Lecture 4: March 29, 2010

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4.1 Lecture overview

In this lecture we will concern ourselves with the existence of a Nash equilibrium and the price of anarchy and stability in several game classes. We will:

- Define a class of games called *congestion games* and we'll show the existence of a pure Nash Equilibirum in any congestion game.
- Define a class called *potential games* and we'll study the existence of pure equilibrium in those games. (Actually the two classed are equivalent)
- Study two variants of a *Network Creation* game (unfair and fair), and analyze the price of anarchy (when a Nash equilibrium exists)

4.2 Congestion Games

4.2.1 Example

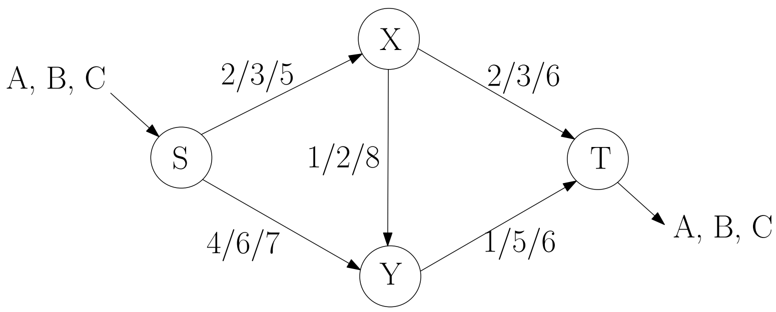

Figure 4.1: Example of a congestion game

Let us start with an illustrative example of a congestion game. Players A,B and C have to go from point S to T using road segments SX,XY,...etc. (See Figure 4.1) Numbers on

[1]Partially based on 2006 scribe notes by Hila Pochter, Yaakov Hoch, Omry Tuval, and 2004 scribe notes by Nir Yosef and Ami Koren

edges denote the cost that each player pays for using that egde, the actual cost for each player is a function of the number of players using that road segment (i.e. a *discrete delay* function). For example: if segment SX is used by a two player, those two players will pay 3 "money points" each for using that egde, if three players will use it than each one of them will pay 5. The total cost for a player is the sum of the costs on all segments he/she uses.

4.2.2 Congestion game - Definition

A congestion model $(N,M,(A_i)_{i\in N},(c_j)_{j\in M})$ is defined as follows:

- $N = \{1, 2, ..., n\}$ denote the set of n players.
- $M = \{1, 2, ..., m\}$ denote the set of m resources.
- For $i \in N$, let A_i denote the set of strategies of player i, where each strategy $a \in A_i$ is a non empty subset of the resources.
- For $j \in M$, $c_j \in R^n$ denote the vector of costs, where $c_j(k)$ is the cost related to resource j, if there are exactly k players using that resource.

Let $A = \times_{i\in N} A_i$ be the set of all possible joint actions. For any $\vec{a} \in A$ and for any $j \in M$, let $n_j(\vec{a})$ be the number of players using resource j, assuming $\vec{a}$ is the current joint action, i.e. $n_j(\vec{a}) = | \{i \mid \ j \in a_i\} |$. The cost function for player i is $u_i(\vec{a}) = \sum_{j\in a_i} c_j(n_j(\vec{a}))$.

Remark 4.1 *In our example we used* Routing with unspllitable flow*. This game can be easy viewed as a congestion game. The resources are the edges $M = E$, the possible strategies are the possible routes for player i: $A_i = P_i$ and the cost of the edge is it's latency, i.e. $c_e(k) = l_e(k)$*

4.2.3 Pure Nash equilibrium

We establish that every congestion game has a pure Nash equilibrium.

Theorem 4.2 *Every finite congestion game has a pure Nash equilibrium.*

Proof: Let $\vec{a} \in A$ be a joint action.
Let Φ: $A \to R$ be a potential function defined as follows: $\Phi(\vec{a}) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(\vec{a})} c_j(k)$
Consider the case where a single player changes its strategy from a_i to b_i (where $a_i, b_i \in A_i$).
Let Δu_i be the change in its cost caused by the the change in strategy:
$\Delta u_i = u_i(b_i, \vec{a}_{-i}) - u_i(a_i, \vec{a}_{-i}) = \sum_{j\in b_i - a_i} c_j(n_j(\vec{a}) + 1) - \sum_{j\in a_i - b_i} c_j(n_j(\vec{a}))$.
(explanation: change in cost = cost related to the use of new resources minus cost related to use of those resources which are not in use anymore due to strategy change)

Let $\Delta\Phi$ be the change in the potential caused by the change in strategy:
$\Delta\Phi = \Phi(b_i, \vec{a}_{-i}) - \Phi(a_i, \vec{a}_{-i}) = \sum_{j \in b_i - a_i} c_j(n_j(\vec{a}) + 1) - \sum_{j \in a_i - b_i} c_j(n_j(\vec{a}))$
(explanation: immediate from potential function's definition).
Thus we can conclude that for a single player's strategy change we get: $\Delta\Phi = \Delta u_i$.

