Subdivision for C^1 Surface Interpolation

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Abstract

Interpolatory subdivision schemes generating surfaces from initial control nets with the topology of a regular grid are introduced and analyzed. The schemes are based upon the butterfly-scheme and are shown to have components with continuous first order derivatives. The generalization of these schemes for the application to general control nets is also discussed.

1 Introduction

The design of surfaces in CAGD starts with a set of points with connectivity relations between them, termed control points. The simplest connectivity is that of a regular grid, namely each point has two indices: p_{ij}, $i = 1, \ldots, N_1$, $j = 1, \ldots, N_2$. Thus the four points $p_{ij}, p_{i+1,j}, p_{i,j+1}$ and $p_{i+1,j+1}$ constitute a "face", and each point p_{ij} has four neighbors $p_{i,j\pm1}$, $p_{i\pm1,j}$. Another possible topology of the connectivity relations is that of a triangular grid, where each face is determined by three control points and each pair of control points can belong to at most two triangles. In this case the control polyhedron consists of the planar faces determined by the triangulation. The most general type of connectivity relation is that of a set of "faces" with a variable number of vertices. The set of control points together with the connectivity relations constitute the control net.

Given the control net it is then required to construct a smooth surface approximating it. Thus to each "face" there corresponds a patch which is determined by the vertices of that "face" and perhaps by several of their direct neighbors. As a general principal the methods of construction are required to be local, so that changes in a control point affect only a limited number of patches. One common approach to the design of curves and surfaces is the parametric representation in terms of a family of smooth compactly supported piecewise polynomials, with the control points as coefficients. Strongly related to this approach is the use of subdivision processes, which is the topic of this paper. In particular we discuss here the case of regular grids.

First we review the relevant analysis of subdivision schemes on regular grids in $\mathbb{R}^s$. We state the results without proofs. The proofs can be found in the review paper [D]. Then, in the next section, we present some new interpolatory schemes for surfaces in $\mathbb{R}^3$, and use the tools provided by the above analysis to investigate the smoothness properties of these schemes. The new schemes presented and analyzed here are extensions of known schemes, the tensor-product of the 4-point scheme [DGL1] and the butterfly scheme [DGL3]. The main features of these schemes are their symmetry relative to a regular grid, and/or the simple geometric neighborhood relationships of their masks. In Section 3 we describe the application of the new schemes to general (non-regular) control nets. For further work on surface-surface intersection and offsets of surfaces generated by the butterfly scheme see [K] and [Sh]. A C^2 extension of the butterfly scheme is presented in [L].

Given a set of control points at level 0, with the topology of a regular grid $P^0 = \{p^0_\alpha \mid \alpha \in \mathbb{Z}^s\}$, a Binary Subdivision Scheme (BSS), recursively defines new sets of points, $P^k = \{p^k_\alpha \mid \alpha \in \mathbb{Z}^s\}$ at level $k \geq 0$, formally by

$$P^k = S^k_a P^0 \ . \tag{1.1}$$

The scheme S_a is given by

$$p^{k+1}_\alpha = \sum_{\beta \in \mathbb{Z}^s} a_{\alpha - 2\beta} p^k_\beta \ , \qquad \alpha \in \mathbb{Z}^s \ . \tag{1.2}$$

Throughout this work we use the multi-index notation. In particular $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$, $x = (x_1, \dots, x_s) \in \mathbb{R}^s$, $x^\alpha = \prod_{i=1}^s x_i^{\alpha_i}$, $|\alpha| = \sum_{i=1}^s |\alpha_i|$, $x^n = \prod_{i=1}^s x_i^n$, $n \in \mathbb{Z}$ and $\mathbf{0} = (0, \dots, 0) \in \mathbb{Z}^s$. In practice the set of initial control points $\{p_\alpha^0 : \alpha \in J_0\}$ is finite. Here we assume for the sake of simplicity that the control points are defined for all $\mathbb{Z}^s$.

