamma + \big(x+2t-(i+1)h\big)\eta\\ &= \big(x+2t-(i+1)h\big)(\gamma-\eta),\end{aligned}$$

so that

$$|\Delta_t^2(g_h,x)| \le |\Delta_h^2(g,ih)|\,. \tag{2.10}$$

Finally,
3. We have $ih < x < (i+1)h < x+t < (i+2)h < x+2t$. Then, instead of (2.9) we have,

$$\begin{aligned} g_h(x) &= \big(x-(i+1)h\big)\gamma + g_h\big((i+1)h\big)\,,\\ g_h(x+t) &= \big(x+t-(i+1)h\big)\eta + g_h\big((i+1)h\big)\,,\\ g_h(x+2t) &= \big(x+2t-(i+2)h\big)\mu + g_h\big((i+2)h\big)\,. \end{aligned}$$

Hence, we obtain

$$\begin{aligned}\Delta_t^2(g_h,x) &= \big(x-(i+1)h\big)\gamma - 2\big(x+t-(i+1)h\big)\eta + \big(x+2t-(i+2)h\big)\mu\\ &\quad + g_h\big((i+2)h\big) - g_h\big((i+1)h\big)\\ &= \big(x-(i+1)h\big)(\gamma-\eta) + \big(x+2t-(i+2)h\big)(\mu-\eta)\,.\end{aligned}$$

Thus, we conclude that

$$|\Delta_t^2(g_h,x)| \le |\Delta_h^2(g,ih)| + |\Delta_h^2(g,(i+1)h)|\,,$$

which together with (2.10) completes the proof of (2.8). $\square$

Corollary 2.2 *For $h=\frac{\pi}{n}$, $n \ge \frac{2\pi}{\pi-\omega}$, we have*

$$\omega_2(g_h,h;\mathbb{R}) \le c\,\max\left\{\Omega_2(h), \frac{h\omega_1(f,h;\omega)}{(\pi-\omega)^2}\right\}, \tag{2.11}$$

where c is an absolute constant.

Proof. In view of Lemma 2.1 we only have to estimate $\Delta_h^2(g,ih)$, $-p \le i \le 2n-p-1$. First, note that, by our construction, if $i=p-1$ or $i=2n-p-1$, then $\Delta_h^2(g,ih)=0$, and for $-p \le i < p-1$, (2.11) readily follows from (1.1) and (2.1).

Now, by (2.3), for $p \le i < n-1$, we obtain

$$\Delta_h^2(g,ih) = \delta_{i+1} - \delta_i = -\Omega_2(h)\alpha,$$

and (2.11) follows by (2.7).

Similarly, we get (2.11) for $n \le i < 2n-p-1$. This completes the proof. $\square$

3 Comonotone approximation

Given $2s$, $s \in \mathbb{N}$, fixed points $Y=\{y_1,\dots,y_{2s}\}$, in $[-\pi,\pi)$, such that $-\pi \le y_{2s} < y_{2s-1} < \cdots < y_1 < \pi$, we extend it to a sequence in $\mathbb{R}$, by $y_{i+2sk} := y_i - 2k\pi$, $1 \le i \le 2s$, $k \in \mathbb{Z}$. We denote by $\Delta^{(1)}(Y)$ the set of all 2π periodic functions $g \in C(-\infty,\infty)$ that change monotonicity at the points of Y and are nondecreasing in $[y_{2s+1}, y_{2s}]$. Thus, $f \in \Delta^{(1)}(Y)$ is nondecreasing in $[y_{2i+1}, y_{2i}]$, $-\infty < i < \infty$, and is nonincreasing otherwise.

Denote by

$$E_n^{(1)}(g,Y) := inf_{T_n \in \mathbb{T}_n \cap \Delta^{(1)}(Y)} \|g-T_n\|_{[-\pi,\pi]},$$

the degree of best comonotone trigonometric approximation to g.

It was proved in [3] that

$$E_n^{(1)}(g,Y) \le c(s)\omega_2\big(g,\pi/n;\mathbb{R}\big), \quad n \ge N(Y), \tag{3.1}$$

where $c(s)$ depends only on s, but $N(Y)$ depends on $\min_{1\le i\le 2s}\{y_i - y_{i+1}\}$.

The functions g_h constructed in Section 2, have two changes of monotonicity, at i_1h and i_2h. By our construction $(i_1+2n-i_2)h > 2\omega$, so that by virtue of (2.6), we conclude that the above minimum is

$$\min\{(i_2-i_1)h, (i_1+2n-i_2)h\} \ge \min\{\pi-\omega, 2\omega\} =: m(\omega). \tag{3.2}$$

If we denote $Y_2 := \{i_1h, i_2h\}$, then it follows by (3.1) and the previous restriction on n, that for g_h, where $h=\frac{\pi}{n}$,

$$E_n^{(1)}(g_h,Y_2) \le c(s)\omega_2\big(g_h,\pi/n;\mathbb{R}\big), \quad n \ge \max\{\frac{2\pi}{\pi-\omega}, N(m(\omega))\} =: N_1(\omega). \tag{3.3}$$

4 Proofs of the main results

We begin with a lemma.