That's an interesting result: We can start from an arbitrary joint action $\vec{a}$, and at each step let one player apply his best response (thus reducing his own cost). That means, that at each step, Φ is reduced (identically). Since Φ can accept a finite amount of values, it will eventually reach a local minima. At this point, no player can achieve any improvement, therefore we reach a Nash equilibrium. $\square$

Remark 4.3 Φ *is actually an* exact potential function *as we will define shortly.*

4.3 Potential games

4.3.1 Potential functions

Let $G =< N, (A_i), (u_i) >$ be a game where $A = \times_{i \in N} A_i$ is the collection of all pure strategy actions in G.

Definition A function Φ: $A \to \mathbb{R}$ is an *exact potential* for game G if
$\forall_{\vec{a} \in A} \forall i \in N \, \forall b_i \in A_i \ \Delta\Phi = \Delta u$
i.e: $\Phi(b_i, \vec{a_{-i}}) - \Phi(a_i, \vec{a_{-i}}) = u_i(b_i, \vec{a_{-i}}) - u_i(a_i, \vec{a_{-i}})$

Definition A function Φ: $A \to \mathbb{R}$ is a *weighted potential* for game G if
$\forall_{\vec{a} \in A} \forall_{a_i, b_i \in A_i} \ \Delta\Phi = \omega_i \Delta u$
i.e: $\Phi(b_i, \vec{a_{-i}}) - \Phi(a_i, \vec{a_{-i}}) = \omega_i(u_i(b_i, \vec{a_{-i}}) - u_i(a_i, \vec{a_{-i}})) = \omega_i \Delta u_i$
Where $(\omega_i)_{i \in N}$ is a vector of positive numbers (weight vector).

Definition A function Φ: $A \to \mathbb{R}$ is an *ordinal potential* for a minimum game G if
$\forall_{\vec{a} \in A} \forall_{a_i, b_i \in A_i} \ \Delta u_i < 0 \Rightarrow \Delta\Phi < 0$
i.e: $(u_i(b_i, \vec{a_{-i}}) - u_i(a_i, \vec{a_{-i}}) < 0) \Rightarrow (\Phi(b_i, \vec{a_{-i}}) - \Phi(a_i, \vec{a_{-i}}) < 0)$
(Intuition: when a player decreases his cost, the potential function also decreases).

Remark 4.4 *Considering the above definitions, it can be seen that the first two definitions are special cases of the third.*

4.3.2 Potential games

Definition A game G is called an *ordinal potential game* if it has an an ordinal potential function.

Theorem 4.5 *Every finite ordinal potential game has a pure equilibrium.*

Proof: Analogous to the proof of Theorem 4.2: Given an initial strategy vector, each time a player changes strategy and reduces his cost, the potential function also decreases. since this is a finite game, the potential function can have a finite set of values and therefore the process of successive improvements by players must reach a local minima of the potential function. No improvements (by any player) are possible at this point, and therefore this is a pure equilibrium. □

Remark 4.6 *Any congestion game (as defined earlier) is an exact potential game. The proof of Theorem 4.2 is based on this property of congestion games.*

4.3.3 Examples

4.3.3.1 Exact potential game (party affiliation game)

Consider an undirected graph $G = (V, E)$ with a weight function $\vec{\omega}$ on its edges. In this game the players are the vertices and the goal is to partition the vertices set V into two distinct subsets D_1, D_2 (where $D_1 \cup D_2 = V$):
For every player i, choose $s_i \in \{-1, 1\}$ where choosing $s_i = 1$ means that $i \in D_1$ and $s_i = -1$ means that $i \in D_2$. The weight on each edge denotes how much the corresponding vertices 'want' to be on the same set. Thus, define the value function of player i as $u_i(\vec{s}) = \sum_{j \neq i} \omega_{i,j} s_i s_j$. (A player 'gains' $\omega_{i,j}$ for players that are in the same set with him, and 'loses' for player in the other set. Note that $\omega_{i,j}$ can be negative.) Each player tries to maximize its utility function.

On the example given in Figure 4.3, $U_1 = +3 + 5 - 6 = 2$ and $U_2 = -6 + 0 - 2 = -8$. it can be seen that players 1,2 and 4 have no interest in changing their strategies, However, player 3 is not satisfied, it can increase his profit by changing his set to D_1(meaning changing his decision to $+1$).
Using $\Phi(\vec{s}) = \sum_{j<i} \omega_{i,j} s_i s_j$ as our potential function, let us consider the case where a single player i changes its strategy (shifts from one set to another):
$\Delta u_i = \sum_{j \neq i} \omega_{i,j} s_i s_j - \sum_{j \neq i} \omega_{i,j} (-s_i) s_j = 2 \sum_{j \neq i} \omega_{i,j} s_i s_j = \Delta(\Phi)$
which means that Φ is an exact potential function, therefore we conclude that the above game is an exact potential game.

