Regularity of generalized Daubechies wavelets reproducing exponential polynomials with real-valued parameters

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A B S T R A C T

We investigate non-stationary orthogonal wavelets based on a non-stationary interpolatory subdivision scheme reproducing a given set of exponentials with real-valued parameters. The construction is analogous to the construction of Daubechies wavelets using the subdivision scheme of Deslauriers and Dubuc. The main result is the existence and smoothness of these Daubechies type wavelets.

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1. Introduction

In the fundamental paper [9] G. Deslauriers and S. Dubuc investigated a subdivision scheme based on polynomial interpolation of odd degree $2n - 1$. An important result is the existence of the basic limit function $\Phi^{D_{2n}}$ for the subdivision scheme, and a deep analysis and new methods were developed to determine its order of regularity. The present work originated in the desire to study a new concept of *multivariate* subdivision scheme in *the spirit of Deslauriers and Dubuc*. According to the Polyharmonic Paradigm introduced in [17], polyharmonic interpolation on parallel hyperplanes (or on concentric spheres) provides an interesting genuine generalization of the one-dimensional polynomial interpolation and generates a natural multivariate subdivision scheme. This *polyharmonic subdivision scheme* will be discussed in a forthcoming paper [11] and it is shown there how to reduce it to an infinite family of one-dimensional *non-stationary*

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subdivision schemes reproducing exponential polynomials of special type. It is a remarkable fact that the notion of non-stationary subdivision is so essential for the multivariate subdivision.

In a parallel direction, the study of non-stationary subdivision schemes reproducing general exponential polynomials was initiated in 2003 in [13], where the main results of Deslauriers and Dubuc were generalized. In particular, the existence of a basic limit function was established and its order of regularity was estimated. The subdivision scheme depends on a given vector of numbers $(\lambda_1, \ldots, \lambda_n)$ (with possible repetitions); this scheme reproduces the linear space $E(\lambda_1, \ldots, \lambda_n)$ spanned by all exponential polynomials $t^s e^{\lambda_j t}$, with integers s satisfying $0 \leqslant s \leqslant \mu_j - 1$, where μ_j is the number of occurrences of λ_j in the vector $(\lambda_1, \ldots, \lambda_n)$. The scheme is characterized by a family of symbols $\{a^{[k]}(z)\}_{k \in \mathbb{Z}}$, where $a^{[k]}(z)$ defines the refinement rule at level k.

It is well known that subdivision schemes and wavelet analysis are intimately related subjects. Indeed, Daubechies wavelets are closely related to the subdivision schemes of Deslauriers and Dubuc: the autocorrelation function of the Daubechies scaling function is equal to the basic limit function $\Phi^{D_{2n}}$ of the Deslauriers–Dubuc scheme (see [7,32], or Section 5 below).

The purpose of the present paper is to provide an analogous construction of generalized Daubechies wavelets based on non-stationary subdivision schemes reproducing the space $E(\lambda_1, \ldots, \lambda_n)$ for a given real vector $(\lambda_1, \ldots, \lambda_n)$. In Section 5 we provide the construction of the non-stationary Daubechies masks as appropriate square roots of masks of non-stationary Deslauriers–Dubuc type subdivision schemes, and we give a rigorous proof for the existence of the generalized Daubechies scaling function.

The main result of the paper is contained in Section 6: an estimate of the regularity of the generalized Daubechies scaling function, and consequently, an estimate of the regularity of the generalized Daubechies wavelets. An important technical tool is here the concept of *asymptotically equivalent* subdivision schemes developed in [15]. Given the real-valued parameters $\lambda_1, \ldots, \lambda_n$ we prove the existence of non-stationary Daubechies masks leading to a subdivision scheme reproducing the corresponding space $E(\lambda_1, \ldots, \lambda_n)$, *which is asymptotically equivalent to the classical subdivision scheme generating Daubechies scaling functions.* The proof of this important fact depends on an appropriate choice of the Daubechies masks at each level. With this choice we are able to estimate from below the order of smoothness of the generalized Daubechies scaling function: we make this by comparing the rate of decay of the Fourier transform of the classical Daubechies scaling function with that of the generalized Daubechies scaling function. Let us emphasize that this method provides an estimate from below for the smoothness but it does not give us a proof of *our conjecture* that *the order of regularity of the new Daubechies type wavelets reproducing the space* $E(\lambda_1, \ldots, \lambda_n)$, *is equal to the order of regularity of the classical Daubechies wavelets reproducing polynomials of degree* $\leqslant n - 1$. Let us mention that the regularity results of the present paper are essential for estimating the regularity of the multivariate polyharmonic subdivision schemes and the corresponding multivariate wavelets in [11].

There is a substantial overlap between the present article and the work of C. Vonesch, T. Blu and M. Unser in [35]. Our approach is written from the standpoint of subdivision schemes which leads in a very natural way to the corresponding filters. The article [35] favorizes the standpoint of wavelet and filter bank design and it is shown there that many results and techniques of classical multiresolution analysis, briefly MRA, carry over to non-stationary MRA (except that all filters and scaling functions depend on the scale of the multiresolution). The authors in [35] used the fundamental results in [15] for introducing non-stationary Daubechies-type wavelets reproducing a family of exponentials $\{e^{\lambda_j t}\}_{j=1}^n$ even for complex-valued parameters $\{\lambda_j\}$. However, there are some key differences with the present paper which we now review. Firstly, the authors in [35] did not discuss the regularity of the corresponding wavelets which is of high importance for theoretical and practical aspects of the wavelets. Secondly, it was not known to the authors of [35] that for *symmetric and real-valued* exponential parameters $\{\lambda_j\}$ the shortest-possible symbols are always positive on the unit circle, a result which has already been proved in [28] in 1996. In [35] it was shown by examples that the symbol is in general not positive for complex parameters. They concluded that in this general setting it may be necessary to introduce symbols that have longer support than in the standard Deslauriers–Dubuc scheme, so

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that they are positive and thus amenable to spectral factorization. Moreover [35] also discusses biorthogonal wavelets and contains a wealth of additional information about non-stationary MRA. Let us also mention that non-stationary MRA and Wavelet Analysis which reproduce exponentials have been discussed in [8]. These classes appear in a natural way in the context of multivariate polyspline wavelets, see [17]. Finally let us mention that there are other systems of Daubechies-like non-stationary wavelets in the literature and we refer to the work of M. Berkolaiko and I. Novikov in [1], see also [33].

The paper is organized as follows: In Section 2 we provide an introduction to non-stationary subdivision schemes and to the concept of asymptotically equivalent subdivision schemes developed in [15] which is a key ingredient in our proofs of the regularity results. Section 3 is devoted to subdivision schemes for exponential polynomials and we give an estimate from below of the order of regularity of the basic limit function. In Section 4 we briefly review the topic of non-stationary multiresolution analysis. In Section 5 we construct the generalized Daubechies scaling functions via subdivision schemes. Section 6 contains the main results about the order of regularity of the generalized Daubechies scaling functions. The short Section 7 provides the construction of the generalized Daubechies wavelets using a non-stationary MRA.