In applications to surface design $\alpha = (i,j) \in \mathbb{Z}^2$ and $p_\alpha^k \in \mathbb{R}^3$. Since we consider schemes where each component of the surface is a scalar function generated by the same BSS, it is sufficient to analyze control points in $\mathbb{R}^1$. The analysis of the BSS determines the smoothness of each component of the generated surface. Therefore, we consider scalar sets of control points $f^k = \{f_\alpha^k \in \mathbb{R} \mid \alpha \in \mathbb{Z}^s\}$ generated by the BSS

$$f_\alpha^{k+1} = \sum_{\beta \in \mathbb{Z}^s} a_{\alpha - 2\beta} f_\beta^k \ , \qquad \alpha \in \mathbb{Z}^s \ . \tag{1.3}$$

In the surface case $s = 2$, there are 4 rules in (1.3) depending on the parity of each component of the vector $\alpha \in \mathbb{Z}^2$. Thus, defining $E_2 = \{\gamma : \gamma_i \in \{0,1\} \ , \ i = 1,2\}$, namely E_2 consists of the extreme points of $[0,1]^2$, we get for each $\gamma \in E_2$ a different rule:

$$f_{\gamma+2\alpha}^{k+1} = \sum_{\beta \in \mathbb{Z}^2} a_{\gamma+2\alpha-2\beta} f_\beta^k = \sum_{\beta \in \mathbb{Z}^2} a_{\gamma+2\beta} f_{\alpha-\beta}^k \ , \qquad \gamma \in E_2 \ , \ \alpha \in \mathbb{Z}^s \ . \tag{1.4}$$

For general s the number of rules is 2^s, as the cardinality of the set E_s of the extreme points of $[0,1]^s$. The set $\{a_{\gamma+2\alpha} \mid a_{\gamma+2\alpha} \neq 0 \ , \ \alpha \in \mathbb{Z}^s\}$ for $\gamma \in E_s$ is termed the mask of the rule corresponding to $\gamma \in E_s$.

Definitions. **a.** A binary subdivision scheme S_a is a *convergent subdivision scheme* if for every set of control point $f^0 = \{f_\alpha^0 \in \mathbb{R} \mid \alpha \in \mathbb{Z}^s\}$, there exists a continuous function $f \in C(\mathbb{R}^s)$ such that

$$\lim_{\ell < k \to \infty} |(S_a^k f^0)_{2^{k-\ell}\alpha} - f(2^{-\ell}\alpha)| = 0 \ , \qquad \alpha \in \mathbb{Z}^s \ , \ \ell \in \mathbb{Z}_+ \ , \tag{1.5}$$

and such that for some initial data the above function $f \not\equiv 0$. The function f is denoted by $S_a^\infty f^0$. **b.** S_a is a C^ν *BSS* , if $S_a^\infty f^0 \in C^\nu(\mathbb{R}^s)$. **c.** S_a is a *uniformly convergent BSS* , if for any bounded domain $\Omega \subset \mathbb{R}^s$ and $\varepsilon > 0$ there exists $K(\varepsilon, \Omega)$ such that

$$|(S_a^k f^0)_\alpha - f(2^{-k}\alpha)| < \varepsilon \ , \quad , k > K(\varepsilon, \Omega) \ , \qquad \alpha \in \mathbb{Z}^s \cap 2^k\Omega \ . \tag{1.6}$$

This is equivalent to the requirement that for all $f^0 \in \ell^\infty(\mathbb{Z}^s)$,

$$\lim_{k\to\infty} \|S_a^k f^0 - f(\frac{\cdot}{2^k})\|_\infty = 0 \ ,$$

where $f(\frac{\cdot}{2^k})$ denotes the sequence $\{f(\frac{\alpha}{2^k}) \mid \alpha \in \mathbb{Z}^s\}$.

A simple necessary condition for S_a to be uniformly convergent is the following:

Proposition 1. *Suppose S_a is a uniformly convergent BSS, then*

$$\sum_{\alpha\in\mathbb{Z}^s} a_{\gamma-2\alpha} = 1 \ , \qquad \gamma \in E_s \ . \tag{1.7}$$