4.3.3.2 Weighted potential game

Consider the following load balancing congestion model $(N, M, (\omega_i)_{i \in N})$ with M identical machines, N jobs and $(\omega_i)_{i \in N}$ weight vector ($\omega_i \in \mathbb{R}^+$). The load on a machine is defined as the sum of weights of the jobs which use it: $L_j(\vec{a}) = \sum_{i:\ a_i = j} \omega_i$ where $\vec{a} \in [1, 2, ..., M]^N$ is a joint action.

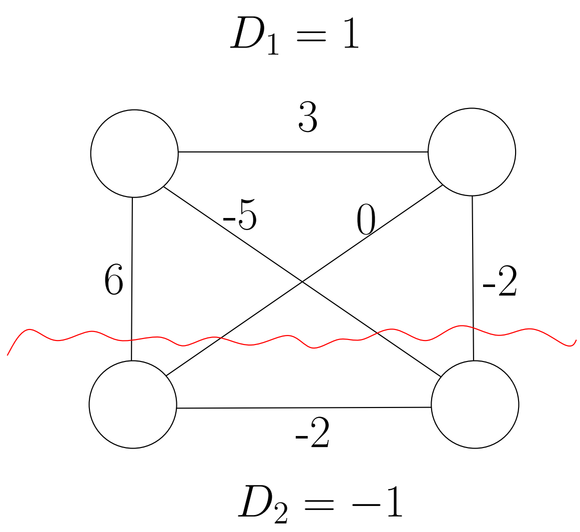

Figure 4.2: Example for an exact potential game

Let $u_i(\vec{a}) = L_{a_i}(\vec{a})$ denote the cost function of player i. We would like to define a potential function whose change in response to a single player's strategy change will be correlated with the change in the player's cost function.
The potential function is defined as follows: $\Phi(\vec{a}) = \frac{1}{2}\sum_{j=1}^{M} L_j^2(\vec{a})$. Consider the case where a job i shifts from its selected machine, M_1, to another machine M_2 (where M_1 and M_2 are two arbitrary machines):
Let Δu_i be the change in its cost caused by the strategy change:
$\Delta u_i = u_i(M_2, \vec{a_{-i}}) - u_i(M_1, \vec{a_{-i}}) = L_2(\vec{a}) + \omega_i - L_1(\vec{a})$.
(Explanation: change in job's load = load on new machine minus load on the old machine)
Let $\Delta\Phi$ be the change in the potential caused by the strategy change:

$$\begin{aligned}
\Delta\Phi &= \Phi(M_2, \vec{a_{-i}}) - \Phi(M_1, \vec{a_{-i}}) \\
&= \frac{1}{2}[(L_1(\vec{a}) - \omega_i)^2 + (L_2(\vec{a}) + \omega_i)^2 - L_1^2(\vec{a}) - L_2^2(\vec{a})] \\
&= \omega_i(L_2(\vec{a}) - L_1(\vec{a})) + \omega_i^2 = \omega_i(L_2(\vec{a}) + \omega_i - L_1(\vec{a})) \\
&= \omega_i \Delta u_i
\end{aligned}$$

Therefore, we can conclude that load balancing on identical machines is a weighted potential game.

Lemma 4.7 *For every game G such that for every joint action $\vec{a} \in A$ any path of best-response actions is finite (i.e a path of best-responses gets to equilibrium), there exists an* ordinal potential function Φ.

Proof: Let $\Phi(\vec{a})$ be the length of the longest possible improvement path (path of best-response actions) in the game G starting from $\vec{a}$. The function Φ is well defined because of the property of G assumed in the lemma.
Consider an improvement step from $\vec{a_1} \in A$ to $\vec{a_2} \in A$. In contradiction, assume that $\Phi(\vec{a_2}) \geq \Phi(\vec{a_1})$. Therefore from $\vec{a_2}$ there exists an improvement path of length $1 + \Phi(\vec{a_2})$ which is a contradiction to $\Phi(\vec{a_1})$ being the *longest* improvement path starting from $\vec{a_1}$. This shows that $\Phi(\vec{a_2}) < \Phi(\vec{a_1})$, and that means Φ is an *ordinal potential function.* □

4.4 Connection Game (General Cost Sharing)

We have a graph $G = (V, E)$. Each edge e has a price $C(e)$. Each player i has two nodes s_i and t_i that he wants to connect. Each player's action is a vector $p_i \in \mathbb{R}^m$, $(p_i(e_1), p_i(e_2), ..., p_i(e_m)) \geq 0$ meaning player i offers $p_i(e)$ for the edge e. Let's denote by p the joint action of the players, and $G(p) = (V, E_p)$ is the graph resulting from the players' strategies, where $e \in E_p$ iff $\sum_i p_i(e) \geq C(e)$. Player i's cost function $C_i(p)$ is equal to ∞ if s_i and t_i are not connected and otherwise it is $\sum_{e \in E_p} p_i(e)$. The player's aim is to minimize this cost (yet to have s_i connected to t_i) We define the social cost to be $C(p) = \sum_i c_i(p) = \sum_{e \in E_p} c(e)$.