Finally, we introduce some notations: $\mathbb{N}_0$ denotes the set of all natural numbers including zero, $\mathbb{Z}$ denotes the set of all integers and we denote by $2^{-k}\mathbb{Z}$ the grid $\{j/2^k \colon j \in \mathbb{Z}\}$. By $C^\ell(\mathbb{R})$ we denote the set of all functions $f : \mathbb{R} \to \mathbb{R}$ which are ℓ-times continuously differentiable where $\ell \in \mathbb{N}_0$. The Fourier transform of an integrable function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\hat{f}(\omega) = \int\limits_{-\infty}^{\infty} f(x)e^{-ix\omega}\,dx.$$

2. Basics in non-stationary subdivision schemes

In this section we briefly recall notations and definitions used in non-stationary subdivision schemes for functions on the real line. The formal definition of a subdivision scheme is the following:

Definition 1. A (non-stationary) **subdivision scheme** S_0 is given by a family of sequences $(a_j^{[k]})_{j\in\mathbb{Z}}$ of real numbers indexed by $k \in \mathbb{N}_0$, called the *masks at level k*, such that $a_j^{[k]} \neq 0$ only for finitely many $j \in \mathbb{Z}$. Given a sequence of numbers $f^0(j)$, $j \in \mathbb{Z}$, one defines inductively a sequence of functions $f^{k+1} : 2^{-(k+1)}\mathbb{Z} \to \mathbb{C}$ by the rule

$$f_j^{k+1} := f^{k+1}\left(\frac{j}{2^{k+1}}\right) = \sum_{l\in\mathbb{Z}} a_{j-2l}^{[k]} f^k\left(\frac{l}{2^k}\right) \quad \text{for } j \in \mathbb{Z}.$$

If for each $k \in \mathbb{N}_0$ the masks $(a_j^{[k]})_{j\in\mathbb{Z}}$ are identical the scheme is said to be **stationary**. The subdivision scheme is called **interpolatory** if for all $k \geqslant 0$ and $j \in \mathbb{Z}$ there holds

$$f_{2j}^{k+1} = f_j^k.$$

An important tool in subdivision schemes is the *symbol* $a^{[k]}$ of the subdivision scheme, defined by

$$a^{[k]}(z) := \sum_{j\in\mathbb{Z}} a_j^{[k]} z^j$$

which is a Laurent polynomial since we assume that the sequence $a_j^{[k]}$, $j \in \mathbb{Z}$, has finite support. We identify the subdivision scheme S_0 by its masks $(a_j^{[k]})_{j\in\mathbb{Z}}$, $k \in \mathbb{N}_0$, or its symbols $a^{[k]}$, $k \in \mathbb{N}_0$. It is clear that a subdivision scheme is stationary if and only if $a^{[k]}(z) = a(z)$, for all $k \in \mathbb{N}_0$. Moreover the scheme is

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interpolatory if and only if $a_{2j}^{[k]} = \delta_{0,j}$, for all $j \in \mathbb{Z}$ and $k \in \mathbb{N}_0$; here $\delta_{0,j}$ is the Kronecker symbol. In terms of the symbol $a^{[k]}$ it is easy to prove that this is equivalent to the identity

$$a^{[k]}(z) + a^{[k]}(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}.$$

Definition 2. Let $\ell \in \mathbb{N}_0$. A subdivision scheme S_0 is called C^ℓ-**convergent** if for any bounded initial sequence $\{f_j^0\}_{j \in \mathbb{Z}}$ there exists $F \in C^\ell(\mathbb{R})$ such that

$$\lim_{k \to \infty} \sup_{j \in \mathbb{Z}} \{ |F(j2^{-k}) - f_j^k| \} = 0.$$

The function F is called the **limit function of the subdivision scheme** for the initial sequence $\{f_j^0\}_{j \in \mathbb{Z}}$. The limit function for the initial data function $f_j^0 = \delta_{0j}$ is called the **basic limit function** of the scheme and it is denoted by Φ_0.

A subdivision scheme is called **convergent** if it is just C^0-convergent. A central problem in the theory of subdivision schemes is to estimate the order of regularity of a limit function whenever it exists. Let us remark that for an *interpolatory* scheme the limit function F in Definition 2 is easily computed for dyadic numbers $t = \frac{j}{2^k}$ with $k \geq 0$ and $j \in \mathbb{Z}$ through the values f_j^k, and convergence of the scheme asks whether it has a continuous extension over the whole $\mathbb{R}$.

Remark 3. It is easy to see that for a convergent subdivision scheme S_0 with symbols $a^{[k]}(z) := \sum_{j=-N}^{N} a_j^{[k]} z^j$ for all $k \in \mathbb{N}_0$, the support of the basic limit function Φ_0 is contained in $[-N, N]$.

In the non-stationary case the following concept is of importance:

Definition 4. Let S_0 be a subdivision scheme given by the masks $(a_j^{[k]})_{j \in \mathbb{Z}}$. For any natural number $m \in \mathbb{N}_0$ define a new subdivision scheme S_m by means of the masks of level k:

$$a^{[k],m}(z) := a^{[k+m]}(z) \quad \text{for all } k \in \mathbb{N}_0.$$

The following result is proved in [14]:

Theorem 5. *If S_0 is a convergent subdivision scheme then S_m is convergent for any $m \in \mathbb{N}_0$.*
The basic limit function of this scheme is denoted by Φ_m.

The following result is well-known, see e.g. the proof of Theorem 2.1 in [6], for further results see also [30].

Proposition 6. *Let S_0 be a subdivision scheme with symbols $a^{[k]}(z)$ for each level $k \in \mathbb{N}_0$, such that*

(i) *$a_j^{[k]} = 0$ for all $|j| > N$ and all $k \in \mathbb{N}_0$ for some fixed integer $N > 0$,*
(ii) *there is a constant $M > 0$ such that $|a^{[k]}(e^{i\omega})| \leq M$ for all $k \in \mathbb{N}_0$ and*

$$\sum_{k=0}^{\infty} \left| \frac{1}{2} a^{[k]}(1) - 1 \right| < \infty.$$

Then the infinite product

$$\prod_{k=1}^{\infty} \frac{1}{2} a^{[k-1]}(e^{i\omega 2^{-k}})$$

converges uniformly on compact subsets of $\mathbb{R}$.

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Note that by Remark 3 the basic limit function Φ_0 of a convergent subdivision scheme satisfying (i) in Proposition 6 has compact support. Hence the continuous function Φ_0 is integrable and square integrable. It follows that the Fourier transform $\widehat{\Phi_0}$ of Φ_0 is well-defined, continuous and square integrable.

Proposition 7. *Let S_0 be a convergent subdivision scheme with symbols $a^{[k]}(z)$ for each level $k \in \mathbb{N}_0$, satisfying* (i) *and* (ii) *in Proposition 6. Then*

$$\widehat{\Phi_0}(\omega) = \prod_{k=1}^{\infty} \frac{1}{2} a^{[k-1]}\left(e^{i\omega 2^{-k}}\right).$$

Definition 8. A subdivision scheme S_0 with masks $(a_j^{[k]})_{j \in \mathbb{Z}}$ for $k \in \mathbb{N}_0$ **reproduces a continuous function** $f : \mathbb{R} \to \mathbb{C}$ at level $k \in \mathbb{N}_0$ if for all $j \in \mathbb{Z}$

$$f\left(\frac{j}{2^{k+1}}\right) = \sum_{l \in \mathbb{Z}} a_{j-2l}^{[k]} f\left(\frac{l}{2^k}\right). \tag{1}$$

We say that S_0 **reproduces f step-wise** if it reproduces f at each level $k \in \mathbb{N}_0$.

Let us note that in [14, p. 31] and [10] a slightly different terminology is used for the above notions.

Theorem 9 below extends Theorem 2.3 in [13] which was formulated only for interpolatory schemes. (We will apply this later to Daubechies schemes which are not interpolatory.) For a proof we refer to an extended version of the present paper in [12]:

Theorem 9. *Let S_0 be a subdivision scheme with masks $(a_j^{[k]})_{j \in \mathbb{Z}}$, $k \in \mathbb{N}_0$. Then for $r = 0, \ldots, \mu - 1$ the functions $f_r(x) = x^r e^{\lambda x}$ are reproduced step-wise by S_0 if and only if for all $k \in \mathbb{N}_0$ there holds*

$$a^{[k]}\left(-\exp\left(-2^{-(k+1)}\lambda\right)\right) = 0 \quad and \quad a^{[k]}\left(\exp\left(-2^{-(k+1)}\lambda\right)\right) = 2$$

and

$$\frac{d^r}{dz^r} a^{[k]}\left(\pm \exp\left(-2^{-(k+1)}\lambda\right)\right) = 0, \quad r = 1, \ldots, \mu - 1. \tag{2}$$

It is easy to see that a function $f : \mathbb{R} \to \mathbb{C}$ is reproduced step-wise by a subdivision scheme if and only if for the data function $f^0(j) := f(j)$ one has

$$f^k\left(\frac{j}{2^k}\right) = f\left(\frac{j}{2^k}\right).$$

From this it is easy to see that a convergent and step-wise reproducing subdivision scheme is reproducing in the following sense:

Definition 10. A convergent subdivision scheme S_0 with masks $(a_j^{[k]})_{j \in \mathbb{Z}}$ for $k \in \mathbb{N}_0$, is reproducing a continuous function $f : \mathbb{R} \to \mathbb{C}$ if the limit function for the data function $f(j)$, $j \in \mathbb{Z}$, is equal to $f(x)$.

