Convexity-preserving Interpolation by Dual Subdivisions Schemes

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Abstract. In this paper, we present convexity-preserving subdivision schemes which are the dual of Chaikin's as well Doo-Sabin's schemes, in one and two dimensions, respectively. The notion of duality is semantically imported to the domain of subdivision schemes. The primary schemes act on points whereas the dual schemes, which contain the same subdivision rules, are applied upon their dual counterparts, i.e., lines or surfaces. When the dual schemes operate upon closed convex polygons and polyhedra, they generate smooth curves and under some restrictions, smooth surfaces. Keywords: subdivision, duality, convexity.

§1. Introduction

The notion of duality is a very helpful tool in geometric analyses. The basic principle is that any symmetry between entities in the axioms of a mathematical structure results in a corresponding symmetry in the theorems and constructions relating to those entities. Duality of points and lines in 2-D and of points and planes in 3-D are well-known examples. All theorems of incidence occur in dual pairs within the confines of the projective plane. For example, 'two points determine a line' is the dual axiom to 'two lines determine a point'. The interface between duality and subdivision can be achieved in the following way: a primary scheme S takes a set of points in the plane or in space and defines new points by the subdivision rule. The dual scheme S', uses the same the rules of S, whereas the objects applied upon are the *dual* objects to points, i.e. lines or planes. We shall be analyzing the dual Chaikin and the dual Doo-Sabin subdivision schemes. Our objects of study are closed, convex polygons and polyhedra. It is known from the theory of convexity, that every convex body is the intersection of the half-planes or half-spaces, whose boundaries form its supporting lines or supporting planes.

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§2. The Dual Chaikin Scheme

Given is a strictly convex data set $\{v_i^0\} \in I\!R^3, i = 0, ..., M$ with z component identical to zero, such that the origin O lies within the convex hull of this set. We apply the convex linear combinations $\{\frac{3}{4}, \frac{1}{4}\}$ onto $\{l_i^0\}$, the set of lines defined by v_i^0 and v_{i+1}^0 s.t.

$$v_M^0 \equiv v_0^0, \quad l_i^0 \equiv l_i^0(x), \quad x \in I\!R^3,$$

$$l_i^0(v_i^0) = l_i^0(v_{i+1}^0) = 0.$$

The explicit form of the lines is given by

$$l_i^0(x) = \langle N_i^0, x \rangle - \langle N_i^0, v_i^0 \rangle = 0,$$

$$N_i^0 = \frac{(v_{i+1}^0 - v_i^0) \times e_3}{\|v_{i+1}^0 - v_i^0\|},$$

where

$$e_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The *dual* to Chaikin's primary scheme ('corner-cutting') is given by

$$l_{2i}^{k+1} = \frac{3}{4} l_i^k + \frac{1}{4} l_{i-1}^k$$

$$l_{2i+1}^{k+1} = \frac{3}{4} l_i^k + \frac{1}{4} l_{i+1}^k$$

$$i = 0, ..., 2^k M - 1, \quad k = 1, 2, ..., \quad l_{2^{k+1}M}^{k+1} = l_0^{k+1}.$$

The convex hull of the initial strictly convex data set forms a closed polygon, which we denote by P^0. At the first stage, the dual scheme operates on the set of lines defining the edges of P^0, $\{l_i^k\}$. A new set of lines is created, $\{l_i^{k+1}\}$, whose intersections define the new vertices added to the old polygon. The process continues ad infinitum. For each finite stage k, we denote the corresponding polygon by P^k. P^{k+1} has twice as many vertices than P^k. The interpolation property of the dual process is equivalent to the preservation of the centroids of the edges of the polygon in the primary scheme. During the application of the dual Chaikin process, the set of the normal vectors to the edges of P^k, N^k, evolves according to Chaikin's *primary* algorithm. Hence, the limit curve formed by the normal vectors is a *smooth quadratic B-spline curve.* At each finite stage, the data determined by the normal vectors remains strictly convex due to the preservation of order of the directions of the normals throughout the process.
The above dual Chaikin scheme is a special case of the C^1 convexity-preserving interpolation method suggested and analysed in [5]. Further work on interpolatory convexity-preserving schemes for curves are found in [6] and [8]. The analysis of the dual Chaikin scheme is presented as an introduction to the generalization of dual schemes to surfaces in the next sections.

