Piecewise Uniform Subdivision Schemes

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Abstract. Convergence and smoothness of piecewise uniform subdivision schemes are studied. Such a scheme is defined in terms of a sequence of uniform stationary binary schemes, each applied in a prescribed interval. Necessary and sufficient conditions are derived for the combined scheme to generate C^m limit functions, when the uniform schemes generate C^m limit functions. For the case of interpolatory schemes these conditions reduce to the analysis of the spectrum of a finite matrix.

§1. Introduction

One of the methods for constructing curves and surfaces in CAGD is by Binary Subdivision Schemes - BSS. Starting with a set P^0 of control points in $\mathbb{R}^3$ a stationary BSS, S_a, recursively defines new sets of points, $P^1 = S_a P^0$, $P^2 = S_a P^1$ and in general $P^k = S_a^k P^0$. A point of P^k is defined by a finite linear combination of points in P^{k-1}, with two different rules, or masks, in the curve case (univariate), and four masks in the surface case (bivariate). Such a scheme is called uniform since the coefficients of these masks are fixed. In each iteration the number of points is roughly doubled in case of curves, and is quadrupled in case of surfaces. Some BSS serve as tools for computing spline curves and surfaces (see [3], [7,8]) while other useful schemes introduced in CAGD converge to non-standard limits. For example see de Rahm [21], Catmull and Clark [4], Doo and Sabin [12], Dubuc [13], Deslauriers and Dubuc [11], Dyn, Gregory and Levin [15,16,17]. The convergence analysis of uniform BSS has been developed considerably in the last few years in a series of works [5], [9,10], [14], [13], [15,17], [18], [19,20].

The analysis of BSS for surface design is well developed for the case of a square grid topology of control points [5], [18], [14]. The case of general

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topology has been approached at by several authors, [4], [12], [1,2], [16], [22]. Stationary schemes for general topology can be analyzed almost everywhere by the square grid theory. Some useful necessary conditions for the convergence and the smoothness of the limit surface at the extraordinary points are known. Yet, the analysis of smoothness near an extraordinary point is not fully understood. Moreover, C^2 schemes for general topology are still not known.

In this work we discuss the analogue setting of extraordinary points in the univariate case, namely that of a piecewise uniform scheme defined in terms of a sequence of uniform schemes, each one operating on a different interval of $\mathbb{R}$. The boundary points of these intervals serve as a model of univariate extraordinary points. Given a sequence of C^m uniform schemes, we seek necessary and sufficient conditions for the combined scheme to be C^m. The sufficient conditions presented in this work were derived in 1992 jointly with John Gregory. The necessity part has been added reecently.

A univariate uniform BSS S_a is defined by a set of real coefficients $\{a_j : j \in \mathbb{Z}\}$ with a finite number of non-zero a_j's; the points in $P^k = \{p_j^k : j \in \mathbb{Z}\}$, $k \geq 0$, are recursively defined by

$$p_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{i-2j} p_j^k \ , \ i \in \mathbb{Z} \ . \tag{1}$$

In (1) we find two rules, or masks, for even and for odd i, involving the non-zero coefficients a_j with even and with odd j's respectively. One of the simplest examples is Chaikin's algorithm [6] which converge to a quadratic B-spline curve

$$p_{2i}^{k+1} = \frac{3}{4} p_i^k + \frac{1}{4} p_{i+1}^k \ , \ p_{2i+1}^{k+1} = \frac{1}{4} p_i^k + \frac{3}{4} p_{i+1}^k \ . \tag{2}$$

The scheme (1) is called interpolatory if $a_{2i} = \delta_{i,0}$, $i \in \mathbb{Z}$, since then $P^k \subset P^{k+1}$, and therefore $P^0 \subset P^k$, $\forall k \in \mathbb{Z}_+$.