Remark 11. In the definition of a convergent scheme some authors require bounded data.

An important concept for the investigation of non-stationary subdivision schemes is the following notion introduced in [15]:

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Definition 12. Two subdivision schemes S_a and S_b with masks $(a_j^{[k]})_{j \in \mathbb{Z}}$ and $(b_j^{[k]})_{j \in \mathbb{Z}}$ for $k \in \mathbb{N}_0$, respectively, are called **asymptotically equivalent** if

$$\sum_{k=0}^{\infty} \sum_{j \in \mathbb{Z}} |a_j^{[k]} - b_j^{[k]}| < \infty. \tag{3}$$

We will also say that the two masks are asymptotically equivalent.

Remark 13. The following simple observation will be very useful in our further considerations: If for some constant $C > 0$ the two subdivision schemes S_a and S_b satisfy the exponential estimate

$$\max_{j \in \mathbb{Z}} |a_j^{[k]} - b_j^{[k]}| \leqslant C \cdot 2^{-k} \quad \text{for all } k \in \mathbb{N}_0, \tag{4}$$

then they satisfy (3), and hence, they are asymptotically equivalent.

Suppose that for some integer $N > 0$, the masks $(a_j^{[k]})_{j \in \mathbb{Z}}$, $k \in \mathbb{N}_0$, and $(b_j^{[k]})_{j \in \mathbb{Z}}$, $k \in \mathbb{N}_0$, have support in the set $\{-N, \ldots, N\}$, i.e. $a_j^{[k]} = b_j^{[k]} = 0$ for $|j| > N$. Then it is easy to see that $a^{[k]}$, $k \in \mathbb{N}_0$, and $b^{[k]}$, $k \in \mathbb{N}_0$, satisfy estimate (4) if and only if for any $R > 1$ there exists $D > 0$ such that

$$|a^{[k]}(z) - b^{[k]}(z)| \leqslant D \cdot 2^{-k} \tag{5}$$

for all $k \in \mathbb{N}_0$ and for all $z \in \mathbb{C}$ with $1/R \leqslant |z| \leqslant R$.

The following elementary result provides a sufficient method for constructing asymptotically equivalent subdivision schemes. As one of the referees pointed out to us, a short proof depends on Vieta's formulas for expressing the coefficients of a polynomial by it roots and the fact that the product of several sequences $\{s_k^{(i)}\}_{k \in \mathbb{N}}$ with the property $|s_k^{(i)} - 1| = O(2^{-k})$, has the same property. Alternatively the reader may consult Theorem 14 in [12].

Theorem 14. *Let $m \in \mathbb{N}_0$ and assume that $p^{[k]}(z)$ and $p(z)$ are polynomials of degree m for each $k \in \mathbb{N}_0$ defined as*

$$p^{[k]}(z) = c^{[k]} \prod_{j=1}^{m} \left(z - \alpha_j^{[k]} \right) \quad \text{and} \quad p(z) = c \prod_{j=1}^{m} (z - \alpha_j)$$

for some complex numbers $c^{[k]}$, c, and $\alpha_j^{[k]}$ and α_j, for $j = 1, \ldots, m$, and $k \in \mathbb{N}_0$. Suppose that there exists a constant $D_m > 0$ such that for all $k \in \mathbb{N}_0$ and $j = 1, \ldots, m$

$$\left| \alpha_j^{[k]} - \alpha_j \right| \leqslant D_m 2^{-k} \quad \text{and} \quad \left| c^{[k]} - c \right| \leqslant D_m 2^{-k}. \tag{6}$$

Then $p^{[k]}(z)$, $k \in \mathbb{N}_0$, and $p(z)$, $k \in \mathbb{N}_0$, are close by an exponential estimate of the type (4), hence, by Remark 13 they are asymptotically equivalent.

The next result is a direct consequence of the fact that a simple root of a polynomial is a locally Lipschitz function of its coefficients:

Theorem 15. *Assume that $p^{[k]}(z)$ and $p(z)$ are polynomials of degree $\leqslant m$ for each $k \in \mathbb{N}_0$ and assume that for each $R > 0$ there exists a constant $C_m(R)$ such that for all $k \in \mathbb{N}_0$ and $|z| \leqslant R$*

$$\left| p^{[k]}(z) - p(z) \right| \leqslant C_m(R) 2^{-k}. \tag{7}$$

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If α is a simple zero of $p(z)$ then there exists $k_0 \in \mathbb{N}_0$ and a constant $\rho > 0$ such that for each natural number $k \geqslant k_0$ there exists a zero $\alpha^{[k]}$ of $p^{[k]}(z)$ with

$$\left| \alpha^{[k]} - \alpha \right| \leqslant \rho 2^{-k} \quad \text{for all } k \geqslant k_0.$$

3. Subdivision schemes based on exponential interpolation and regularity of the basic limit function

The classical $2n$-point Deslauriers–Dubuc subdivision scheme is defined via interpolation of polynomials of degree $2n - 1$, see [9]. We shall denote its symbol by $D_{2n}(z)$ which has the form

$$D_{2n}(z) = \sum_{|j| \leqslant 2n-1} p_j z^j. \tag{8}$$

The scheme of Deslauriers and Dubuc is interpolatory, i.e., $p_{2j} = \delta_{0,j}$, for all $j \in \mathbb{Z}$, or equivalently

$$D_{2n}(z) + D_{2n}(-z) = 2, \quad \text{for all } z \in \mathbb{C} \setminus \{0\}. \tag{9}$$

According to [9] the symbol $D_{2n}(z)$ satisfies condition

$$\frac{d^j}{dz^j} D_{2n}(-1) = 0, \quad \text{for } j = 0, \ldots, 2n - 1. \tag{10}$$

Together, conditions (10) and (9) constitute a linear system which uniquely determines the symbol $D_{2n}(z)$ of the Deslauriers and Dubuc scheme and it can be written in the form

$$D_{2n}(z) = \left(\frac{1+z}{2} \right)^{2n} b_{D_{2n}}(z).$$

Let us mention that condition (10) means that polynomials of degree $\leqslant 2n - 1$ are reproduced by the subdivision scheme.

Now we turn to subdivision schemes for exponential polynomials. Let L be the linear differential operator given by

$$L = \left(\frac{d}{dx} - \lambda_0 \right) \cdots \left(\frac{d}{dx} - \lambda_N \right).$$

Complex-valued solutions f of the equation $Lf = 0$ are called *L-polynomials* or *exponential polynomials* or just *exponentials*. We shall denote the set of all solutions of $Lf = 0$ by $E(\lambda_0, \ldots, \lambda_N)$ which is a linear span of the following set of functions:

$$\left\{ x^k e^{\lambda_j x} : 0 \leqslant j \leqslant N, \ 0 \leqslant k \leqslant \mu_j - 1 \right\};$$

here μ_j is the multiplicity of λ_j in the vector $(\lambda_0, \ldots, \lambda_N)$, i.e. the number of times the value λ_j occurs in $(\lambda_0, \ldots, \lambda_N)$.

Throughout this article we shall assume that $\lambda_0, \ldots, \lambda_N$ are real numbers. This assumption implies that $E(\lambda_0, \ldots, \lambda_N)$ is a space spanned by an **extended Chebyshev system**, cf. [31,24]. In particular, for any pairwise distinct points $t_0, \ldots, t_N$ and data values $y_0, \ldots, y_N$, there exists a unique element $p \in E(\lambda_0, \ldots, \lambda_N)$ with $p(t_j) = f_j(t_j)$ for $j = 0, \ldots, N$, i.e. p is interpolation exponential polynomial.