Let $F_k(z) = \sum_{\alpha\in\mathbb{Z}^s} f_\alpha^k z^\alpha$, $z \in \mathbb{R}^s$, be a formal generating function associated with the control points f^k. The relation (1.3) can be written formally in terms of the generating functions by

$$F_{k+1}(z) = a(z)F_k(z^2) \ , \tag{1.8}$$

where $a(z)$ is the characteristic polynomial of S_a

$$a(z) = \sum_{\alpha\in\mathbb{Z}^s} a_\alpha z^\alpha \ . \tag{1.9}$$

The characteristic polynomials of subdivision schemes with compactly supported masks belong to the class $\mathcal{L}$ of Laurent polynomials. It is convenient to present the convergence analysis of a BSS in terms of its characteristic polynomial. The formalism of the generating functions allows us to compute the masks of the iterated schemes S_a^k, $k > 1$, given the mask of S_a. From (3.4) it follows that

$$F_{k+L}(z) = a(z)a(z^2)\ldots a(z^{2^{L-1}})F_k(z^{2^L}) = a^{[L]}(z)F_k(z^{2^L}) \ . \tag{1.10}$$

Comparing coefficients of equal powers of z on both sides we get 2^{sL} different rules mapping f^k to f^{k+L}, determined by the coefficients of

$$a^{[L]}(z) = \prod_{j=0}^{L-1} a(z^{2^j}) = \sum_{\alpha} a_{\alpha}^{[L]} z^{\alpha} \ . \tag{1.11}$$

These rules have the form

$$f_{\gamma+2^L\alpha}^{k+L} = \sum_{\beta\in\mathbb{Z}} a_{\gamma+2^L\beta}^{[L]} f_{\alpha-\beta}^{k} \ , \qquad \gamma \in \{0,1,\ldots,2^L-1\}^s \ , \tag{1.12}$$

and hence the norm of S_a^L is given by

$$\|S_a^L\|_\infty = \max\left\{ \sum_{\beta\in\mathbb{Z}} |a_{\gamma+2^L\beta}^{[L]}| \ : \ \gamma \in \{0,1,\ldots,2^L-1\}^s \right\} . \tag{1.13}$$

In the univariate case, $s = 1$, the necessary condition (1.7) implies that the characteristic polynomial of the BSS can be written as

$$a(z) = 2(1+z)q(z) \ . \tag{1.14}$$

The polynomial $q(z)$ determines a BSS S_q mapping the divided differences of f^k onto those of f^{k+1}.

Theorem 2. *A univariate BSS is uniformly convergent if and only if there exists $L \in \mathbb{Z}_+$ such that $\|S_q^L\|_\infty < 1$. This condition is necessary and sufficient for S_q to be contractive. Furthermore, the BSS S_a is a C^1 BSS if S_{2q} is uniformly convergent.*

In the multivariate case, the necessary condition (1.7) also implies the existence of a related divided difference scheme. However, in the general case this scheme has a matrix mask, and its analysis is more complicated (see [H] and [D]). In this work we present schemes with a special structure that allows a simpler convergence and smoothness analysis [CDM], which is a direct extension of that in the univariate case. The BSS discussed in this work have characteristic polynomials of the form

$$a(z) = \prod_{i=1}^{\ell} (z^{\theta^{(i)}} + 1)^{\nu_i} q(z) \ , \quad \theta^{(1)}, \ldots, \theta^{(\ell)} \in \mathbb{Z}^s \ , \ \nu_1, \ldots, \nu_\ell \in \mathbb{Z}_+ \ , \tag{1.15}$$

and the relevant analysis is the content of the following Theorems (see also [D]):

Theorem 3. *Let the characteristic polynomial of S_a be of the form*

$$a(z) = q(z) \prod_{i=1}^{s} (z^{\theta^{(i)}} + 1) \ , \tag{1.16}$$

where $q \in \mathcal{L}$ and $\theta^{(1)}, \ldots, \theta^{(s)} \in \mathbb{Z}^s$ satisfy

$$|\det(\theta^{(1)}, \ldots, \theta^{(s)})| = 1 \ . \tag{1.17}$$

Also, let S_{Δ_i} be the BSS corresponding to the characteristic polynomials $a(z)(z^{\theta^{(i)}} + 1)^{-1}$, $i = 1, \ldots, s$. Then the BSS S_a is uniformly convergent if and only if for some $L \in \mathbb{Z}_+$

$$\|S_\Delta^L\|_\infty = \max_{1 \le i \le s} \|S_{\Delta_i}^L\|_\infty < 1 \ . \tag{1.18}$$

Theorem 4. *Let S_a be a convergent BSS with a characteristic polynomial $a(z) = (z^\theta + 1)^\nu 2^{-\nu} q(z)$, $\theta \in \mathbb{Z}^s$, $\nu \in \mathbb{Z}_+$, $q \in \mathcal{L}$. If the BSS S_q converges uniformly, then for all initial data f^0*

$$\partial_\theta^\nu S_a^\infty f^0 \in C(\mathbb{R}^s) \ , \tag{1.19}$$

where $\partial_\theta f = \lim_{t \to 0} (f(\cdot + t\theta) - f)/t$.