Theorem 2.1 *Let the elements of the set of normal vectors to the initial convex polygon,* $N_i^0 \quad i = 1, .., M$*, be of length 1. Furthermore, assume that*

$$O \notin \cup_i \ Conv \, \{N_{i-1}^0, N_i^0, N_{i+1}^0, N_{i+2}^0\}. \quad (*)$$

Then,

$$\lim_{k \to \infty} \max_i \alpha_i^k = 0,$$

whereas α_i^k *denotes the angle between* N_i^k *and* N_{i+1}^k.

Proof. Condition (*) ensures that the limit curve obtained by the application of Chaikin's scheme to the polygon prescribed by N^0 (i.e., a quadratic B-spline curve) is a *closed* curve which contains the origin. Continuity and smoothness of the limit curve are therefore guaranteed. The set of normal vectors generated in the dual Chaikin process is *dense*. Density means that for each chosen normal-vector direction in the plane, there exists a finite stage in dual Chaikin process in which an arbitrarily close normal is created. A simple calculation shows that the convergence rate of the distance between adjacent normal vectors is bounded above by the convergence rate of a vanishing geometric series. Furthermore, as the normal vectors are bounded away from zero, the required result is immediately obtained. □

Theorem 2.2 *Under the conditions of Theorem 2.1, the lengths of the edges of* P^k *tend to zero.*

Proof. This follows directly from the theory of duality (see e.g., [7]) or it can be shown directly as in [9].

Corollary 2.3 *The limit boundary curve obtained by the application of the dual Chaikin scheme on a convex polygon is strictly convex.*

Next, we discuss the continuity property of the boundary curve. At each finite stage k, P^k is convex and it is bounded by the "mean polygon", which is the polygon formed by the intersection of the lines which are the algebraic mean of the initial lines. This is easy to see as the local behaviour of the scheme around each vertex is described by the cyclic matrix C

$$C = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ & \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

It holds that

$$\lim_{n \to \infty} C^n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ & \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Theorem 2.4. *A monotonously incresing sequence of bounded and convex polyhedra converges to a convex polyhedron, hence with continuous boundary.*

The proof is given in [9]. One notes that the dual Chaikin operating upon an initial closed convex polygon generates a sequence of closed, non-degenerate, convex polygons. This sequence is monotonously increasing (i.e., $P^{k+1} \supset P^k$) and bounded by the mean polygon. By Corollary 2.3 and Theorem 2.4, we obtain that the limit body generated by the dual Chaikin process is a strictly convex, closed body with a continuous boundary. The last step is to show that the limit curve is C^1. For this purpose, we shall define halfcones.

Definition 2.5. *Given a convex curve C, one defines for each point $p \in C$, the halfcone of p, which is the set of points on outwards halflines through p, which are perpendicular to the supporting lines through p.*

If p is a regular point of the curve C, it has a unique supporting line, hence a unique normal vector (equivalently, $H(p)$ is degenerate). We also define order on the curve for appropriate orientation.

Lemma 2.6. *Given an orientated closed convex curve C, it holds that*

$$H(A) \cap H(B) = \emptyset \quad \forall A, B \in C \quad s.t. \quad A \neq B.$$

Proof. Assume otherwise, then there $\exists A, B \in C, \quad A \neq B \quad$, s.t.

$$H(A) \cap H(B) \neq \emptyset.$$

Let $D \in H(A) \cap H(B)$. Now consider the triangle formed by A, D, B. The supporting lines at A and B must be over the edge AB. Otherwise, they cut the curve C. We obtain that $\angle DBA + \angle DAB > \pi$, which is impossible in the Euclidean space. $\square$