Given real numbers $\lambda_0, \ldots, \lambda_{2n-1}$ one can define the *subdivision scheme based on interpolation in* $E(\lambda_0, \ldots, \lambda_{2n-1})$: the new value $f^{k+1}(j/2^{k+1})$ is computed by constructing the unique function $p_j^k \in$

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$E(\lambda_0, \ldots, \lambda_{2n-1})$ interpolating the previous data $f^k((j + l)/2^k)$ for $l = -n + 1, \ldots, n$, and putting $f^{k+1}(j/2^{k+1}) = p_j^k(j/2^{k+1})$ (see [28] for details). Then the symbols of this scheme are of the form

$$a^{[k]}(z) = \sum_{|j| \leqslant 2n-1} a_j^{[k]} z^j$$

and since the scheme is interpolatory one has

$$a^{[k]}(z) + a^{[k]}(-z) = 2, \quad z \in \mathbb{C} \setminus \{0\}. \tag{11}$$

Due to the interpolatory definition of the subdivision it is clear that each function $f \in E(\lambda_0, \ldots, \lambda_{2n-1})$ is reproduced step-wise by the scheme, and Theorem 9 implies that

$$\frac{d^s}{dz^s} a^{[k]}\left(-\exp\left(-2^{-(k+1)}\lambda_j\right)\right) = 0, \quad s = 0, \ldots, \mu_j - 1, \tag{12}$$

where μ_j is the *multiplicity* of λ_j. Hence the subdivision scheme based on interpolation in $E(\lambda_0, \ldots, \lambda_{2n-1})$ is completely characterized by (11) and (12). In the terminology of [13] this is the *even-order, symmetric and minimal rank scheme reproducing* $E(\lambda_0, \ldots, \lambda_{2n-1})$. Note that the Deslauriers–Dubuc scheme is a special case by taking $\lambda_0 = \cdots = \lambda_{2n-1} = 0$, reproducing the space of algebraic polynomials Π_{2n-1}. Let us put

$$z_j^{[k]} = \exp\left(-2^{-(k+1)}\lambda_j\right), \quad j = 0, \ldots, 2n - 1.$$

By means of (12) we can write

$$a^{[k]}(z) = \left(\prod_{j=1}^{2n} \frac{z + z_j^{[k]}}{2}\right) b^{[k]}(z) \tag{13}$$

which defines the important functions $b^{[k]}(z)$ to be used later on.

Definition 16. For given real numbers $\lambda_0, \ldots, \lambda_{2n-1}$ let us denote by $S_0^{\Lambda_{2n}}$ the subdivision scheme with symbols $a^{[k]}$, $k \in \mathbb{N}_0$, satisfying (11) and (12).

In the rest of the paper we shall use the notation given in Definition 16. According to Definition 4, for any natural number $m \in \mathbb{N}_0$, $S_m^{\Lambda_{2n}}$ is a subdivision scheme given by the symbols

$$a^{[k],m}(z) = a^{[k+m]}(z) \quad \text{for } k \in \mathbb{N}_0. \tag{14}$$

Remark 17. It is easy to see that the subdivision scheme $S_m^{\Lambda_{2n}}$ is again an *even-order, symmetric and minimal rank scheme reproducing* $E(\lambda_0/2^m, \ldots, \lambda_{2n-1}/2^m)$.

Many properties of the subdivision scheme $S_m^{\Lambda_{2n}}$ for exponential polynomials can be derived from its polynomial counterpart, the Deslauriers–Dubuc scheme. The key to these results is found in the following observation in [13, Theorem 2.7].

Proposition 18. *The subdivision scheme $S_m^{\Lambda_{2n}}$ and the 2n-point Deslauriers–Dubuc subdivision scheme are asymptotically equivalent in the sense of Definition* 12. *Even stronger statement holds: there exists a constant $C > 0$ such that*

$$\sum_{|j| \leqslant 2n-1} \left|p_j - a_j^{[k],m}\right| \leqslant C 2^{-k} \quad \text{for all } k \in \mathbb{N}_0, \tag{15}$$

where p_j denotes the jth coefficient of the mask of D_{2n}, see (8).

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The asymptotic equivalence follows from the last inequality, by applying Remark 13.

Deslauriers and Dubuc showed in [9] that their scheme D_{2n} for $n \geqslant 1$ is C^0-convergent implying the existence of a basic limit function which will be denoted in the following by $\Phi^{D_{2n}}$. Furthermore, one can find sufficient conditions for C^ℓ-convergence in [9].

By Remark 17 we may apply Theorem 2.10 in [13] and obtain the following result:

Theorem 19. *Let $\ell \in \mathbb{N}_0$. If the Deslauriers–Dubuc subdivision scheme of order $2n$ is C^ℓ-convergent then the subdivision scheme $S_m^{\Lambda_{2n}}$ is C^ℓ-convergent as well.*

According to the last theorem the subdivision scheme $S_m^{\Lambda_{2n}}$ has a basic limit function which will be denoted in the following by $\Phi_m^{\Lambda_{2n}}$.

A function $f : \mathbb{R}^d \to \mathbb{C}$ is called *Lipschitz function of order $\alpha \in (0,1)$* (or *Hölder function of order α*) if there exists a number $L > 0$ such that for all $x, y \in \mathbb{R}^d$

$$\big|f(x) - f(y)\big| \leqslant L|x - y|^\alpha.$$

The set of all Lipschitz functions of order α is denoted by $Lip(\alpha)$. Lemma 7.1 in [9, p. 56] provides a sufficient condition for a function to belong to the space $Lip(\alpha)$ in terms of Fourier transform:

Lemma 20. *Let $f : \mathbb{R} \to \mathbb{C}$ be an integrable function whose Fourier transform is $g(\xi)$. We assume that $|\xi|^{\ell+\alpha} g(\xi)$ is integrable where $\ell \in \mathbb{N}_0$ and $\alpha \in [0,1)$. If $\alpha = 0$ then f is ℓ times continuously differentiable. If $\alpha \neq 0$, then $f^{(\ell)}$ is a Lipschitz function of order α.*

We prove one of the main results of the paper.

Theorem 21. *Let $\alpha \in [0,1)$ and $\ell \in \mathbb{N}_0$ and assume that the basic limit function $\Phi^{D_{2n}}$ of the $2n$-point Deslauriers–Dubuc scheme satisfies for some $\varepsilon > 0$ and $C > 0$ the inequality*

$$\big|\widehat{\Phi^{D_{2n}}}(\omega)\big| \leqslant C\big(|\omega| + 1\big)^{-\ell-1-\alpha-\varepsilon}$$

for all $\omega \in \mathbb{R}$. Then the basic limit function $\Phi_m^{\Lambda_{2n}}$ of the scheme $S_m^{\Lambda_{2n}}$ (defined by the symbols in formula (14)) has its ℓth derivative in $Lip(\alpha)$.

Proof. In formula (13) we have defined the function $b^{[k]}(z)$. Let $D_{2n}(z) = (\frac{1+z}{2})^{2n} b_{D_{2n}}(z)$ be the symbol of the Deslauriers–Dubuc scheme. It was shown in [13, formula (2.32)] that $b^{[k]}$ is asymptotically equivalent to $b_{D_{2n}}$, by proving an estimate as in (5): there exists a constant $B > 0$ such that

$$\big|b^{[k]}\big(e^{i\omega}\big) - b_{D_{2n}}\big(e^{i\omega}\big)\big| \leqslant B \cdot 2^{-k} \tag{16}$$

for all $\omega \in \mathbb{R}$ and for all $k \in \mathbb{N}_0$. As an intermediate step we consider the non-stationary scheme S_m^c defined by the symbols

$$c_m^{[k]}(z) = \left(\frac{z+1}{2}\right)^{2n} b^{[m+k]}(z). \tag{17}$$

In the proof of Theorem 2.10 in [13] it is shown that the scheme S_m^c has a basic limit function denoted by Φ_m^c. Let $\Phi_m^{\Lambda_{2n}}$ be the basic limit function of $S_m^{\Lambda_{2n}}$. By Proposition 7

$$\widehat{\Phi_m^{\Lambda_{2n}}}(\omega) = \prod_{k=1}^{\infty} \frac{1}{2} a^{[m+k-1]}\big(e^{i\omega 2^{-k}}\big).$$