Lemma 4.1 *There exists a constant $C = C(\omega)$ such that for every nonlinear function $f \in C[-\omega, \omega]$ and all $h \le \omega^2$, we have*

$$h\omega_1(f, h; \omega) \le C \frac{\omega_1(f, \frac{\omega}{3}; \omega)}{\omega_2(f, 2\omega; \omega)} \omega_2(f, h; \omega) . \tag{4.1}$$

Proof. We quote [10, 3.6(11)] with $l = 2\omega$ to obtain the estimate,

$$\omega_1(f, h; \omega) \le C(f, \omega)\omega_2(f, \sqrt{h}; \omega), \quad h \le \omega^2 .$$

Hence,

$$\begin{aligned} h\omega_1(f, h; \omega) &\le C(f, \omega) h\omega_2(f, \sqrt{h}; \omega) \\ &\le C(f, \omega) h \left(\frac{1}{\sqrt{h}} + 1 \right)^2 \omega_2(f, h; \omega) \le C(f, \omega)\omega_2(f, h; \omega) . \end{aligned}$$

A closer look at the proof of [10, Theorem 3.9] shows that

$$C(f, \omega) = C(\omega) \frac{\omega_1(f, \frac{4\omega}{3}; \omega)}{\omega_2(f, 2\omega; \omega)} ,$$

and the proof of (4.1) is complete. □

Proof of Theorem 1.1. Fix $n \ge \max\{\frac{\pi}{\omega^2}, N_1(\omega)\} =: N(\omega)$, and let $h = \frac{\pi}{n}$. Then, by (3.3), there exists a trigonometric polynomial $T_n \in \mathbb{T}_n$, nondecreasing in $[-\omega, \omega]$, such that

$$\|g_h - T_n\|_{[-\omega,\omega]} \le c(s)\omega_2\big(g_h, \pi/n; \mathbb{R}\big).$$

Hence, it follows by (2.11) and Lemma 4.1 that,

$$\|g_h - T_n\|_{[-\omega,\omega]} \le C(\omega) \frac{\omega_1(f, \frac{\omega}{3}; \omega)}{\omega_2(f, 2\omega; \omega)} \omega_2(f, \pi/n; \omega).$$

Now, by (2.1),

$$\|f - T_n\|_{[-\omega,\omega]} \le \|f - g_h\|_{[-\omega,\omega]} + \|g_h - T_n\|_{[-\omega,\omega]} \le C(\omega) \frac{\omega_1(f, \frac{\omega}{3}; \omega)}{\omega_2(f, 2\omega; \omega)} \omega_2(f, \pi/n; \omega).$$

This completes the proof. □

In order to prove Theorem 1.2, we need two lemmas (compare the first with [16, Lemma 4]).

Lemma 4.2 *Given $b > \frac{1}{\omega}$, there exist a function $f \in \Delta^{(1)}[-\omega,\omega]$ and a quadratic polynomial p, such that*

$$p'(\omega) < 0 \quad \text{and} \quad b\|f-p\| \le |p'(\omega)|.$$

Proof. Set $\pi(x) := \omega - 2/b - x$ and

$$\varphi(x) := \begin{cases} \pi(x), & x \in [-\omega, \omega - 2/b], \\ 0, & x \in (\omega - 2/b, \omega]. \end{cases}$$

Let $f(x) := \int_{-\omega}^{x} \varphi(u)du$ and $p(x) := \int_{-\omega}^{x} \pi(u)du$.

Clearly, $f \in \Delta^{(1)}[-\omega,\omega]$ and $p'(\omega) = -2/b$, while

$$|f(x) - p(x)| = \left| \int_{-\omega}^{x} (\varphi(u) - \pi(u))du \right| \le \int_{\omega-2/b}^{\omega} (u + 2/b - \omega)du = \frac{2}{b^2}.$$

This completes the proof. □

It is well known that the sets of functions $\{1, x, x^2, \cos x, \sin x, \dots, \cos nx, \sin nx\}$ and $\{1, x, \cos x, \sin x, \dots, \cos nx, \sin nx\}$ are Chebyshev systems in $[-\omega,\omega]$. Denote

$$\mathbf{H}_n := \operatorname{span}\{1, x, x^2, \cos x, \sin x, \dots, \cos nx, \sin nx\},$$
$$\mathbf{K}_n := \operatorname{span}\{1, x, \cos x, \sin x, \dots, \cos nx, \sin nx\},$$

and let C_n be the Chebyshev polynomial, associated with $\mathbf{H}_n$, in $[-\omega,\omega]$. Namely, following [1, Sections 3.1 and 3.3] (see Theorem 3.1.6, (3.3.2) and 3.3E.2a there)

$$C_n(x) = c\big(x^2 - a_0 - b_0 x - \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)\big),$$

where c is a normalizing factor so taken that $\|C_n\|_{[-\omega,\omega]} = 1$ and $C_n(\omega) = 1$, and there exist $2n+3$ points $-\omega = x_1 < x_2 < \cdots < x_{2n+2} < x_{2n+3} = \omega$, such that

$$C_n(x_i) = -C_n(x_{i+1}) = (-1)^{i+1}\|C_n\|_{[-\omega,\omega]}.$$

Observe that $C_n' \in \mathbf{K}_n$ and vanishes at x_i, $1 < i < 2n+3$, that is, $2n+1$ zeros. Since C_n' has at most $2n+1$ zeros, it follows that $C_n'(\omega) \ne 0$, thus $C_n'(\omega) > 0$.