Theorem 2.7 *The limit boundary curve of the dual Chaikin scheme applied to a strictly convex data set satisfying the conditions of Theorem 2.1 is C^1.*

Proof By Theorem 2.4, the limit boundary curve of the the limit body generated by the dual Chaikin scheme is convex. We shall denote this curve by P^∞. Hence, there is a supporting line through each of its points. To show that the limit curve is differentiable, one has to show that through each point $p \in P^\infty$, there is a *unique* supporting line or equivalently, $H(p)$ is degenerate. One distinguishes between two cases:

1. $\exists i, k$ such that $p = v_i^k$. In this case, the sequence of set of lines defining p converges in the limit to a unique supporting line given by the arithmetic mean of the lines intersecting at p.

2. The more intricate case occurs when p is not a vertex attained at any finite stage of the dual Chaikin scheme. Assume that there is a point $p \in P^\infty$, such that H(p) is non-degenerate. Let us denote the head-angle of H(p) by θ.

By Theorem 2.1,

$$\lim_{k\to\infty} \max_i \alpha_i^k = 0$$

Hence, $\exists A, B$ with unique normals, N_A, N_B, respectively, on P^∞, such that the triple (A, p, B) is ordered and $\angle(N_A, N_B) < \theta/2$. Therefore, either N_A or N_B intersects $H(p)$. This contradicts Lemma 2.6, and thus the proof of Theorem 2.7 is complete. $\square$

§3. The Dual Doo-Sabin Scheme

The Doo-Sabin scheme operates on general control nets by "cutting-off" the corners of a polyhedral configuration, whereas the faces need not be planar. A face of valency n (defining n edges, n vertices, $\{v_i^k\}_{i=1}^n$) is replaced by a new face of n vertices $\{v_j^{k+1}\}_{j=1}^n$, defined by

$$v_j^{k+1} = \sum_{i=1}^n a_{i,j} v_i^k$$

with

$$a_{j,j} = \frac{n+5}{4n},$$

$$a_{i,j} = \frac{3 + 2\cos\frac{2\pi(i-j)}{n}}{4n} \text{ for } i \neq j. \quad (**)$$

Additional new faces are formed by joining the new vertices with new edges (see e.g., [3]). This is a generalization of the 1-D algorithm of Chaikin's "corner-cutting" scheme. The new net contains more vertices and smaller faces than the original mesh. After a few iteration steps, the bulk of the structure formed by the control net will contain faces arranged in a regular lattice (i.e., with valency 4) almost everywhere. Quadratic B-splines tensor products represent the regular regions. Carrying out the subdivision scheme on an arbitrary control net yields a limit surface, which, usually, has no analytic representation but is C^1.

Returning now to the notion of duality, we would like to investigate the *dual* process of the Doo-Sabin scheme. In other words, we shall use the Doo-Sabin scheme on the dual objects to points in the space, namely upon planes. Our objects of study are closed convex polyhedra. Let us denote the initial polyhedron by B^0, its set of vertices by $\{v_q^0\}$ and its set of planar faces by $\{F_q^0\}$. The vertices of B^0 are defined by the intersection of planar faces,

$$v_q^0 = \cap_{j=1}^{n_q^0} F_{q,j}^0,$$

where n_q^0 is the valency of v_q^0 and $F_{q,j}^0, \quad j = 1, 2, .., n_q^0$ are the faces intersecting at v_q^0. The normal representation of the planes $\{P_q^0\}$ corresponding to $\{F_q^0\}$ is given by

$$P_q^0(x) = \langle N_q^0, x\rangle - \langle N_q^0, v_q^0\rangle = 0.$$

Each plane can be viewed as a vector in $I\!R^4$

$$\begin{pmatrix} N_i^0 \\ d_i^0 \end{pmatrix}, \text{ where } d_i^0 := \langle N_i^0, v_i^0\rangle.$$