An equivalent formulation to (1) is in terms of multiplication of the data vector P^k by a bi-infinite matrix, denoted also by S_a, of the form

$$S_a = \{a_{i-2j}\}_{i,j \in \mathbb{Z}} \ .$$

The analysis of BSS can be reduced to the scalar case, namely to initial sets of control points $f^0 = \{f_j^0 \in \mathbb{R} \ : \ j \in \mathbb{Z}\}$. This is equivalent to analyzing one component of the vector valued function in $\mathbb{R}^3$, representing the curve in a parametric form. We say, that S_a is a (uniformly) convergent subdivision scheme if for every set of control points $f^0 = \{f_j^0 \in \mathbb{R} : \ j \in \mathbb{Z}\}$ there is a continuous function f on $\mathbb{R}$ such that for any interval $[a,b] \subset \mathbb{R}$

$$\lim_{k \to \infty} \sup_{2^{-\ell} j \in [a,b]} \left| (S_a^k f^0)_{2^{k-\ell} j} - f(2^{-\ell} j) \right| = 0 \ , \tag{3}$$

and for some initial set of control points this f is not identically zero. We denote f in (3) by $S_a^\infty f^0$, and call it a limit function of S_a or a function generated by S_a. We also introduce the notion of the basic limit function of a scheme S_a,

$$\phi = S_a^\infty\{\delta_{0,j}\} \ .$$

Any limit function of the subdivision process can be expressed as

$$S_a^\infty f^0 = \sum_{j\in\mathbb{Z}} f_j^0 \phi(\cdot - j) \ . \tag{4}$$

The scheme S_a is called C^m if its basic limit function, and hence all functions generated by it, is C^m.

The interesting questions in the analysis of subdivision schemes are the conditions for convergence, and for the basic limit function to be C^m, formulated in terms of the masks of the scheme. Also interesting is the structure of the shift invariant space

$$\Phi = span\{\phi(\cdot - j) \ , \ j \in \mathbb{Z}\} \ .$$

In particular, are there polynomials in Φ, what is the approximation power of it and are $\{\phi(\cdot - j) \ , \ j \in \mathbb{Z}\}$ linearly independent. Let $[L, U]$ denote the support of ϕ. We say that the subdivision scheme S_a is non-singular if it generates non-zero limit functions from non-zero initial data. It is called strongly non-singular if $\sum_{j\in\mathbb{Z}} f_j^0 \phi(x-j) \equiv 0$ in $[0,1]$ implies $f_j^0 = 0$, $j = -U+1, \ldots, -L$.

A necessary condition for the convergence of a uniform BSS to C^m limit functions is given in [18] for interpolatory schemes and in [5] for the general case. This result states that S_a is C^m only if the shift invariant space Φ contains Π_m, the space of all polynomials of degree not exceeding m. The analysis of piecewise uniform schemes given here relies strongly on the above necessary condition and some of its consequences. In the next section we provide a new simple proof of this result.

§2. Necessary conditions for C^m uniform schemes

The results in this section are based on a recent result of Warren [23]. This result applies to non-uniform schemes given in terms of a bi-infinite two slanted matrix S with the property that

$$S_{i,j} \neq 0 \implies i - 2j \in K = \{L, \ldots, U\} \ , \ L \leq 0 \ , \ U \geq 0 \ . \tag{5}$$

The operation of the scheme on a vector of data f^k is multiplication by the matrix S. The notion of convergence is as in (3) with S_a replaced by S.

Theorem 1. *[23] Let S be a convergent BSS not necessarily uniform, which generates C^m limit functions, and let v be an eigenvector of S corresponding to an eigenvalue λ, namely $Sv = \lambda v$. Then:*

(i) If $|\lambda| \geq 2^{-m}$, either $S^\infty v \equiv 0$ or $S^\infty v = x^i$ for some $0 \leq i \leq m$ and $\lambda = 2^{-i}$. Also $\lambda = 2^{-i}$ cannot have a generalized eigenvector u satisfying $Su = \lambda u + v$.