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Similarly, $\widehat{\Phi^{D_{2n}}}(\omega) = \prod_{k=1}^{\infty} \frac{1}{2} a(e^{i\omega 2^{-k}})$ and we see that

$$\frac{\widehat{\Phi_m^c}(\omega)}{\widehat{\Phi^{D_{2n}}}(\omega)} = \prod_{k=1}^{\infty} \frac{b^{[m+k-1]}(e^{i\omega 2^{-k}})}{b_{D_{2n}}(e^{i\omega 2^{-k}})} \tag{18}$$

$$= \prod_{k=1}^{\infty} \left(1 + \frac{b^{[m+k-1]}(e^{i\omega 2^{-k}}) - b_{D_{2n}}(e^{i\omega 2^{-k}})}{b_{D_{2n}}(e^{i\omega 2^{-k}})} \right). \tag{19}$$

It is well known [9] that the trigonometric polynomial $b_{D_{2n}}(e^{-i\omega 2^{-k}})$ does not vanish on the unit circle, hence the denominator in (19) satisfies

$$\left| b_{D_{2n}}(e^{i\omega 2^{-k}}) \right| \geqslant \delta > 0 \tag{20}$$

for some $\delta > 0$. Using (20) and (16) it is straightforward to prove that the infinite product in (18) is uniformly bounded for $\omega \in \mathbb{R}$, and we obtain

$$\left| \frac{\widehat{\Phi_m^c}(\omega)}{\widehat{\Phi^{D_{2n}}}(\omega)} \right| \leqslant M_m.$$

Recalling that each factor $\frac{1+z}{2}$ induces convolution with a B-spline of order 0 (denoted as usual by B_0) the above inequality can be written as

$$\left| \widehat{\Phi_m^c}(\omega) \right| = \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2n} \prod_{k=1}^{\infty} \left| b^{[m+k-1]}(e^{i\omega 2^{-k}}) \right| \leqslant M_m \left| \widehat{\Phi^{D_{2n}}}(\omega) \right|$$

$$\leqslant M_n C \left(|\omega| + 1 \right)^{-\ell - 1 - \alpha - \varepsilon}. \tag{21}$$

Comparing (13) and (17) we see that in order to prove the theorem we need to replace each factor $\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$ by the Fourier transform of the basic limit function generated by the subdivision scheme with symbols $\{\frac{z+z_j^{[k]}}{2}\}$. This should be done with care since $\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$ vanishes at infinite number of points on $\mathbb{R}$. We employ another observation from [15], that the basic limit function Φ_0 of the scheme with symbols $\{\frac{z+z_j^{[k]}}{2}\}$ is an exponential B-spline of order 0,

$$B_0^j(\omega) := \begin{cases} e^{\lambda_j x} & \text{for } x \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Adding such factors, $\{\frac{z+z_j^{[k]}}{2}\}_{j=1}^{2n}$ to the symbols $\{c^{[k]}\}$ results in repeated convolutions of $\widehat{\Phi_m^c}$ with $\{B_0^j\}_{j=1}^{2n}$. Since $\widehat{B_0^j}(x)$ decays as $\frac{1}{|x|}$ for $|\omega| \to \infty$, each convolution adds one power to the decay power α in (21). Hence, after $2n$ convolutions we obtain

$$\widehat{\Psi_m}(\omega) := \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2n} \prod_{k=1}^{\infty} a^{[m+k-1]} b(e^{-i\omega 2^{-k}})$$

$$= \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2n} \widehat{\Phi_m^{\Lambda_{2n}}}(\omega) = O\left((|\omega| + 1)^{-\ell - 1 - \alpha - 2n} \right).$$

It follows by Lemma 20 that the $(2n + \ell)$th derivative of Ψ_m is $Lip(\alpha)$. Now we are ready to return to $\widehat{\Phi_m^{\Lambda_{2n}}}$ by removing the factors $(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}})^{2n}$ from $\widehat{\Psi_m}(\omega)$. We need to show that each factor removed implies that the

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order of the derivative which is in $Lip(\alpha)$ is reduced by one. To show this we consider $g = B_0 * f$, where f is a function of compact support and g' is in $Lip(\alpha)$. It follows that

$$g'(s) = f(s) - f(s-1).$$

Summing the above relations over all numbers $s \in \{t+j\}_{j=0}^{N}$, for large enough N, we obtain

$$f(t) = \sum_{j=0}^{N} g'(t+j),$$

implying that f is $Lip(\alpha)$. Since $\hat{g}(\omega) = \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \hat{f}(\omega)$, we have just shown that the consequence of removing a factor $\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$ is that the order of the derivative which is in $Lip(\alpha)$ is reduced by one. Removing $2n$ such factors yields the desired result, namely, that the ℓth derivative of $\{\Phi_m^{\Lambda_{2n}}\}$ is in $Lip(\alpha)$. □

Remark 22. In [11,18–21] motivated by the multivariate polyharmonic subdivision and wavelets on parallel hyperplanes, an explicit expression for the polynomial $b^{[k]}(z)$ is found and used, for the case of parameters given by $\lambda_j = \xi$, $j = 0, 1, \ldots, n - 1$, for some real $\xi \geqslant 0$, and $\lambda_j = -\xi$, for $j = n, \ldots, 2n - 1$. The classical Deslauriers–Dubuc case corresponds to $\xi = 0$.

4. Non-stationary multiresolution analysis

The concept of a multiresolution analysis, introduced by S. Mallat and Y. Meyer, is an effective tool to construct wavelets in a simple way from a given scaling function φ, see e.g. [2,7]. *Non-stationary multiresolution analysis* was introduced in [8] by C. de Boor, R. DeVore and A. Ron. A systematic investigation of non-stationary MRA can be found in [29], see also [1]. For convenience of the reader we recall here the definition for the univariate case:

Definition 23. A non-stationary multiresolution analysis consists of a sequence of closed subspaces V_m, $m \in \mathbb{Z}$, in $L^2(\mathbb{R})$ satisfying

 (i) $V_m \subset V_{m+1}$ for all $m \in \mathbb{Z}$,
 (ii) the intersection $\bigcap_{m \in \mathbb{Z}} V_m$ is the trivial subspace $\{0\}$,
 (iii) the union $\bigcup_{m \in \mathbb{Z}} V_m$ is dense in $L^2(\mathbb{R})$,
 (iv) for each $m \in \mathbb{Z}$ there exists a function $\varphi_m \in V_m$ such that the family of functions $\{\varphi_m(2^m t - k) \colon k \in \mathbb{Z}\}$ form a Riesz basis of V_m.

The function φ_m in condition (iv) is called a *scaling function* for V_m. The requirement (iv) means that for each $f \in V_m$ there exists a unique sequence $(c_k)_{k \in \mathbb{Z}}$ in $l^2(\mathbb{Z})$ (i.e., $\sum_{k=-\infty}^{\infty} |c_k|^2 < \infty$) such that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \varphi_m(2^m t - k)$$

with convergence in $L^2(\mathbb{R})$ and

$$A_m \sum_{k=-\infty}^{\infty} |c_k|^2 \leqslant \left\| \sum_{k=-\infty}^{\infty} c_k \varphi_m(2^m t - k) \right\|^2 \leqslant B_m \sum_{k=-\infty}^{\infty} |c_k|^2$$

for all $(c_k)_{k \in \mathbb{Z}}$ in $l^2(\mathbb{Z})$ with $0 < A_m \leqslant B_m < \infty$ constants independent of $f \in V_m$.

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The *wavelet space* W_m is the unique subspace such that $V_m \oplus W_m = V_{m+1}$ for $m \in \mathbb{Z}$ and W_m is orthogonal to V_m. Then W_k and W_m are orthogonal subspaces for $k \neq m$ and conditions (ii) and (iii) imply that

$$L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m.$$

Definition 24. A multiresolution analysis is called orthonormal if in condition (iv) the functions $t \mapsto 2^{m/2}\varphi_m(2^m t - k)$ for $k \in \mathbb{Z}$ are an orthonormal basis of V_m.

The general aim in non-stationary wavelet analysis is to find a sequence of functions $\psi_m \in L^2(\mathbb{R})$, $m \in \mathbb{Z}$, such that the set of functions

$$\psi_{m,k}(x) = 2^{m/2}\psi_m(2^m x - k)$$

with $m, k \in \mathbb{Z}$, is an orthonormal basis of $L^2(\mathbb{R})$.