In this paper we consider interpolatory subdivision schemes, namely schemes in which $P^k \subset P^{k+1}$, $k \in \mathbb{Z}_+$. In these schemes the rule (1.3) for $\gamma = (0,0)$ has the form $a_{2\alpha} = \delta_{\alpha \mathbf{0}}$, $\alpha \in \mathbb{Z}^s$. Special properties of such rules where considered in [DL]. In particular every convergent interpolatory BSS is uniformly convergent.

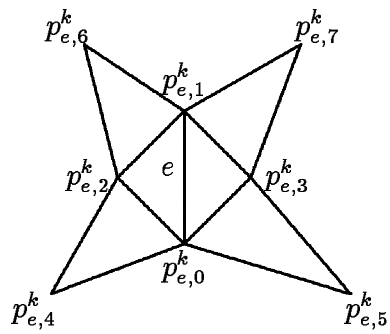

Figure 1 - *The neighborhood relations in the butterfly scheme.*

Example - The butterfly scheme.

The example we present here is an extension of the 4-point interpolatory BSS [DGL1] to the case of surfaces defined by control points with the topology of general triangulations [DGL3]. This scheme can be analyzed within the setting of the square-grid topology, with the exclusion of a fixed set of irregular points.

Given a set of control points $\{p_i^k\}$ which comprise the vertices of a triangulation T^k, the scheme associates with each edge $e \in T^k$ a new point q_e^k defined according to the rule

$$q_e^k = \tfrac{1}{2}(p^k_{e,0} + p^k_{e,1}) + 2\omega(p^k_{e,2} + p^k_{e,3}) - \omega \sum_{j=4}^{7} p^k_{e,j} \ , \tag{1.20}$$

where the locations of the points $p^k_{e,j}$ relative to the edge e in T^k are depicted in Figure 1. The configuration of points in Figure 1. suggests the name "butterfly scheme". The $(k+1)$th set of control points is $P^{k+1} = P^k \cup \{q_e^k \mid e \in T^k\}$, and the triangulation T^{k+1} consists of the edges connecting q_e^k, $e \in T^k$, to $p^k_{e,0}$, $p^k_{e,1}$ and to the points associated with the four edges forming with e two triangles in T^k.

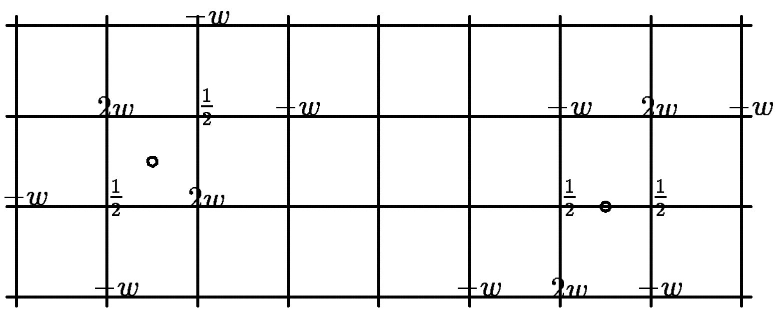

Figure 2 - *Masks of two rules of the butterfly scheme on a three directional grid.*

The application of the butterfly scheme to a control net with the regular square grid topology is made by associating the control points to the vertices of a three directional mesh, with directions $(0,1)$, $(1,0)$, $(1,1)$. The masks of two of the resulting rules are depicted in Figure 2. The third rule, for the point corresponding to the center of a vertical edge, is obtained from the second rule by symmetry. The fourth rule leaves the points corresponding to the vertices as they are.