(ii) If $0 < |\lambda| < 2^{-m}$, then $(S^\infty v)^{(i)}(0) = 0$, $i = 0, , \ldots, m$.

(iii) All the non-zero eigenvalues of S are eigenvalues of the finite matrix $S|_E = \{S_{i,j}\}_{i,j\in E}$, where E is any index set containing $-K$. The corresponding generalized eigenvectors of S are uniquely determined as extensions of the corresponding ones of $S|_E$.

The proof of this result is based on the observation that for v and u as above

$$S^\infty v(x) = \lambda S^\infty v(2x) \ ,$$

and that

$$S^\infty u(x) = \lambda S^\infty u(2x) + S^\infty v(2x) \ .$$

The continuity of the derivatives of order $\leq m$ at zero of $S^\infty v$ and $S^\infty u$ implies the consequences of the theorem.

The application of the above result to uniform C^m schemes yields the following necessary conditions.

Theorem 2. *Let S_a be a uniform scheme which generates C^m limit functions. Then $\Pi_m \in \Phi$. Moreover there exists bi-infinite vectors v_i , $i = 0, \ldots, m$, such that*

$$S_a v_i = 2^{-i} v_i \ , \quad S_a^\infty v_i = x^i \ , \quad i = 0, \ldots, m \ . \tag{6}$$

In case S_a is non-singular then $v_i = p_i|_{\mathbb{Z}}$ with p_i a monic polynomial of degree i , $i = 0, \ldots, m$, and $\Pi_m|_{\mathbb{Z}}$ is an invariant subspace of S_a .

Proof: Let f^0 be any initial data vector for the subdivision process. It follows that only the finite data set $f^0|_{-K}$ contributes to $S_a^\infty f^0|_{[-1,1]}$. By part (i) of Theorem 1 there is a set $N \subset \{0, \ldots, m\}$ such that (6) holds for $i \in N$, and $v_i|_{-K}$ determines the limit function x^i in $[-1, 1]$ for $i \in N$. We would like to show that $N = \{0, \ldots, m\}$. Expanding $f^0|_{-K}$ in the eigenvectors of $S_a|_{-K}$ we can write

$$f^0|_{-K} = \sum_{i\in N} d_i v_i|_{-K} + u + w \ ,$$

where u is in the span of the generalized eigenvectors of $S_a|_{-K}$ corresponding to non-zero eigenvalues with module less than 2^{-m} , and w is in the null space of $S_a|_{-K}$. If $j \in \{0, \ldots, m\} \setminus N$ then

$$(S_a^\infty f^0)^{(j)}(0) = 0 \ .$$

This follows from the definition of N, from parts (ii),(iii) of Theorem 1, and since

$S_a|_{-K}(w) = 0$, which implies that w does not contribute to the limit function on $[-\frac{1}{2}, \frac{1}{2}]$. For a uniform scheme the point 0 is not a special point, and by the same argument we have $(S_a^\infty f^0)^{(j)}(x) = 0$ at all diadic points. Since $(S_a^\infty f^0)^{(j)}$ is continuous, it must be identically zero for any f^0, which is impossible since $S_a^\infty \delta = \phi$ is non-zero and compactly supported. Thus (6) must hold, and $\Pi_m \in \Phi$.

In case the functions $\{\phi(\cdot - i) \ , \ i \in \mathbb{Z}\}$, are linearly independent then v_0 is the only initial data generating the constant function 1. That v_0 is identically 1 follows from the fact that masks of convergent uniform schemes satisfy $\sum_{i\in\mathbb{Z}} a_{2i} = \sum_{i\in\mathbb{Z}} a_{2i+1} = 1$. Now, the initial data $\Delta^i v_i/i!$ where Δ is the forward difference operator ($\Delta u = u_{\cdot+1} - u$ for vectors and $\Delta f = f(\cdot + 1) - f$ for functions), generates the limit function $\Delta^i x^i/i! = 1$, by the linearity of the scheme. Thus $\Delta^i v_i/i! = v_0$, implying that v_i coincides with the restriction to $\mathbb{Z}$ of a monic polynomial of degree i. Since $\{v_i\}_{i=0,\ldots,m}$ is a basis of $\Pi_m|_{\mathbb{Z}}$ it thus follows that $\Pi_m|_{\mathbb{Z}}$ is invariant under S_a. ■

The above result can be proved in the multivariate setting by the same method.