Important examples of non-stationary MRA occur in the context of cardinal exponential-spline wavelets which generalizes the work of C.K. Chui and J.Z. Wang about cardinal spline wavelets in [5,3,4]. The interested reader may consult [27,31] for the theory of exponential splines and [8,17,22,23,26,34] for the construction of wavelets in this context.

5. The scaling functions for generalized Daubechies wavelets

Daubechies wavelets ψ are orthonormal wavelets with compact support and certain degree of smoothness, see e.g. [2,16,25]. Using the concept of an orthonormal MRA it suffices to construct a suitable scaling function φ. It is well known that the Fourier transform $\widehat{\varphi}$ of the scaling function φ should be of the form

$$\widehat{\varphi}(\omega) = \prod_{k=1}^{\infty} H(2^{-k}\omega)$$

where $H(\omega)$ is a trigonometric polynomial with real coefficients and $H(0) = 1$ satisfying the equation

$$\left|H(\omega)\right|^2 + \left|H(\omega + \pi)\right|^2 = 1.$$

This leads to the question which *non-negative* trigonometric polynomials $q(\omega)$ satisfy an equation of the type

$$q(\omega) + q(\omega + \pi) = 1 \quad \text{and} \quad q(0) = 1. \tag{22}$$

There are many explicit solutions of (22). For example, if n is a natural number then the trigonometric polynomial

$$q_n(\omega) = 1 - c_n \int_0^\omega (\sin t)^{2n-1}\, dt$$

with $c_n := \int_0^\pi \sin^{2n-1} t\, dt$ satisfies Eq. (22). By the Fejér–Riesz lemma one can find a (non-unique) trigonometric polynomial $H(\omega)$ such that

$$q_n(\omega) = \left|H(\omega)\right|^2. \tag{23}$$

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We call a Laurent polynomial $H(\omega)$ with real coefficients and $H(0) = 1$ satisfying (23) a *Daubechies filter of order n*. The *Daubechies scaling function* φ^H for the Daubechies filter $H(\omega)$ is then defined by

$$\widehat{\varphi^H}(\omega) = \prod_{k=1}^{\infty} H\big(2^{-k}\omega\big).$$

Let us emphasize that I. Daubechies has shown more (see e.g. [7, p. 210] or [36]): the regularity of the wavelet and the scaling function imply that the symbol $H(\omega)$ must contain a factor $(1 + e^{i\omega})^n/2^n$. Hence $q_n(\omega)$ is of the form

$$q_n(\omega) = \frac{(1 + e^{i\omega})^{2n}}{2^n} F_{2n-1}(\omega)$$

where $F_{2n-1}(\omega)$ is a suitable trigonometric polynomial with real coefficients which can be determined by Bezout's theorem from (22). Indeed, it follows from these considerations that

$$q_n(\omega) = D_{2n}\big(e^{i\omega}\big)$$

where D_{2n} is the symbol of the Deslauriers–Dubuc subdivision scheme, a fact which is already mentioned by Daubechies in her book [7, Section 6.5] giving credit to this observation to M.J. Shensa in [32], see [7, p. 210]. *Hence the Deslauriers–Dubuc scheme leads in a very natural and direct way to the construction of the Daubechies scaling function and therefore, by MRA-methods, to Daubechies wavelets.*

Now we want to use this concept for introducing Daubechies type wavelets for exponential polynomials. In this setting we have some additional freedom which is interesting for applications: we may choose real numbers $\lambda_0, \ldots, \lambda_{n-1}$ and we shall construct *Daubechies type wavelets reconstructing the space* $E(\lambda_0, \ldots, \lambda_{n-1})$. In the case of Daubechies wavelets this corresponds to the fact that the Daubechies wavelet reproduces polynomials of degree $\leqslant n - 1$.

We shall write shortly $\Lambda_0 = (\lambda_0, \ldots, \lambda_{n-1})$ and define $\lambda_{n+j} := -\lambda_j$ for $j = 0, \ldots, n - 1$. We consider now the subdivision scheme based on interpolation in $E(\lambda_0, \ldots, \lambda_{2n-1})$. According to Definition 16 the subdivision scheme $S_0^{\Lambda_{2n}}$ has the symbols

$$a^{[k]}(z) = \prod_{j=0}^{2n-1} \frac{z + z_j^{[k]}}{2} b^{[k]}(z) \quad \text{with } z_j^{[k]} := \exp\big(-2^{-(k+1)}\lambda_j\big) \tag{24}$$

for $k \in \mathbb{N}_0$ and $j = 0, \ldots, 2n - 1$. For subdivision, a crucial role plays the result by Micchelli in [28, Proposition 5.1], according to which the symbols $a^{[k]}$ satisfy

$$a^{[k]}(z) \geqslant 0, \quad \text{for } |z| = 1,$$

with equality possible only if $z = -1$. By the Fejér–Riesz lemma we can find for each level k a "square root" Laurent polynomial $M^{[k]}(z)$ with *real* coefficients, satisfying

$$a^{[k]}\big(e^{i\omega}\big) = \frac{1}{2}\big|M^{[k]}\big(e^{i\omega}\big)\big|^2 \quad \text{and} \quad M^{[k]}(1) > 0. \tag{25}$$

Note that this implies that

$$a^{[k]}(z) = \frac{1}{2}M^{[k]}(z)M^{[k]}\left(\frac{1}{z}\right)$$

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for all complex $z \neq 0$. Again, there are many Laurent polynomials $M^{[k]}(z)$ which satisfy (25) and all possible choices can be described through suitable subsets of the zero-set of $a^{[k]}(z)$. First we choose the roots $z = -\exp(-\lambda_j/2^{k+1})$ for $j = 0, \ldots, n-1$, in order to obtain step-wise reproduction of the space $E(\lambda_0, \ldots, \lambda_{n-1})$, see Proposition 25 below. Further, we have to choose another $n-1$ roots of the factor $b^{[k]}$ in (24). Since $b^{[k]}$ is symmetric, its $2n-2$ roots come in inverse pairs, say z_i and z_i^{-1}, and as well complex conjugates $\overline{z_i}$ and $\overline{z_i}^{-1}$ if z_i is not real, for i in an index set I_{n-1}. We choose either the set $\{z_i, \overline{z_i}\}$ or the set $\{z_i^{-1}, \overline{z_i}^{-1}\}$ for each $i \in I_{n-1}$, leading to a Laurent polynomial with real coefficients which still has to be normalized so that $M^{[k]}(1) = \sqrt{2a^{[k]}(1)} > 0$. We shall call a sequence of filters $M^{[k]}(z)$, $k \in \mathbf{N}_0$, chosen in this way a *non-stationary Daubechies type subdivision scheme of order n*.

Since $M^{[k]}(1)$ is positive it follows that $1 \leqslant \frac{1}{2}M^{[k]}(1) + 1$. Therefore, $a^{[k]}(1) = \frac{1}{2}M^{[k]}(1)^2$ implies the inequality

$$\left| \frac{1}{2}M^{[k]}(1) - 1 \right| \leqslant \left| \frac{1}{2}M^{[k]}(1) - 1 \right| \left| \frac{1}{2}M^{[k]}(1) + 1 \right| = \left| \frac{1}{2}a^{[k]}(1) - 1 \right|.$$

By inequality (15) (which implies the asymptotic equivalence of $a^{[k]}(z)$ to the Deslauriers–Dubuc scheme in Proposition 18) and equality $\frac{1}{2}D_{2n}(1) = 1$ we infer that there exists $C > 0$ such that for all $k \in \mathbb{N}_0$

$$\left| \frac{1}{2}M^{[k]}(1) - 1 \right| \leqslant C \cdot 2^{-k}. \tag{26}$$

At first we notice the following result:

Proposition 25. *Let $\lambda_0, \ldots, \lambda_{n-1}$ be real numbers. Then there exists $k_0 \in \mathbb{N}_0$ such that the Daubechies type subdivision scheme reproduces step-wise functions in $E(\lambda_0, \ldots, \lambda_{n-1})$ for all levels $k \geqslant k_0$.*