The control net of the *dual graph* consists of a set of vertices $\{P_q^0\}$ and an edge in the dual graph between P_q^0 and P_j^0 exists if and only if P_q^0 and P_j^0 are adjacent faces of B^0. We apply the Doo-Sabin scheme upon the dual graph, defining the equations of the new planes by

$$P_{q,j}^{k+1} = \sum_{i=1}^{n_q^k} a_{i,j} P_{q,i}^k, \quad k \geq 0.$$

The *dual Doo-Sabin scheme* applied on B^0 is defined as the application of the Doo-Sabin scheme on the planes corresponding to the faces of B^0, P_q^0, as given above in the normal form or as points in $I\!R^4$. Applying the primary scheme on the dual graph of planes, we obtain at level k a family of planes $\{P_i^k\}_{i\in I^k}$, whereas I^k is a the index set corresponding to stage k. Each plane P_i^k defines a halfspace, $H_i^k := \{x \mid P_i^k(x) \geq 0\}$. We now define $B^k := \cap_{i\in I^k} H_i^k$. The dual scheme is applied ad infinitum and we consider the limit body $\lim_{k\to\infty} B^k$, whose characteristics we shall investigate. A different approach to restart the process after each subdivision step.

There are intersections of new planes around an old vertex and above an old face. In general, the intersection of the planes above an old face does not yield a vertex but a polygonal line, which we name a "generalized vertex". The "generalized vertex" is regarded as a vertex in the sense that we apply the weights $a_{i,j}$ around it as if it were a real vertex. The dual Doo Sabin scheme is interpolatory in the sense that the vertices of B^0, $\{v_i^0\}$, are preserved all along the process. Real vertices "born" throughout the process are preserved as well. As for the "generalized vertices", these are not preserved, but the polygonal lines gradually become shorter and seem to collapse into a point in the limit (see part 4). One notes that the existence of "generalized vertices" is equivalent to the existence of corresponding nonplanar faces in the dual graph, and the Doo-Sabin scheme is well defined on such a graph. Next, we will show that the set of normal vectors near each real vertex or 'generalized vertex' converges to a single normal vector, which defines a unique supporting plane at the corresponding vertex. But before, we define influence domains.

Definition 3.1 *Given is the set of unit normal vectors to the initial polyhedron, N^0. We perform the primary Doo-Sabin process upon it. For each N_i^k we define the influence domains of order j recursively*

$$M_{i,0}^k := N_i^k$$

$$M_{i,j}^k := M_{i,j-1}^k \cup \{ \textit{ all normals having a common edge with vertices in } M_{i,j-1}^k \}.$$

Theorem 3.2 *Let the normal vectors to the initial polyhedron be on a unit sphere around the origin O, such that*

$$0 \notin \cup_i \; Conv\, \{M_{i,2}^0\}.(***)$$

Then,

$$\lim_{n\to\infty} \max_i \alpha_i^k = 0,$$

whereas α_i^k is the angle between pairs of normal vectors adjacent planes in $\{P_i^k\}$

Proof. Performing the primary scheme upon the set of vertices of the normal vectors results in a quadratic B-spline tensor product (except near extraordinary points). The limit surface is C^1 and closed, hence the set of normals N^k gets denser as $k \to \infty$, where density is defined in the same sense as in the one-dimensional case. Similarly to the one-dimensional case, it is easy to see that the distance between neighbouring normal vectors tends to zero. Condition (***) assures that $\infty > \|N_i^k\| > c > 0$, thus, the angle between the normal vectors must tend to zero, too. □

In full analogue to the one-dimensional case, the matrix describing the unfolding of the normal vectors from stage 0 to k near each vertex (or "generalized vertex") v_i^k, denoted by VN^0 and VN^k, respectively, is given by the cyclic matrix D, whose components are the coefficients given in (**)

$$D = \begin{pmatrix} a_{i,1} & a_{i,2} & .. & a_{i,n_i^k} \\ a_{i,n_i^k} & a_{i,1} & .. & a_{i,n_i^k-1} \\ .. & .. & .. & .. \\ a_{i,2} & a_{i,3} & .. & a_{i,1} \end{pmatrix}$$