§3. Smoothness of piecewise uniform subdivision schemes

The analysis of a piecewise uniform scheme can be reduced to that of a scheme determined by two uniform schemes each applied on a different side of a contact point. This is the case whenever the supports of the masks of all the uniform schemes are jointly bounded. After enough iterations each boundary point of an interval where a uniform scheme is applied is affected only by the two schemes applied in its neighboring intervals.

Let S_{a^+} and S_{a^-} be two finitely supported C^m BSS with coefficients $\{a_j^+ \ : \ j \in \mathbb{Z}\}$ and $\{a_j^- \ : \ j \in \mathbb{Z}\}$ respectively. We denote the basic limit functions of the above schemes by ϕ^+ and ϕ^-, with supports (L^+, U^+) and (L^-, U^-) respectively. Let us assume that $L^\pm \leq 0$ and $U^\pm \geq 0$.

We define the combined subdivision scheme S by :

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j}^+ f_j^k \ , \ i \in \mathbb{Z}_+ \tag{7}$$

and

$$f_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j}^- f_j^k \ , \ i \in \mathbb{Z}_- \ , \tag{8}$$

where $\mathbb{Z}_+ = \{i \ : i \geq 0\}$ and $\mathbb{Z}_- = \{i \ : i < 0\}$. By the finiteness of the supports of S_{a^+} and S_{a^-}, there exists an integer $M > 0$, such that the limit function generated by S in the interval $I_k^+ = 2^{-k}[M, 2M]$ is given by

$$F_k^+(x) = \sum_{i\in J^+} f_i^k \phi^+(2^k x - i) \ , \tag{9}$$

and in $I_k^- = -I_k^+$ by

$$F_k^-(x) = \sum_{i \in J^-} f_i^k \phi^-(2^k x - i) \ , \tag{10}$$

where $J^+ = \{M - U^+ + 1, \ldots, 2M + L^+ - 1\}$, and $J^- = \{-2M - U^- + 1, \ldots, -M + L^- - 1\}$. Note that $\cup_{k=0}^{\infty} I_k^+ = (0, 2M]$, and $\cup_{k=0}^{\infty} I_k^- = [-2M, 0)$, and thus the limit function is C^m at any point of these intervals. After showing that the scheme is C^0, to show that the combined scheme is C^m it is sufficient to prove that, for any initial data f^0, the derivatives up to order m from the left and the right of the origin tend to the same finite limits.

To follow the functions $F_k^{\pm}$ generated by the process we define the matrix A which maps the values $f^k|_{J^+ \cup J^-}$ determining $F_k^{\pm}$ onto the values $f^{k+1}|_{J^+ \cup J^-}$ determining $F_{k+1}^{\pm}$. That is, the matrix A is defined on the set of indices $J = \{-2M - U^- + 1, \ldots, 2M + L^+ - 1\}$, by

$$A_{ij} = a_{i-2j}^+ \ , \quad i \in J \cap \mathbb{Z}_+ \ ,$$

$$A_{ij} = a_{i-2j}^- \ , \quad i \in J \cap \mathbb{Z}_- \ ,$$

mapping $u^k = f^k|_J$ onto u^{k+1}. By part (iii) of Theorem 1, any eigenvector of A corresponding to a non-zero eigenvalue can be extended to an eigenvector of S with the same eigenvalue. Also the non-zero eigenvalues of A are the non-zero eigenvalues of the smaller matrix $\{A_{i,j}\}_{i,j \in \widetilde{J}}$, $\widetilde{J} = \{-n, \ldots, n-1\}$, with $2n$ greater or equal the maximal number of non-zero elements in one column of A. The corresponding eigenvectors are the restriction of those of A to the index set $\widetilde{J}$.