The butterfly characteristic polynomial can be put in a factored form

$$a_\omega(z_1,z_2) = 2^{-1}(1+z_1)(1+z_2)(1+z_1z_2)z_1^{-1}z_2^{-1}(1-\omega b(z_1,z_2)) \ , \qquad (1.21)$$

where

$$b(z_1,z_2) = 2z_1^{-2}z_2^{-1} + 2z_1^{-1}z_2^{-2} - 4z_1^{-1}z_2^{-1} - 4z_1^{-1} - 4z_2^{-1} + 2z_1^{-1}z_2 +$$

$$2z_1z_2^{-1} + 12 - 4z_1 - 4z_2 - 4z_1z_2 + 2z_1^2z_2 + 2z_1z_2^2 \ .$$

The polynomial $a_\omega(z_1,z_2)$ is of the form (1.15) with $\theta^{(1)} = (1,0)$, $\theta^{(2)} = (0,1)$, $\theta^{(3)} = (1,1)$. By Theorems 3 and 4, in order to check if S_{a_ω} is C^1 one should check the contractivity of each one of the schemes $S_{d_\omega^{(i)}}$, $i = 1,2,3$, where

$$d_\omega^{(1)}(z_1,z_2) = 2(1+z_2)^{-1}(1+z_1z_2)^{-1}a_\omega(z_1,z_2) \ ,$$

$$d_\omega^{(2)}(z_1, z_2) = 2(1+z_1)^{-1}(1+z_1z_2)^{-1}a_\omega(z_1, z_2) ,$$

$$d_\omega^{(3)}(z_1, z_2) = 2(1+z_1)^{-1}(1+z_2)^{-1}a_\omega(z_1, z_2) .$$

The scheme S_{a_ω} can be regarded as a perturbation of the scheme S_{a_0}, which generates the linear C^0 box-spline. As shown in [DLM], $\|S^2_{d_\omega^{(i)}}\|_\infty < 1$, $i = 1, 2, 3$, for small enough $\omega > 0$. Here we present a specific ω-interval for which it has been verified that S_{a_ω} is C^1. The norms $\|S^L_{d_\omega^{(i)}}\|_\infty$ for $L = 1, 2, 3$, are presented in [H] as functions of the parameter ω. It is concluded there that the butterfly scheme S_{a_ω} is C^1 for $0 < \omega < 0.096$ at least. It has also been found that the minimal norm is attained near $\omega = \frac{1}{16}$. The role of this specific value of ω will be clarified below.

Bibliographical notes.

Subdivision schemes with non-standard limit functions were first analyzed in [R] and in recent years in many works, e.g. [CDM], [DL1], [D2], [DD], [DGL1-3], [DL1-3], [DLL], [DLM], [MP1-2]. The main sources for the material in this paper are [CDM], [DGL1-3], [DL3], [DLM], [D]. More details of the analysis and the applications of the schemes presented here can be found in [H].

2 New C^1 BSS related to the butterfly schemes

A family of cubic reproducing schemes.

The butterfly scheme with $\omega = \frac{1}{16}$ on a regular grid, is an interpolatory BSS which reproduces cubic polynomials. The proof of this fact is simply by checking, and is related to the same property of the univariate 4-point scheme [DGL1]. The property of cubic reproduction immediately implies that for $f \in C^4(\mathbb{R}^s)$, $\|f(x) - S_a^\infty f(\frac{\cdot}{2^k})\|_\infty = O(h^4)$, $h = 2^{-k}$, even though $S_a^\infty f(\frac{\cdot}{2^k}) \in C^1$. In the following we present a perturbation of S_{a_ω}, with

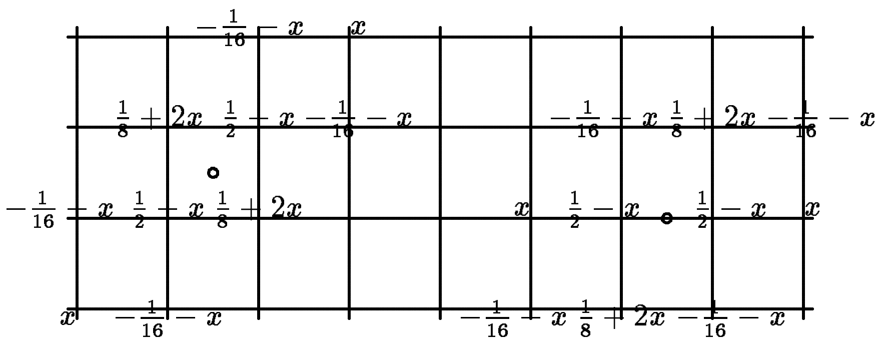

Figure 3 - *Masks of two rules of the scheme* $S_{\tilde{a}_x}$.