Remark 4.8 *Notice that in a Nash equilibrium the players will pay exactly the cost of each edge bought in G(p) and every one of them will have a path from s_i to t_i in G(p).*

Theorem 4.9 *A pure Nash equilibrium does not always exist for in a network creation game.*

Proof: Let's look at the following game in Figure 4.4.

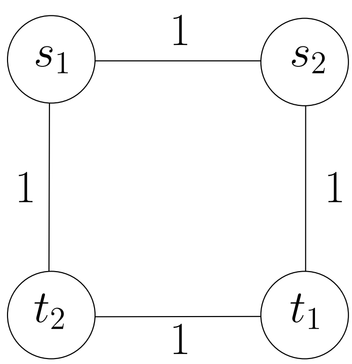

Figure 4.3: Network Creation Game with no Nash equilibrium (taken from [2])

- In every NE the players will buy exactly 3 edges.
- Without loss of generality assume that the edges bought are $(s_1, s_2), (s_1, t_2), (t_1, s_2)$.

- Only player 1 pays for (s_2, t_1) (he's the only player who needs it).
- Only player 2 pays for (s_1, t_2) (he's the only player who needs it).
- Without loss of generality, suppose player 1 pays (at least ϵ) for (s_1, s_2). Player 1 can change his strategy and
  - not pay for (s_1, t_1) and (s_1, s_2) gaining $(1 + \epsilon)$.
  - buy (t_1, t_2) paying 1.
- Thus there is no pure Nash equilibrium

$\square$

Remark 4.10 *In the above proof, the problem is that player 1 ignores the fact that in the resulting network player 2 has no motivation to continue paying for* (s_1, t_2). *This is a serious weakness of the Nash equilibrium concept: it ignores the fact that other players can and might react to a certain player changing his strategy.*

We now define a social cost function $C(p) = \sum c_i(p)$ and assume we are given a game in which there exists a Nash equilibrium.

1. $PoA \leq N$: Every player i, given the other players actions, the cost of connecting s_i to t_i is at most the cost of connecting them regardless of the other players. Which in turn is at most the total cost of the optimal solution. So every player pays at most OPT and the total cost in a NE is at most $N \cdot OPT$.

2. $PoA \geq N$: Consider Figure 4.4.

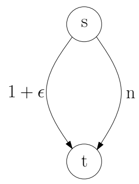

Figure 4.4: POA $= \frac{N}{1+\epsilon}$ (taken from [2])

This is a network of a single source single sink network creation game with N players. Assume that all the players want to connect the the top vertex s to bottom vertex t. In the social optimum solution all the players buy together the left edge from s to t. Each player pays only $\frac{1+\epsilon}{N}$. The social cost in this case is $1 + \epsilon$. Now look at the worst case Nash equilibrium. In this case each player pays his part in buying the right edge thus each player pays 1. (notice that this is indeed equilibrium since each player will pay more if he decides to buy the left edge alone). The social cost in this case is N. Therefore for $\epsilon \to 0$ we get $PoA \geq N$.

Definition [Price of Stability]

$$PoS = \min_{p \in PNE} \frac{C(p)}{OPT(p)}$$

In the previous example the PoS is 1 since the optimum is a Nash equilibrium. Now we will show a case in which the PoS is high. Consider Figure 4.5.