Proof. Let $z_j^{[k]} = \exp(-\lambda_j/2^{k+1})$. By construction $M^{[k]}(z)$ has a zero at $-z_j^{[k]}$ of multiplicity μ_j, the number of times λ_j occurs in $(\lambda_0, \ldots, \lambda_{n-1})$, hence

$$\frac{d^s}{dx^s} M^{[k]}\big(-z_j^{[k]}\big) = 0 \quad \text{for } s = 0, \ldots, \mu_j - 1 \text{ and } j = 1, \ldots, n-1. \tag{27}$$

By (25) and the fact that $\{a^{[k]}\}$ reproduces step-wise functions in $E(\lambda_0, \ldots, \lambda_{2n-1})$ we conclude that

$$\frac{1}{2}\big|M^{[k]}\big(z_j^{[k]}\big)\big|^2 = a^{[k]}\big(z_j^{[k]}\big) = 2.$$

Since $z_j^{[k]}$ is real and $M^{[k]}(z)$ has real coefficients it follows that $M^{[k]}(z_j^{[k]})$ is real so $M^{[k]}(z_j^{[k]}) = 2$ or -2. Since $z_j^{[k]}$ converges to 1 for $k \to \infty$ and $M^{[k]}(z_j^{[k]})$ converges to $M^{D_{2n}}(1) > 0$ there exists $k_0 \in \mathbf{N}_0$ such that $M^{[k]}(z_j^{[k]}) > 0$ for all $k \geqslant k_0$ and $j = 1, \ldots, n-1$. Hence $M^{[k]}(z_j^{[k]}) = 2$ for all $k \geqslant k_0$. From (25) we infer that for real x

$$\frac{d^s}{dx^s} a^{[k]}(x) = \sum_{r=0}^{s} \binom{s}{r} \frac{d^r}{dx^r} M^{[k]}(x) \cdot \frac{d^{s-r}}{dx^{s-r}} M^{[k]}(x).$$

For $s = 1$ this means that $0 = M^{[k]}(z_j^{[k]})\frac{d}{dx}M^{[k]}(z_j^{[k]}) + \frac{d}{dx}M^{[k]}(z_j^{[k]}) \cdot M^{[k]}(z_j^{[k]})$. Since $M^{[k]}$ has real coefficients and $z_j^{[k]}$ is real we conclude that $\frac{d}{dx}M^{[k]}(z_j^{[k]}) = 0$. Inductively we obtain that $\frac{d^s}{dx^s}M^{[k]}(z_j^{[k]}) = 0$ for $s = 1, \ldots, \mu_j - 1$. $\square$

Proposition 26. *The product $\prod_{k=1}^{\infty} \frac{1}{2}M^{[k-1]}(e^{i\frac{\omega}{2^k}})$ converges.*

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Proof. By construction $M^{[k]}(e^{i\omega})$ has real coefficients and $M^{[k]}(1) > 0$. Since $a^{[k]}(e^{i\omega}) \geqslant 0$ we infer from (11) that $|a^{[k]}(e^{i\omega})| \leqslant 2$, and therefore $|M_k(e^{i\omega})| \leqslant 2$. Moreover, it follows from (26) that

$$\sum_{k=1}^{\infty} \left| \frac{1}{2} M^{[k]}(1) - 1 \right| \leqslant C \sum_{k=1}^{\infty} 2^{-k}.$$

Proposition 6 finishes the proof. $\square$

6. Regularity of the generalized Daubechies scaling function

In order to obtain asymptotic equivalence for the non-stationary Daubechies subdivision scheme we have to choose the filter $M^{[k]}(z)$ with more care:

Theorem 27. *Let $M^{D_{2n}}$ be the Daubechies filter of order n as defined above and let $\lambda_0, \ldots, \lambda_{n-1}$ be real numbers. Then there exists a non-stationary Daubechies type subdivision scheme $\{M^{[k]}(z)\}$ which is asymptotically equivalent to $M^{D_{2n}}$. Moreover, there exists a natural number k_0 such that this scheme reproduces the space $E(\lambda_0, \ldots, \lambda_{n-1})$ step-wise for all $k \geqslant k_0$.*

Proof. Note that $M^{D_{2n}}(z)$ has n zeros at -1 and $n-1$ other zeros, say $\alpha_1, \ldots, \alpha_{n-1}$ which are of course zeros of the factor $b_{D_{2n}}(z)$ of the Deslauriers–Dubuc symbol $D_{2n}(z) = (\frac{1+z}{2})^{2n} b_{D_{2n}}(z)$. It is well-known that $\alpha_1, \ldots, \alpha_{n-1}$ are pairwise different and simple zeros of $D_{2n}(z)$. Recall that by Proposition 18, $a^{[k]}(z)$ is asymptotically equivalent to the symbols $D_{2n}(z)$ by means of an estimate of the type (4). Then $z^{2n-1}a^{[k]}(z)$ are polynomials and $z^{2n-1}a^{[k]}(z)$ is obviously asymptotically equivalent to $z^{2n-1}D_{2n}(z)$ by means of a similar estimate. By Theorem 15 there exists a constant $C > 0$ and a zero $\alpha_j^{[k]}$ of $a^{[k]}(z)$ such that $|\alpha_j^{[k]} - \alpha_j| \leqslant C2^{-k}$ for all $k \in \mathbb{N}_0$ and $j = 1, \ldots, n-1$. Take $k_0 \in \mathbb{N}_0$ large enough so that:

(i) for each $k \geqslant k_0$ the balls $|z - \alpha_j| \leqslant C2^{-k}$ have empty intersection with the unit circle,
(ii) they are pairwise disjoint for $j = 1, \ldots, n-1$, and
(iii) they have empty intersection with the x-axis if α_j is a non-real zero.

Then for each $k \geqslant k_0$ there is for given α_j exactly one zero $\alpha_j^{[k]}$ with

$$\left| \alpha_j^{[k]} - \alpha_j \right| \leqslant C2^{-k} \quad \text{for } j = 1, \ldots, n-1, \tag{28}$$

leading to a *unique choice* for $M^{[k]}(z)$ for $k \geqslant k_0$. Further the leading coefficient $c^{[k]}$ of the polynomial $z^{2n-1}M^{[k]}(z)$ is determined by the equation

$$M^{[k]}(1) = c^{[k]} \prod_{j=0}^{n-1} (1 + z_j^{[k]}) \cdot \prod_{j=1}^{n-1} (1 - \alpha_j^{[k]}). \tag{29}$$

Let c be the leading coefficient of $z^{2n-1}D_{2n}(z)$. By (26) and (28) and (29) it is easy to see that there exists $D > 0$ such that

$$\left| c^{[k]} - c \right| \leqslant D2^{-k}$$

for all $k \in \mathbb{N}_0$. By Theorem 14 the subdivision scheme defined by the symbols $z^{2n-1}M^{[k]}(z)$, $k \in \mathbb{N}_0$, is asymptotically equivalent to the scheme defined by the symbol $z^{2n-1}M^{D_{2n}}$, by means of an exponential estimate (4). This implies that $M^{[k]}(z)$, $k \in \mathbb{N}_0$, is asymptotically equivalent to $M^{D_{2n}}$ by means of a similar estimate. This ends the proof. $\square$

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Remark 28. Assume that $M^{D_{2n}}$ is the Daubechies filter such that all zeros $\neq -1$ have absolute value bigger than 1. Then one can define $M^{[k]}(z)$ in the last theorem by the condition that all its non-trivial zeros have absolute value bigger than 1.

It thus follows, from the theory of asymptotically equivalent schemes in [15], that the scheme with symbols $\{M^{[k+m]}(z),\ k \in \mathbb{N}_0\}$ defines continuous basic limit functions $\{\varphi_m^{\Lambda_0}(\cdot)\}$. Proposition 7 shows that

$$\widehat{\varphi_m^{\Lambda_0}}(\omega) = \prod_{k=1}^{\infty} \frac{M^{[m+k-1]}(e^{i\frac{\omega}{2^k}})}{2}. \tag{30}$$

In particular, we have

$$\left|\widehat{\varphi_m^{\Lambda_0}}(\omega)\right|^2 = \widehat{\Phi_m^{\Lambda_{2n}}}(\omega) \tag{31}$$

where $\Phi_m^{\Lambda_{2n}}$ is the basic limit function of $S_m^{\Lambda_{2n}}$.