$\omega = \frac{1}{16}$, which defines a family of cubic reproducing schemes, with a free parameter x. The characteristic polynomials for this family are $\tilde{a}_x$ where

$$\tilde{a}_x(z_1, z_2) = a_{\frac{1}{16}}(z_1, z_2) + xp(z_1, z_2) \ , \tag{2.1}$$

with

$$p(z_1, z_2) = (1 + z_1)(1 + z_2)(1 + z_1 z_2)(1 - z_1)^2(1 - z_2)^2(1 - z_1 z_2)^2 z_1^{-3} z_2^{-3} \ .$$

The masks of two rules of the BSS $S_{\tilde{a}_x}$ for the mid-cell and mid-edge points, are shown in Figure 3.

The relevant characteristic polynomials for C^1 analysis are

$$2(1 + \theta^{(i)})^{-1}(1 + \theta^{(j)})^{-1}\tilde{a}_x \ , i \neq j \ ,$$

where $\theta^{(1)} = (1, 0)$, $\theta^{(2)} = (0, 1)$, $\theta^{(3)} = (1, 1)$. By the symmetry of these polynomials, it suffices to check the contractivity of the BSS corresponding to one of them, say S_{d_x}, where

$$d_x = 2(1 + z_1)^{-1}(1 + z_2)^{-1}\tilde{a}_x \ .$$

We have used Mathematica to estimate $\|S^L_{d_x}\|_\infty$. It has been established that

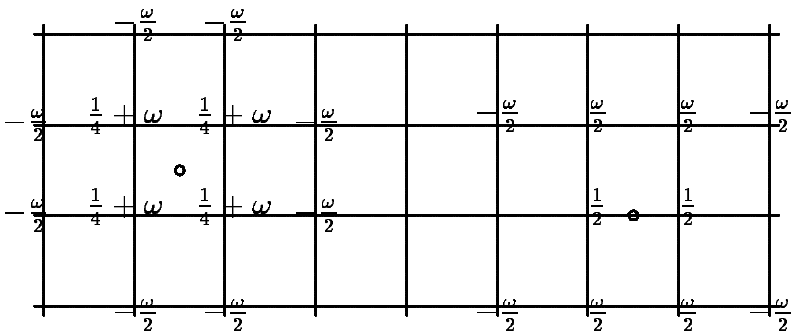

Figure 4 - *Masks of two rules of the averaged-butterfly scheme* S_{A_ω}.

$\|S^2_{d_x}\|_\infty < \|S^2_{d_0}\|_\infty < 1$ for $x \in [-0.03, 0)$,
$\|S^3_{d_x}\|_\infty < \|S^3_{d_0}\|_\infty < 1$ for $x \in [-0.035, 0)$.

These results suggest a faster convergence of the divided differences to the appropriate derivatives of the limit function of these BSS with $x > 0$ as compared to the BSS with $x = 0$.

The averaged-butterfly scheme.

The butterfly scheme for a regular grid, as presented in Figure 2, is based on a three-directional grid, where the third direction is chosen as $(1,1)$. As a result of this, the scheme is not symmetric relative to the regular grid. A symmetric scheme can be obtained by averaging two butterfly schemes, corresponding to the two choices, $(1,1)$ and $(-1,1)$, of the third direction. The masks of two rules of the averaged-butterfly scheme, corresponding to a mid-cell and a mid-edge points, are shown in Figure 4:

The characteristic polynomial A_ω corresponding to this new scheme is

$$A_\omega(z_1, z_2) = \frac{1}{2}[a_\omega(z_1, z_2) + a_\omega(z_1^{-1}, z_2)] \ .$$

As in the case of the butterfly scheme, the new scheme also reproduces cubic polynomials for $\omega = \frac{1}{16}$. Here, the characteristic polynomial A_ω is not divisible by $(1 + z_1 z_2)$. Instead, it is divisible by $(1 + z_1)^2(1 + z_2)^2$, which is also sufficient for C^1 analysis. The scheme S_{A_ω} can be viewed as a perturbation of the scheme S_{A_0}, which generates the tensor-product of the linear B-spline. The relevant characteristic polynomials for the C^1 analysis are

$$d_\omega^{(1)}(z_1, z_2) = 2(1 + z_2)^{-2} A_\omega(z_1, z_2) \ ,$$

$$d_\omega^{(2)}(z_1, z_2) = 2(1 + z_1)^{-1}(1 + z_2)^{-1} A_\omega(z_1, z_2) \ .$$

Numerical computations of the norms $\|S^5_{d_\omega^{(1)}}\|_\infty$ and $\|S^5_{d_\omega^{(2)}}\|_\infty$ show that these norms are less than 1, and hence that the averaged-butterfly scheme is C^1, at least for $\omega \in (0, 0.1363]$.