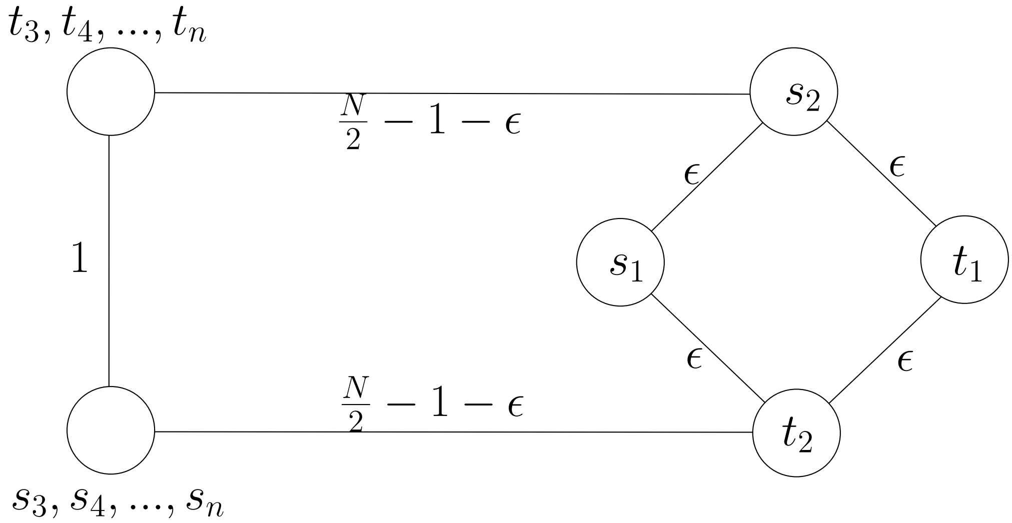

Figure 4.5: $PoS \approx N - 2$ (taken from [2])

The social optimal cost is $1 + 3\epsilon$, when the players buy the leftmost path and 3 of the ϵ edges in the square on the right. However the lowest cost achieved in an equilibrium is $N - 2 + \epsilon$, when the players buy the two edges with cost $\frac{N}{2} - 1 - \epsilon$ and three ϵ edges in the right square. Note that there is no other Nash equilibrium due to the square on the right which we have shown that does not admit a Nash equilibrium.

4.5 Network Creation Game with fair cost sharing

We will consider a modification of the previous game. Instead of allowing the players to directly set the cost, players will choose the edges and the cost will be divided equally between all players participating in an edge. The strategies for player i are $a_i \in A_i$ where $a_i \subseteq E$. Define

$$n_e(a) = |\{i \; : \; e \in a_i\}|$$

The cost of edge e to the player choosing it is $c_e(a) = \frac{c(e)}{n_e(a)}$
The total cost for player i is: $c_i(a) = \sum_{e \in a_i} \frac{c(e)}{n_e(a)}$
The social cost is $C(a) = \sum_i c_i(a)$
We can define the game as a congestion game with $c_e(k) = \frac{c(e)}{k}$.

$$u_i(a) = \sum_{e \in a} c_e(n_e(a))$$

since this is a congestion game it is an exact potential game and thus always have a pure Nash equilibrium !

Remark 4.11 *The previous example about PoA still holds, thus $PoA \leq N$.*

Next, we show that the price of stability is only logarithmic.

Theorem 4.12

$$PoS \leq H(N) = \sum_{l=1}^{N} \frac{1}{l}$$

Proof:

$$\Phi(a) = \sum_{e \in E} \sum_{l=1}^{n_e(a)} \frac{c(e)}{l} = \sum_{e} c(e) \cdot H(n_e(a))$$

Consider $a^* = argmin_a \Phi(a)$ which is obviously a Nash equilibrium. Let a_{opt} be the optimal solution.

$$H(N) \cdot C(a_{opt}) \geq \sum_{e \in E_{opt}} c(e) \cdot H(n_e(a_{opt})) \geq \Phi(a_{opt}) \geq \Phi(a^*) \geq C(a^*)$$

□

Theorem 4.13

$$PoS \geq H(N) = \sum_{l=1}^{N} \frac{1}{l}$$

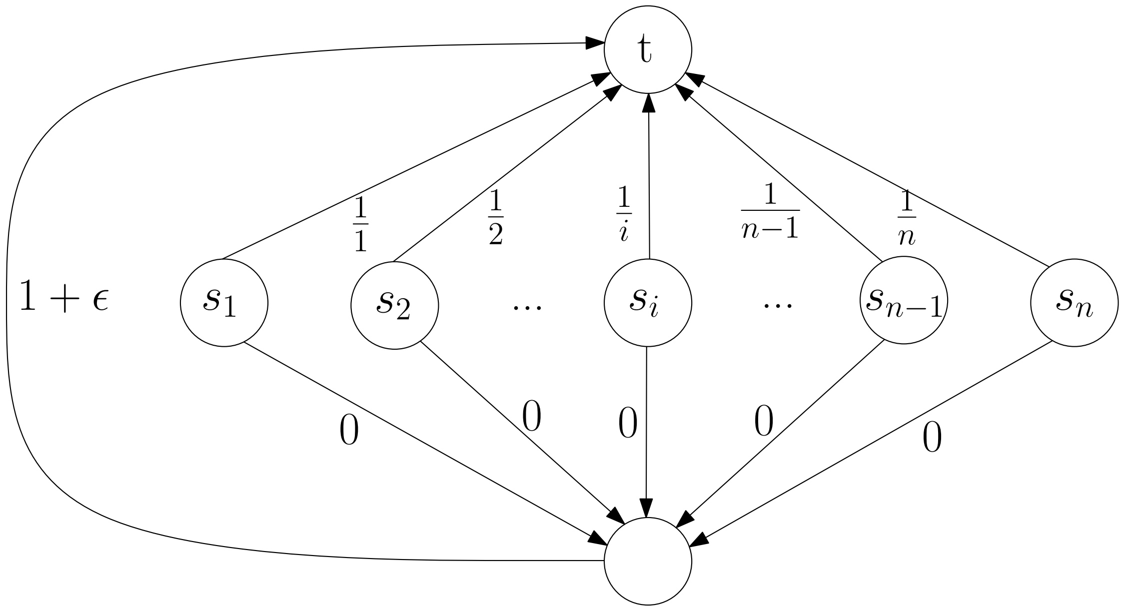

Figure 4.6: $PoS \geq H(N)$ (taken from [3])