The following Theorem states necessary and sufficient conditions for C^m smoothness of the combined scheme (7)-(8). It is stated in terms of the eigenvalues and eigenvectors of the matrix A. If S_{a^+} and S_{a^-} are C^m schemes, then by Theorem 2 the bi-infinite matrices S_{a^+} and S_{a^-} have eigenvectors v_i^+ and v_i^- respectively, with eigenvalues $\lambda_i = 2^{-i}$, such that $S_{a^\pm}^{\infty} v_i^{\pm} = x^i$, $i = 0, \ldots, m$.

Theorem 3. *Let S_{a^+} and S_{a^-} be two finitely supported C^m subdivision schemes. Let $T = [-U^+/2, -(L^- + 1)/2] \cap \mathbb{Z}$, and*

$$I = \{i \ : \ v_i^+|_T = v_i^-|_T \ , \ 0 \le i \le m\} \ . \tag{11}$$

The combined scheme S given by (7)-(8) is C^m if

(a) For any $i \in I$ $\lambda = 2^{-i}$ is a simple eigenvalue of A.

(b) All other eigenvalues of A have module less than 2^{-m}.
Moreover, if S_{a^+} and S_{a^-} are strongly non-singular then conditions (a)-(b) are also necessary for S to be C^m.

Proof: Let us first assume that (a) and (b) are satisfied. For $i \in I$ define the concatenated vector v_i by $v_i|_{\mathbb{Z}_-} = v_i^-|_{\mathbb{Z}_-}$ and $v_i|_{\mathbb{Z}_+} = v_i^+|_{\mathbb{Z}_+}$. It follows from (7),(8) and (11) that $u_i = v_i|_J$ is an eigenvector of A with an eigenvalue 2^{-i}. We know that the limit function of S_{a^+} with initial data v_i^+, and of S_{a^-} with initial data v_i^+ is x^i. By (7),(8) and (11)

$$S^k v_i|_{\mathbb{Z}_\pm} = S^k_{a^\pm} v_i^\pm|_{\mathbb{Z}_\pm} \ , \ k \in \mathbb{Z}_+ \ . \tag{12}$$

Thus the combined scheme (7)-(8), starting with the data vector v_i generates $S^\infty_{a^+} v_i^+$ for $x \geq 0$ and $S^\infty_{a^-} v_i^-$ for $x < 0$. Hence, for $i \in I$, the limit function of the combined scheme generated from the initial data u_i, is x^i.

By (a) and (b), a general initial data vector defined on the set J can be written as

$$f^0|_J = \sum_{i \in I} \alpha_i u_i + u^* \ , \tag{13}$$

with

$$\|A^k u^*\|_\infty \leq C(0.5 - \varepsilon)^{mk} \|u^0\|_\infty \ , \tag{14}$$

for some $\varepsilon \in (0, 0.5)$. From the initial data $\sum_{i=0}^m \alpha_i u_i$ S generates on $[-2M, 2M]$ the limit function $\sum_{i=0}^m \alpha_i x^i$, which is evidently a C^m function. Let $\overline{F}_k^\pm$ denote the limit function on $I_k^\pm$ of the combined scheme starting with the initial data vector u^*. It follows from (9) that

$$\frac{d^\ell}{dx^\ell} \overline{F}_k^\pm(x) = 2^{k\ell} \sum_{i \in J^\pm} (A^k u^*)_i \frac{d^\ell \phi^\pm}{dx^\ell}(2^k x - i) \ , \ \ \ell \leq m \ . \tag{15}$$

Now by (14), and since $\phi^\pm$ has m bounded derivatives,

$$\max_{x \in I_k^\pm} \left|\frac{d^\ell}{dx^\ell} \overline{F}_k^\pm(x)\right| \leq \overline{C} 2^{k\ell} (0.5 - \varepsilon)^{mk} \|u^*\|_\infty \quad \to 0 \ \ as \ k \to \infty \ . \tag{16}$$

The limit function of the combined scheme starting with the initial data u^* is clearly C^0 by the fact that $(A^k u^*)_0 \to 0$ as $k \to \infty$, and by using (16) for $\ell = 0$. Further, by using (16) for $\ell = 1, \ldots, m$, it follows that this limit function is C^m at $x = 0$.