A truncated tensor-product scheme.

Working on a regular grid it is natural to consider schemes which are tensor-products of univariate BSS. The univariate model used throughout this work is the 4-point BSS [DGL1], which has the characteristic polynomial

$$a_\omega(z) = \frac{1}{2}(1 + z)^2 z^{-1}(1 - \omega r(z)) \ , \tag{2.2}$$

where $r(z) = 2(1+z^2)(1-z^{-1})^2$. The corresponding bivariate tensor-product scheme has the characteristic polynomial

$$A_\omega(z_1, z_2) = a_\omega(z_1) a_\omega(z_2) \ , \tag{2.3}$$

and it directly inherits the properties of the 4-point scheme. Namely, the scheme S_{A_ω} reproduces cubic polynomials for $\omega = \frac{1}{16}$, and it is C^1 for $\omega \in (0, \frac{\sqrt{5}-1}{8})$ (see [DGL2]).

Motivated by the perturbation theory in [DLM], and aiming to reduce the support of the masks of the BSS, we consider a perturbation of the above tensor-product scheme. The suggested scheme is obtained by removing all the ω^2 terms in A_ω. The resulting characteristic polynomial is

$$A^t_\omega(z_1, z_2) = \frac{1}{4}(1 + z_1)^2(1 + z_2)^2 z_1^{-1} z_2^{-1}(1 - \omega[r(z_1) + r(z_2)]) \ . \tag{2.4}$$

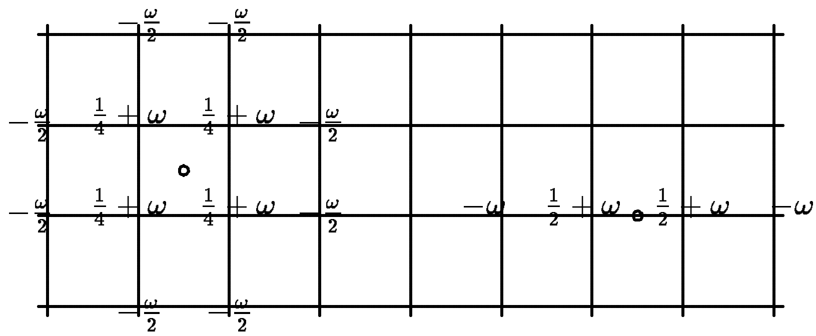

Figure 5 - *Masks of two rules of the truncated tensor-product scheme* $S_{A_\omega^t}$.

The masks of two rules of the truncated tensor-product scheme are shown in Figure 5.

Some basic characteristics of the new scheme $S_{A_\omega^t}$ are: **1.** The rules for new points corresponding to centers of edges coincide with the univariate 4-point scheme. **2.** The fourth rule of $S_{A_\omega^t}$ corresponding to mid-cell points coincides with that of the averaged-butterfly scheme. **3.** It reproduces cubics for $\omega = \frac{1}{16}$ (follows from 1 and 2). **4.** As the other schemes, the scheme $S_{A_\omega^t}$ is reduced to the 4-point scheme in one direction, when the data is constant along the other direction. This implies, among other things, that none of these schemes is C^2 [DGL1]. **5.** The removal of the ω^2 terms from the tensor-product scheme enables the implementation of $S_{A_\omega^t}$ to non-regular configurations of control points (as shown in the next section). **6.** The C^1 analysis here is similar to that of the averaged-butterfly scheme, and it shows that $S_{A_\omega^t}$ is C^1 at least for $\omega \in (0, 0.21]$, which is a range bigger than that for the averaged-butterfly scheme. It is also established in [H] that the second order mixed derivative is continuous at least for $\omega \in (0, 0.122]$.