Proof:

Consider at the following game (Figure 4.7): This is a single target game with N players. In the social optimal solution each player would buy the 0 cost edge from him to the bottom node and the $1+\epsilon$ edge (whose cost will be shared equally between all players).
The social cost in this case is $1+\epsilon$. The only Nash equilibrium that exists in this case is the one in which each player i buys the $\frac{1}{i}$ edge from him to t, which gives us a social cost of $H(N)$. We will show that this is the only Nash equilibrium: Each player i has only 2 ways to connect s_i to t. Let's assume that a group Γ is connecting using the $1+\epsilon$ edge. Let's i be the player with the highest index in the group Γ. Player i would pay in this case $\frac{1+\epsilon}{|\Gamma|}$ and if he chooses to use the $\frac{1}{i}$ edge from him to t he would pay $\frac{1}{i}$. Since $|\Gamma| \leq i$, player i would rather use the $\frac{1}{i}$ edge. Therefore this is not an equilibrium. In this case $PoS \geq \frac{H(N)}{1+\epsilon}$ $\square$

4.6 Strong Equilibrium

4.6.1 Symmetric Games

As we shall see in the next section, a strong equilibrium doesn't always exist. However, for a class of games called *symmetric games* a strong equilibrium is guaranteed to exist.

Definition A *symmetric connection game* is a connection game where all players have the same source s and the same sink t.

Theorem 4.14 *In every symmetric connection game there exists a strong Nash equilibrium.*

Proof: Consider a state of the game where all players fairly share the cost of the shortest path from s to t. We claim that this state is a strong equilibrium. For every coalition of players that may consider moving to a different path, the cost of this path is obviously not less than the price of the shortest path. The price of the new path will be shared by at most n players, so clearly there will be a player of the coalition which will not benefit of the move. $\square$

4.6.2 Fair Cost Sharing

A strong equilibrium doesn't always exist in a connection game with fair cost sharing. Take for example the game described in figure 4.8: Each player has only 2 paths that lead from

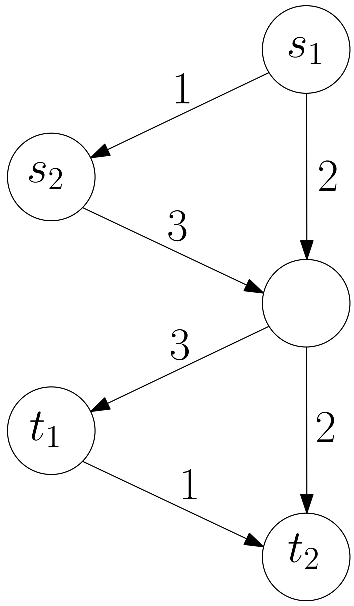

Figure 4.7: No strong equilibrium in fair cost sharing

his source to his sink: a path of length (length is the number of edges) 2, or a path of length 3. We can therefore represent the game as a 2x2 table:

	2	3
2	(5, 5)	(3.5, 5.5)
3	(5.5, 3.5)	(4, 4)

We conclude that this game is equivalent to the prisoner's dillema, and hence (as seen in lecture 2) has no strong equilibrium.

So we've seen that strong equilibrium doesn't always exist. However, when it does exist, we can look at the $SPoA$ and prove a non-trivial upper bound of $H(n) \approx log(n)$ (recall that in the previous section we showed an upper bound of only n for PoA).

Theorem 4.15 *For every fair cost sharing game, if a strong equilibrium exists then $SPoA \leq H(n)$.*

Proof: Let S be a strong equilibrium, and S^* an optimal solution. First, we define a coalition of all players: $\Gamma_n = \{1, 2, ..., n\}$. Because S is a strong equilibrium, there exists a player k that will not benefit from moving to the optimal solution: $u_k(S) \leq u_k(S_{-\Gamma_n}, S^*_{\Gamma_n})$. WLOG, assume $k = n$. We define a new coalition by removing this player $\Gamma_{n-1} = \{1, 2, ..., n-1\}$. We apply the same argument again to get a player (WLOG) $n-1$ such that $u_{n-1}(S) \leq u_{n-1}(S_{-\Gamma_{n-1}}, S^*_{\Gamma_{n-1}})$. We continue this way to define coalitions $\Gamma_k = \{1, 2, ..., k\}$ with $u_k(S) \leq u_k(S_{-\Gamma_k}, S^*_{\Gamma_k})$ for every $0 \leq k \leq n$.