The proof of the necessity of (a)-(b) is based on Theorem 1. By the strong non-singularity of S_{a^+} and S_{a^-} we rule out the possibility of an eigenvector of S generating the zero limit function. Hence, the possibility of a multiple eigenvalue $\lambda_i = 2^{-i}$, $0 \leq i \leq m$, or an eigenvalue $\lambda \neq 2^{-i}$, $|\lambda| > 2^{-m}$, is excluded and (b) must hold. If $Au = \lambda u$ with $\lambda = 2^{-i}$, for some i , $0 \leq i \leq m$, then $S^\infty u = x^i$, and by the strong non-singularity of $S_a^\pm$, $u|_{J^+} = v_i^+|_{J^+}$ and $u|_{J^-} = v_i^-|_{J^-}$. Moreover, by the two-slanted structure of A it follows that the vector u_i coincides with v_i^+ for $i \geq -U^+/2$ and with v_i^- for $i \leq -(L^- + 1)/2$, and thus (a) follows. ■

Corollary 4. *Let S_{a^+} and S_{a^-} be two finitely supported non-singular C^m subdivision schemes, with corresponding matrix A having $|J| - m - 1$ eigenvalues with module less than 2^{-m}, counting multiplicities. The combined scheme S is C^m if*

$$S_{a^+}(p|_{\mathbb{Z}}) = S_{a^-}(p|_{\mathbb{Z}}) \ , \quad \forall p \in \Pi_m \ . \tag{17}$$

In case S_a^+ and S_a^- are strongly non-singular and $|T| = m$, then (17) is also necessary for S to be C^m.

Proof: By Theorem 2 and the non-singularity of $S_a^{\pm}$, the eigenvector of $S_a^{\pm}$ corresponding to the eigenvalue 2^{-i} is the restriction to $\mathbb{Z}$ of a monic polynomial of degree i. Hence by (17) $v_i^+ = v_i^-$, $i = 0, \ldots, m$, the set I in (11) is $\{0, \ldots, m\}$, and $v_i^{\pm}$ is an eigenvector of S corresponding to the eigenvalue 2^{-i} for all $i \in I$. This together with the assumption on the matrix A implies (a)-(b), and the first part of the corollary follows from Theorem 3. For the second part, the assumption on the matrix A together with the second part of Theorem 3 implies that $I = \{0, \ldots, m\}$. Now, the fact that the eigenvector of $S_a^{\pm}$ corresponding to the eigenvalue 2^{-i} is the restriction to $\mathbb{Z}$ of a monic polynomial of degree i, implies that if $|T| \geq i$ then $v_i^+ = v_i^-$. Thus (17) holds for a basis of Π_m and hence for all Π_m .

Corollary 5. *The combined scheme S is C^m if S_{a^+} and S_{a^-} are finitely supported interpolatory C^m schemes and A has $|J| - m - 1$ eigenvalues with module less than 2^{-m}, counting multiplicities.*

Proof: Any interpolatory scheme has the property that if the limit function generated from f^0 is a polynomial p then $f^k = p|_{2^{-k}\mathbb{Z}}$ for all $k \in \mathbb{Z}_+$. Hence an interpolatory C^m scheme has eigenvectors $v_i = x^i|_{\mathbb{Z}}$, $i = 0, \ldots, m$. Thus by Corollary 4 the claim follows. ■

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