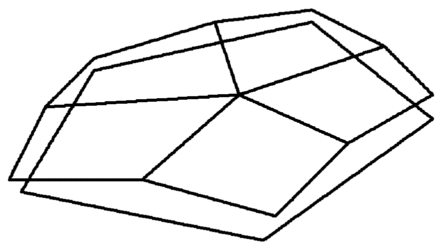

Figure 6 - *Refinement structure of Catmull-Clark algorithm.*

3 Application to general control nets

In this section we demonstrate the application of the averaged-butterfly scheme and the truncated tensor-product scheme with $\omega = \frac{1}{16}$, to three model control polyherda. All models consist of 4-sided faces, and two contain irregular vertices, namely the number of edges meeting at these vertices does not equal 4. So first we define the rules for general control nets.

The first presentation of a subdivision scheme for general control nets is by Catmull and Clark [CC], who device a surface version of the cubic spline BSS. The refinement scheme of a general control net, suggested in [CC], requires three types of rules for defining the new control points:
1. A rule for a new point corresponding to an old control vertex v, with coefficients depending on the number of edges meeting at v. Here we discuss interpolatory schemes, which leave the old vertices unchanged.
2. A rule for a new point corresponding to an edge in the old control net.
3. A rule for a new point corresponding to a "face" in the old control net.
Every m-sided "face" in the old control net is then replaced by m 4-sided "faces", as shown in Figure 6.

Therefore all "faces" are 4-sided in all the control nets generated by the scheme. Also all the new vertices generated by the scheme are regular with the exception of those corresponding to irregular vertices and to m-sided faces with $m \neq 4$ in the initial control net. Thus we can use the BSS as defined for a regular grid for most of the edges. For edges emanating from irregular vertices we design special rules for each of the BSS considered. The masks for these rules are depicted in Figure 8 for the averaged-butterfly scheme, and

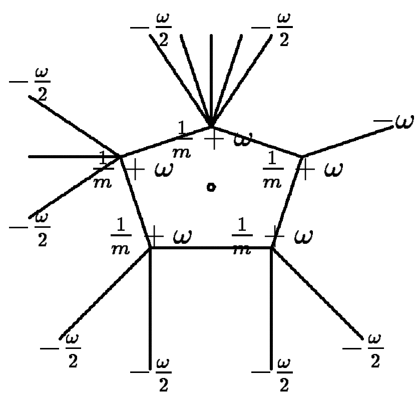

Figure 7 - *The mask for a new mid-point of an m-sided face.*

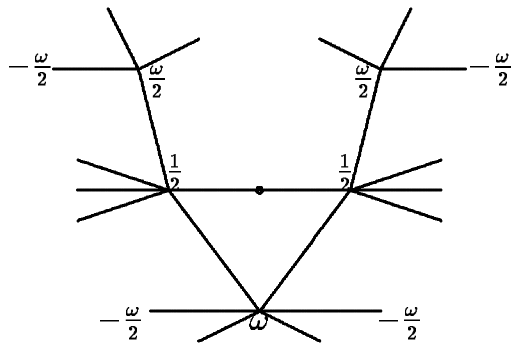

Figure 8 - *The averaged-butterfly mask for a new edge-point.*

in Figure 9 for the truncated tensor-product scheme. In Figure 7 we give a reasonable mask for a new point corresponding to an m-sided "face". This rule is used only in the first application of the scheme, and does not affect the convergence or the smoothness properties of the scheme.

The convergence analysis of subdivision shcemes for general control nets

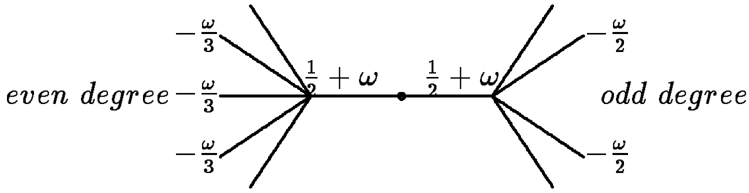

Figure 9 - *Truncated tensor-product mask for a new edge-point.*

involves the analysis of the regular scheme away from irregular vertices, together with local analysis near irregular vertices. Our local analysis near irregular points is based on [DS], where some necessary conditions for the C^1 regularity of the surface near an irregular point are stated. For the schemes tested here these necessary conditions where verified in [H].

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