Next, we define $S(\Gamma_k)$ as the game played only by the players of Γ_k which play according to S, while the players not in Γ_k are not existant in the game. We define $S^*(\Gamma_k)$ in a similiar way. Clearly $u_k(S_{-\Gamma_k}, S^*_{\Gamma_k}) \leq u_k(S^*(\Gamma_k))$, since the players in Γ_k play the same and the number of players is reduced in $S^*(\Gamma_k)$.

Now recall that a fair cost sharing game is a congestion game. Thus we can define the potential function $\Phi(a) = \sum_{e \in E} c(e) \cdot H(n_e(a))$ and it holds that a change Δu_k in the utility of a player equals the change $\Delta\Phi$ in the potential function. Therefore $u_k(S^*(\Gamma_k)) = \Phi(S^*(\Gamma_k)) - \Phi(S^*(\Gamma_{k-1}))$.

Using all the claims we made so far we get:

$$u_k(S) \leq u_k(S_{-\Gamma_n}, S^*_{\Gamma_n}) \leq u_k(S^*(\Gamma_k)) = \Phi(S^*(\Gamma_k)) - \Phi(S^*(\Gamma_{k-1}))$$

Summing up on k we get the desired bound:

$$\begin{aligned} U(S) = \sum_{k=1}^{n} u_k(S) &\leq \sum_{k=1}^{n} \Phi(S^*(\Gamma_k)) - \Phi(S^*(\Gamma_{k-1})) \\ &= \Phi(S^*(\Gamma_n)) - \Phi(S^*(\Gamma_0)) = \Phi(S^*) - \Phi(S^*(\emptyset)) \\ &= \Phi(S^*) = \sum_{e \in E} c(e) \cdot H(n_e(S^*)) \leq H(n) \cdot U(S^*)) \end{aligned}$$

$\square$

4.6.3 General Cost Sharing

As in the previous section, we start with the question of strong equilibrium *existance*. We show (using an example) that even for a game where all players have the same source s, a strong equlibrium doesn't always exist (recall that we proved that if all players have the same source *and* sink, a strong equilibrium does exist).

Consider the single source game of 3 players described in figure 4.9 (edges with no price tag are free, i.e. zero cost):

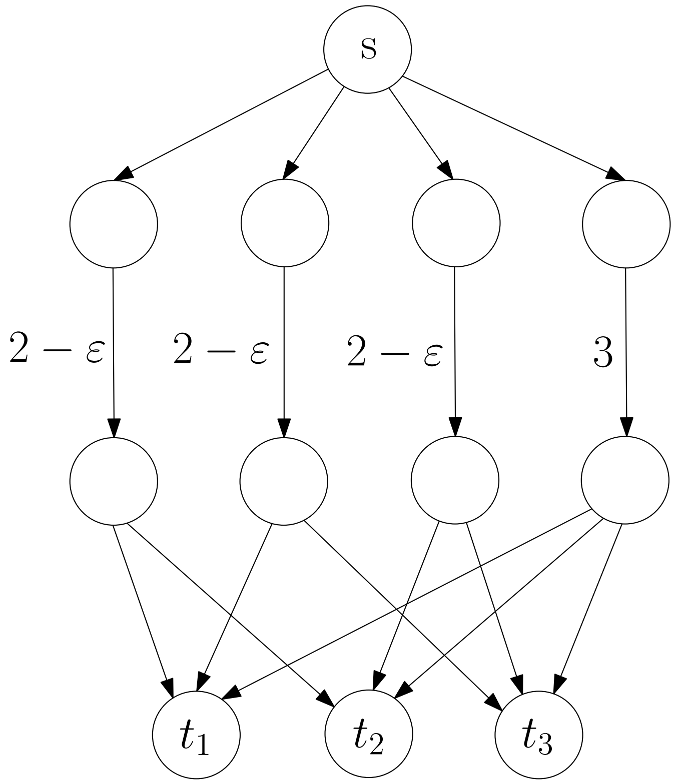

Figure 4.8: No strong equilibrium in single source general cost sharing

- If all 3 players choose the edge of price 3, then there are 2 players which pay together at least 2. They can thus unite and migrate to an edge of price $2-\varepsilon$ and improve their utility.

- If 2 players share an edge of price $2-\varepsilon$, the third player pays at least $2-\varepsilon$ for his edge. All 3 together pay $4-2\varepsilon$ and thus can improve their benefit if all would move to the edge of price 3.