The following lemma is a modification of [1, Theorem 3.3.1] for H_n (see also [15, Theorem 1.10]).

Lemma 4.3 *If $Q \in \mathbf{H}_n$, then*

(4.2) $$|Q'(\omega)| \le |C_n'(\omega)| \|Q\|_{[-\omega,\omega]},$$

Proof. Without loss of generality, we may assume that $\|Q\|_{[-\omega,\omega]} = 1$. Assume, to the contrary, that (4.2) is false, so that

$$|Q'(\omega)| > |C_n'(\omega)|. \tag{4.3}$$

We may assume that $Q'(\omega) > C_n'(\omega)$, for otherwise we take $-Q$. It follows from [1, Theorem 3.3.1] that $Q(\omega) < 1$, for otherwise (4.2) is valid, so that we cannot have (4.3). Take the interval $[x_i, x_{i+1}]$, $1 < i < 2n+3$. Either $C_n(x) - Q(x)$ vanishes at x_i or x_{i+1}, in which case it has a double zero there, or $C_n(x_i) - Q(x_i)$ and $C_n(x_{i+1}) - Q(x_{i+1})$ have opposite signs, so that there is a zero in (x_i, x_{i+1}). Also, we may have $C_n(x_{2n+2}) - Q(x_{2n+2}) = 0$, a double zero, or, if $C_n(x_{2n+2}) - Q(x_{2n+2}) \neq 0$, then in view of our assumption on Q, the polynomial $C_n(x) - Q(x)$ must vanish at $\theta < \omega$. Similarly, we either have a double zero at x_2 or $C_n(x) - Q(x)$ must vanish in $[-\omega, x_2)$. Hence, the number of zeros counting multiplicities, in $[-\omega, \theta]$, is at least $2n+2$, which is the maximum possible number of zeros of $C_n(x) - Q(x)$. By Rolle's theorem, this implies that all zeros of $C_n'(x) - Q'(x)$ are in $(-\omega, \theta)$. However,

$$C_n'(\omega) - Q'(\omega) < 0,$$

and coupled with $C_n(\omega) - Q(\omega) > 0$, this implies that there is a $\zeta \in (\theta, \omega)$ such that $C_n(\zeta) - Q(\zeta) = C_n(\omega) - Q(\omega)$. In turn, we conclude by the mean value theorem that $C_n'(x) - Q'(x)$ vanishes at a point in (ζ, ω), a contradiction. Hence, our assumption (4.3) in wrong, and (4.2) follows. This completes the proof. □

Remark *Comparing* Lemma 4.3 *and* [1, Theorem 3.3.1], *one observes that in the former we did not require that* $Q(\omega) = C_n(\omega)$. *However, our Chebyshev system had the additional property that* 1 *belonged to it, of which we take advantage both by knowing that* $|C_n(\omega)| = 1$, *thus assuming that* $C_n(\omega) = 1$ (*see* [1, 3.3E.2a]), *and in counting the number of zeros of* $C_n - Q$. *Our proof is valid for any Chebyshev system that contains* 1.

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $A > 0$ be arbitrary, and take $B > 8A + 1$ such that $b := B|C_n'(\omega)| > 1/\omega$. Let $f \in \Delta^{(1)}[-\omega, \omega]$ be the function from Lemma 4.2 for b, and let $T_n \in \mathbb{T}_n \cap \Delta^{(1)}[-\omega, \omega]$, be the best monotone approximation to f. Then, for the quadratic polynomial p of Lemma 4.2, we have, by Lemma 4.3,

$$b\|f - p\| \le |p'(\omega)| \le |p'(\omega) - T_n'(\omega)| \le |C_n'(\omega)| \|p - T_n\|_{[-\omega,\omega]}.$$

Hence

$$\begin{aligned} E_n^{(1)}(f, \omega) &= \|f - T_n\|_{[-\omega,\omega]} \ge \|p - T_n\|_{[-\omega,\omega]} - \|f - p_2\|_{[-\omega,\omega]} \\ &\ge (B-1)\|f - p\|_{[-\omega,\omega]} > 8A\|f - p\|_{[-\omega,\omega]} \\ &\ge A\omega_3(f - p, \omega; \omega) = A\omega_3(f, \omega; \omega). \end{aligned}$$

This completes our proof. □

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