It holds that

$$VN^k := D^k * VN^0$$

Independently of the iteration stage we are at, the process is 'fractal', self-similar. The application of the matrix D on the relevant normal vectors will give the new generation of normal vectors. As all eigenvalues of D are smaller than one, except for the unique leading eigenvalue, 1, it holds that

$$\lim_{n\to\infty} D^n = \begin{pmatrix} \frac{1}{n_i^k} & \cdots & \frac{1}{n_i^k} \\ \vdots & \ddots & \vdots \\ \frac{1}{n_i^k} & \cdots & \frac{1}{n_i^k} \end{pmatrix}.$$

Hence, we see that, locally, the normal vectors around each 'generalized vertex' coalesce in the limit to the mean of all normal vectors surrounding the vertex. This normal vector uniquely defines a supporting plane at each vertex of the initial polyhedron as well as at every vertex, real or generalized, created throughout the dual Doo-Sabin process. Also, the polyhedron formed by the intersection of the limit mean tangent planes of the initial polyhedron is a bounding polyhedron of every P^k. Next, We use the 2-D generalization of Theorem 2.4.

Theorem 3.3 *A sequence of convex, monotonously increasing, bounded polyhedra converges to a convex polyhedron.*

Lemma 3.4 When applied to a convex polyhedron, under condition (***) of Theorem 3.2, the dual Doo-Sabin scheme generates a monotonously increasing, bounded sequence of convex polyhedra $\{B^k\}$. Hence $\lim_{k\to\infty} B^k =: B^\infty$ is convex.

Proof. The monotonicity property is obvious from the non-negativity of the weights $a_{i,j}$. $B^k \subseteq H_j^{k+1} \Rightarrow B^{k+1} \subseteq B^k$. Convexity is assured simply by the fact that B^k is the intersection of a finite number of half planes. Local boundedness by the "mean polyhedron" is straightforward. □

Therefore, the dual Doo-Sabin process converges to a convex polyhedron. As a last stage of the analysis, we will show that under a certain set of restrictions, the dual Doo-Sabin produces a C^1 limit boundary. For this purpose, we will need an enlarged version Definition 3.1.

Definition 3.5. For each vertex v_i^k on the boundary of the polyhedron created at stage k of the Dual Doo-Sabin process, B^k, one defines a sequence of a set of faces recursively, which are called influence domains of order j.

$$F_{i,0}^k := \{ \text{ faces of the polyhedron } B^k \text{ intersecting at } v_i^k \}$$

$$F_{i,j}^k := F_{i,j-1}^k \cup \{ \text{ all faces having a common edge with faces in } F_{i,j-1}^k \}$$

$$G_{i,j} := \{ \text{ all planes of the faces in } F_{i,j+1}^k \backslash F_{i,j}^k \}$$

Theorem 3.6. Under the condition (***) of Theorem 3.2 and under the assumption that the Doo-Sabin scheme preserves convexity or, equivalently, that at any finite stage, no face is degenerate, the dual Doo-Sabin scheme produces a limit body with a C^1 limit surface.

Proof. The proof is analogous to the smoothness proof for the 'honeycomb' scheme [5]. One notes that in the dual Doo-Sabin process as well, the data set $F_{i,1}^k, G_{i,1}^k$ contains all the information required to define $F_{2^l}^{k+1}, G_{2^l}^{k+1}$ for every $l \geq 0$. For every face Q in B^{k+1}, there is an index m, such that Q is determined by the set $\{F_{m,1}^k, G_{m,1}^k\}$. As the angle between adjacent normal vectors to the same vertex tend to zero and given that either the primary scheme preserves convexity or that at every finite stage, no face is degenerate, for k large enough, the polyhedron faces $F_{i,1}^k, G_{i,1}^k$ form a **closed** polyhedron, denoted as Δ_i^k. Δ_i^k is a bounding polyhedron for a patch in the limit surface. B^k converges to a polyhedron with continuous boundary, which we denote by B^∞. Just as we did in the one dimensional case, one distinguishes between two cases:

1. There exist k such that v is a real vertex created at a finite level k. In this case there is a unique supporting plane in the limit surface, which is given by the algebraic mean of the initial planes whose intersection defines v.
2. v is not a vertex attained at some finite subdivision level. In this case, by the above argumentation, there exists an index j, such that v is in the influence domain (of order 1) of the data set $F_{i,1}^k, G_{i,1}^k$, and v is in the closure of Δ_i^k. Every supporting plane of B^∞, Π, at v, does not cut B^k. In particular, it does not cut any face of $F_{i,1}^k$. Let n be an outward unit normal vector to Π. Π cuts from Δ_i^k a convex polyhedron Q, whose base is on Π with area w_0^k and the other faces of Q are on $G_{i,1}^k$ with areas w_j^k and the corresponding normals n_j^k. As shown in [5],

$$nw_0^k = \sum n_j w_j^k$$

I.e., n is in C_i^k, the cone formed by the n_j^k. Now assume that there are two supporting planes Π_1, Π_2 at v with corresponding normal vectors n_1, n_2, such that $n_1 \neq n_2$. $n_1, n_2 \in C_i^k$. By Theorem 3.2, the cones C_i^k shrink to a degenerate cone as $k \to \infty$, and thus we must have $n_1 = n_2$. Therefore, there is a unique supporting plane through v. This completes the proof of Theorem 3.6. $\square$

Fig. 1. Dual Chaikin Scheme.

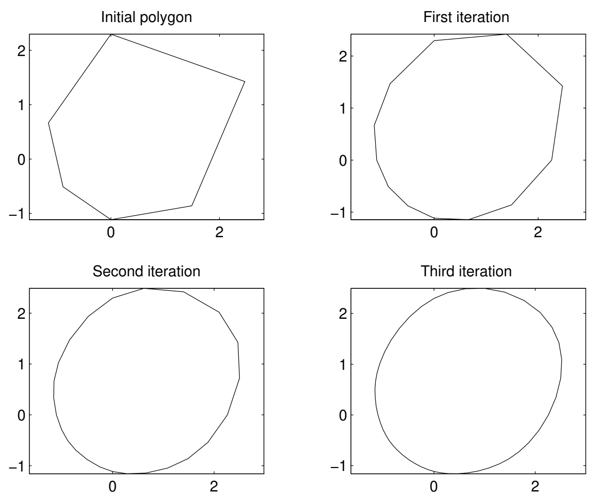
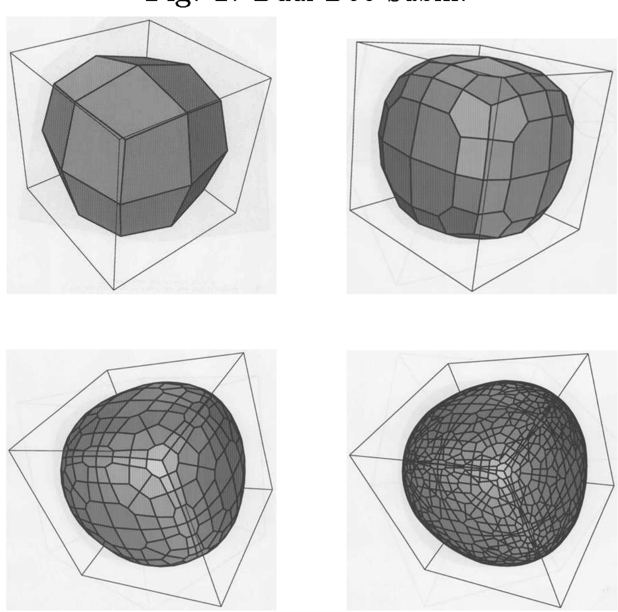


Fig. 1. Dual Doo-Sabin.



§4. Examples

The figures above show the application of the Dual Schemes upon a nonsymmetric hexagon and a polyhedron, respectively.

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