Theorem 29. *Let $M^{D_{2n}}(z)$ be a Daubechies filter of order n and assume that $M^{[k]}(z)$ is as in Theorem 27. Let $\alpha \in [0,1)$ and $\ell \in \mathbb{N}_0$, and assume that the scaling function $\varphi^{D_{2n}}$ of Daubechies defined by the symbol $M^{D_{2n}}(z)$, satisfies for some $\varepsilon > 0$ and $C > 0$ the inequality*

$$\left|\widehat{\varphi^{D_{2n}}}(\omega)\right| \leqslant C\big(|\omega| + 1\big)^{-\ell-1-\alpha-\varepsilon}$$

for all $\omega \in \mathbb{R}$. Then the scaling function $\varphi_m^{\Lambda_0}$ associated to the subdivision scheme $M^{[k+m]}(z),\ k \in \mathbb{N}_0$, has ℓth derivative in $Lip(\alpha)$.

Proof. Let us define $z_j^{[k]} = \exp(-2^{-(k+1)}\lambda_j)$. Then the symbol $M^{[k]}(z)$ can be written as

$$M^{[k]}(z) = \left(\prod_{j=1}^{n} \frac{z + z_j^{[k]}}{2}\right) B^{[k]}(z).$$

Similarly we can write for the Daubechies filter

$$M^{D_{2n}}(z) = \left(\prod_{j=1}^{n} \frac{z + 1}{2}\right) B^{D_{2N}}(z).$$

We apply Proposition 18 and inequality (15) there to obtain that $M^{[k]}(z)$ is asymptotically equivalent to the Daubechies filter $M^{D_{2n}}$. Since $B^{[k]}(z)$ has only simple zeros, it follows from Theorems 15 and 14 that there exists a constant $C > 0$ such that

$$\left|B^{[k]}\big(e^{i\omega}\big) - B^{D_{2N}}(z)\big(e^{i\omega}\big)\right| \leqslant C \cdot 2^{-k},$$

hence, by Remark 13, we prove that $B^{[k]}$ is asymptotically equivalent to $B^{D_{2N}}(z)$. Now one can proceed as in Theorem 21. $\square$

7. Construction and regularity of generalized Daubechies wavelets

The construction will follow the classical pattern in MRA. Below we only outline the main procedure, while for the standard details we refer to [35,12].

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The following fact is straightforward:

Proposition 30. *The functions $2^{m/2}\varphi_m^{\Lambda_0}(2^m \cdot -k)$, $m, k \in \mathbb{Z}$, are orthonormal.*

Remark 31. For completeness sake, let us mention that the numerical stability of our subdivision scheme obviously follows from the above statement, however in [35] the numerical stability for biorthogonal systems follows from the Riesz-basis property.

Definition 32. For each $m \in \mathbb{N}_0$ we define the linear spaces V_m by

$$V_m := \left\{ f \in L^2(\mathbb{R}) \ \Big|\ f(t) = \sum_{j \in \mathbb{Z}} c_j \varphi_m^{\Lambda_0}(2^m t - j),\ \sum_{j \in \mathbb{Z}} |c_j|^2 < \infty \right\}.$$

We remark that we could also define V_m for integers $m \in \mathbb{Z}$ since the symbols $a^{[k+m]}(z)$ and the scaling functions $\Phi_m^{\Lambda_{2n}}$ and $\varphi_m^{\Lambda_0}$ could be defined for all $m \in \mathbb{Z}$.

Proposition 33. *The spaces V_m are nested, i.e., $V_m \subset V_{m+1}$ for all $m \in \mathbb{N}_0$.*

Proof. Using the product representation (30) we obtain

$$\widehat{\varphi_m^{\Lambda_0}}(\omega) = \prod_{k=1}^{\infty} \frac{M^{[m+k-1]}(e^{i\frac{\omega}{2^k}})}{2} = \frac{M^{[m]}(e^{i\frac{\omega}{2}})}{2} \widehat{\varphi_{m+1}^{\Lambda_0}}\left(\frac{\omega}{2}\right). \tag{32}$$

Let us write $M^{[m]}(z) = \sum_{j=-n+1}^{n} \mu_j^{[m]} z^j$ where $\mu_j^{[m]}$ are *real* numbers. Using elementary techniques in Fourier analysis it is easy to see that Eq. (32) is equivalent to the refinement equation

$$\varphi_m^{\Lambda_0}(t) = \sum_{j=-n+1}^{n} \mu_j^{[m]} \varphi_{m+1}^{\Lambda_0}(2t + j). \tag{33}$$

Replacing t by $2^m t$ in (33) we obtain that $\varphi_m^{\Lambda_0}(2^m \cdot) \in V_{m+1}$. Similarly it follows that $\varphi_m^{\Lambda_0}(2^m t - j) \in V_{m+1}$ for each $j \in \mathbb{Z}$. $\quad\square$

Let us write the refinement equation (33) in the form used in MRA, namely

$$\varphi_m^{\Lambda_0}(t) = \sum_{j \in \mathbb{Z}} \mu_{-j}^{[m]} \varphi_{m+1}^{\Lambda_0}(2t - j).$$

The Daubechies type wavelets $\psi_m^{\Lambda_0}$ are now defined in the classical way, namely by

$$\psi_m^{\Lambda_0}(t) = \sum_{j \in \mathbb{Z}} \nu_j^{[m]} \varphi_{m+1}^{\Lambda_0}(2t - j), \tag{34}$$

where the coefficients $\{\nu_k^{[m]}\}$ are related to those in (33) by

$$\nu_j^{[m]} = (-1)^{j+1} \mu_{-1+j}^{[m]}.$$

Then $\psi_m^{\Lambda_0}$ has compact support since $\varphi_{m+1}^{\Lambda_0}$ has compact support. It is a routine exercise to see that the system of functions $\{2^{m/2} \psi_m^{\Lambda_0}(2^m t - r) \colon r \in \mathbb{Z}\}$ is orthonormal.

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It follows from (34) that the smoothness of Daubechies type wavelet $\psi_m^{\Lambda_0}$ is at least as large as that of the scaling function $\varphi_{m+1}^{\Lambda_0}$. Hence one can apply Theorem 29 for an estimate. In this connection the following conjecture seems to be reasonable:

Conjecture. *The smoothness of the Daubechies type wavelets $\psi_m^{\Lambda_0}$ is equal to the smoothness of the classical Daubechies wavelet ψ.*

Finally, let us mention that as usual the wavelet spaces W_m are defined as the orthogonal complement W_m of V_m in V_{m+1}, and that the following identity holds:

$$W_m = \left\{ f \in L^2(\mathbb{R}) \;\middle|\; f(t) = \sum_{j\in\mathbb{Z}} c_j 2^{m/2} \psi_m^{\Lambda_0}\big(2^m t - j\big),\; \sum_{j\in\mathbb{Z}} |c_j|^2 < \infty \right\}.$$

The concept of reproduction for a non-stationary MRA is defined in the following way, see e.g. [35]:

Definition 34. A non-stationary multiresolution analysis $(V_m)_{m\in\mathbb{Z}}$ with compactly supported scaling functions φ_m reproduces a function $f : \mathbb{R} \to \mathbb{C}$ if for each $m \in \mathbb{Z}$ there exist complex coefficients c_m such that

$$f(x) = \sum_{l\in\mathbb{Z}} c_m \varphi_m\big(2^m x - l\big). \tag{35}$$

Let us remark that Definition 34 reduces to the particular case of Definition 10 when the subdivision scheme is interpolatory. It is proved in [35] that condition (27) implies that the MRA $(V_m)_{m\in\mathbb{Z}}$ of the Daubechies subdivision scheme reproduces the space $E(\lambda_0, \ldots, \lambda_{n-1})$ and the wavelets ψ_m have vanishing "exponential" moments in the sense that

$$\int \psi_m^{\Lambda_0}(t) e^{\lambda_j t}\, dt = 0 \quad \text{for all } j = 0, \ldots, n-1.$$

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