So in any case, there is a coalition of players that can change their move and benefit. It follows that a strong equilibrium doesn't exist.

As before, we'll bound the $SPoA$ when it does exist. In fact, we'll show $SPoA = 1$, meaning that a strong Nash equilibrium has an optimal cost. The bound will hold for *single source* cost sharing *only*.

Theorem 4.16 *In a single source general cost sharing game, every strong equilibrium is optimal.*

Proof: Let $p = (p_1, p_2, ..., p_n)$ and let T^* be the tree (with root s, the players' source point) of an optimal solution. T^* is simply the set of edges used by an optimal solution. It must be a tree, otherwise we would have a vertex with 2 edges pointing it, one of them can be discarded, in contradiction to the optimality of T^*.

Denote by T_e^* the subtree of T^* that disconnects from s when e is removed, and let $\Gamma(T_e^*)$ be the set of players whose sink is in T_e^*. We define $P(T_e^*)$ as the total cost (by p) for players in $\Gamma(T_e^*)$, i.e. $P(T_e^*) = \sum_{i \in \Gamma(T_e^*)} c_i(p)$. Lastly, we define the cost for a set of edges E naturally as $C(E) = \sum_{e \in E} c(e)$.

Assume by contrudiction that p is not optimal, i.e., $c(p) > c(T^*)$. We'll show a sub-tree T' such that the set of players $\Gamma(T')$ can change their payments, so each one of them will benefit. This will contradict the fact that p is a strong equilibrium. First we'll show how to build T', then we'll construct the new payments of the coalition players.

We start with building T'. We define an edge e as *bad* if $P(T_e^*) \leq C(T_e^* \cup \{e\})$, meaning the players with sinks in T_e^* pay no more than the cost of the sub-tree T_e^* (including the edge that connects it to the main tree T^*). These players will not be part of the coalition. We define T' as the tree derived from T^* after we remove all bad edges, including their sub-trees: $T' = T^* \setminus \cup_{e \in BAD}(T_e^* \cup \{e\})$. Note that it's enough to remove only the top (closest to the root) edge and its sub-tree from each path containing bad edges. Thus we can assume the trees removed to be disjoint. From the definition of a bad edge it follows that every sub-tree that we removed from T^* didn't decrease the difference between the payments and the cost of the remaining tree. Therefore, for every edge e we have that $P(T'_e) > C(T'_e \cup \{e\})$.

Now that we defined T', we got a coalition $\Gamma(T')$, and we can define new payments $\bar{p}$ for them, such that $c_i(\bar{p}) < c_i(p)$ for every $i \in \Gamma(T')$. Note that the coalition is not empty since $c(p) > c(T^*)$ and thus at least one of the edges the starts from s is not a bad edge.

We define $c_i(\bar{p}, T'_e)$ as the new payment of player i for the sub-tree T'_e, i.e., $c_i(\bar{p}, T'_e) = \sum_{e \in T'_e} c_i(\bar{p}_i(e))$. Note we have not yet defined $\bar{p}$. We will show that for every sub-tree T'_e, $c_i(\bar{p}, T'_e \cup \{e\}) < c_i(p)$, and therefore $c_i(\bar{p}) < c_i(p)$.

We define the new payments $\bar{p}$ by edge, going bottom-up on the tree T'. When we're done assigning payments for a sub-tree T'_e, we have that $\sum_{i \in \Gamma(T'_e)} c_i(\bar{p}, T'_e) = c(T'_e)$, meaning the sum of payments of the coalition members for edges in this sub-tree equals the price of the sub-tree $c(T'_e)$. Recall that $P(T'_e) > C(T'_e \cup \{e\}) = c(T'_e) + c(e)$. We can use this difference to split the cost of the edge as follows: for every $i \in \Gamma(T'_e)$ set $\bar{p}_i(e) = c(e)\frac{\Delta_i}{\sum_{j \in \Gamma(T'_e)} \Delta_j}$ where $\Delta_i = c_i(p) - c_i(\bar{p}, T'_e)$.

With the new payments $\bar{p}$, we have for every $i \in \Gamma(T'_e)$:

$$\begin{aligned}
c_i(\bar{p}, T'_e \cup \{e\}) &= c_i(\bar{p}, T'_e) + \bar{p}_i(e) \\
&= c_i(\bar{p}, T'_e) + c(e)\frac{\Delta_i}{\sum_{j \in \Gamma(T'_e)} \Delta_j} \\
&= c_i(p) + \Delta_i(1 - \frac{c(e)}{\sum_{j \in \Gamma(T'_e)} \Delta_j}) \\
&= c_i(p) + \Delta_i(1 - \frac{c(e)}{P(T'_e) - c(T'_e)}) \\
&< c_i(p)
\end{aligned}$$

This completes the proof. We have shown that if p creates a sub-optimal tree, then it is not a strong equilibrium.

□

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