Computational Game Theory	Spring Semester, 2009/2010

Lecture 6: April 12, 2010

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Partially based on a scribe by Lior Gavish, Andrey Stolyarenko and Asaph Arnon from 2005/6, and on a scribe by Nataly Sharkov and Itamar Nabriski from 2003/4

6.1 Outline

This lecture deals with the existence of Nash Equilibria in general games (i.e, non-zero-sum games). We start with a proof of the existence theorem. This proof uses a fixed-point theorem known as Brouwer's Lemma, which we shall prove in the following section. We then discuss the complexity of finding Nash Equilibrium, and the class PPAD to which it is complete.

6.2 Brouwer's Lemma

Lemma 6.1 *(Brouwer) Let $f : B \to B$ be a continuous function from a non-empty, compact, convex set $B \subset \Re^n$ to itself. Then there is $x^* \in S$ such that $x^* = f(x^*)$ (i.e. x^* is a fixed point of f).*

We will sketch the proof later. First, let us explore some examples:

B	**f(x)**	**Fixed Points**
$[0,1]$	x^2	$0, 1$
$[0,1]$	$1-x$	$\frac{1}{2}$
$[0,1]^2$	(x^2, y^2)	$\{0,1\} \times \{0,1\}$
unit ball (in polar coord.)	$(\frac{r}{2}, 2\theta)$	$(0, \theta)$ for all θ

We shall first show that the conditions are necessary, and then outline a proof in 1D and in 2D. The proof of the general N-D case is similar to the 2D case (and is omitted).

6.2.1 Necessity of Conditions

To demonstrate that the conditions are necessary, we show a few examples:

When B is not bounded: Consider $f(x) = x + 1$ for $x \in \Re$. Then, there is obviously no fixed point.

When B is not closed: Consider $f(x) = x/2$ for $x \in (0, 1]$. Then, although $x = 0$ is a fixed point, it is not in the domain.

When B is not convex: Consider a circle in 2D with a hole in its center (i.e. a ring). Let f rotate the ring by some angle. Then, there is obviously no fixed point.

6.2.2 Proof of Brouwer's Lemma for 1D

Let $B = [0, 1]$ and $f : B \longrightarrow B$ be a continuous function. We shall show that there exists a fixed point, i.e. there is a x_0 in $[0, 1]$ such that $f(x_0) = x_0$. There are 2 possibilities:

1. If $f(0) = 0$ or $f(1) = 1$ then we are done.
2. If $f(0) \neq 0$ and $f(1) \neq 1$. Then define:

$$F(x) = f(x) - x$$

In this case:

$$F(0) = f(0) - 0 = f(0) > 0$$
$$F(1) = f(1) - 1 < 0$$

Thus, we have $F : [0, 1] \longrightarrow \Re$, where $F(0) \cdot F(1) < 0$. Since $f(\cdot)$ is continuous, $F(\cdot)$ is continuous as well. By the *Intermediate Value Theorem*, there exists $x_0 \in [0, 1]$ such that $F(x_0) = 0$. By definition of $F(\cdot)$:

$$0 = F(x_0) = f(x_0) - x_0$$

And thus:

$$f(x_0) = x_0$$

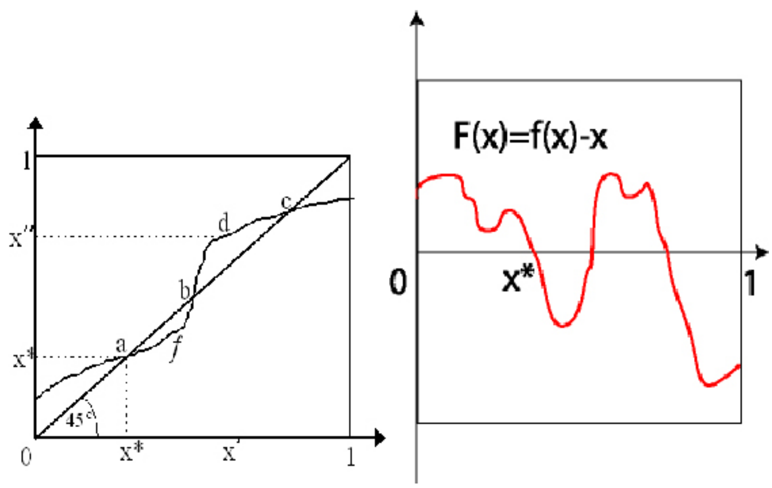


Figure 6.1: A one dimensional fixed point (left) and the function $F(\cdot)$ (right)

The proof for 2 (and higher) dimensions will follow from Spencer's lemma.

6.3 Existence Theorem

Theorem 6.2 *Every finite game has a (mixed-strategy) Nash Equilibrium.*

This section shall outline a proof of this theorem. We begin with a definition of the model, proceed with a statement of Brouwer's Lemma and conclude with the proof.

6.3.1 Model and Notations

Recall that a finite strategic game consists of the following:

- A finite set of players, namely $N = \{1, \ldots, n\}$.
- For every player i, a set of actions $A_i = \{a_{i1}, \ldots, a_{im}\}$.
- The set $A = \otimes_{i=1}^{n} A_i$ of joint actions.
- For every player i, a utility function $u_i : A \to \Re$.

A mixed strategy for player i is a random variable over his actions. The set of such strategies is denoted $\triangle(A_i)$. Letting every player have his own mixed strategy (independent of the others) leads to the set of joint mixed strategies, denoted $\triangle(A) = \otimes_{i=1}^{n} \triangle(A_i)$.

Every joint mixed strategy $p \in \triangle(A)$ consists of n vectors $\vec{p_1}, \ldots, \vec{p_n}$, where $\vec{p_i}$ defines the distribution played by player i. Taking the expectation over the given distribution, we define the utility for player i by

$$u_i(p) = E_{a \sim p}\left[u_i(a)\right] = \sum_{a \in A} p(a) u_i(a) = \sum_{a \in A} \left(\prod_{i=1}^{n} \vec{p_i}(a_i) \right) u_i(a)$$

We can now define a Nash Equilibrium (NE) as a joint strategy where no player profits from unilaterally changing his strategy:

Definition A joint mixed strategy $p^* \in \triangle(A)$ is NE, if for every player $1 \leq i \leq n$ it holds that

$$\forall q_i \in \triangle(A_i) \quad u_i(p^*) \geq u_i(p^*_{-i}, q_i)$$

or equivalently

$$\forall a_i \in A_i \quad u_i(p^*) \geq u_i(p^*_{-i}, a_i)$$

6.3.2 Proof of Existence of Nash Equilibrium

We now turn to the proof of the existence theorem. For $1 \leq i \leq n$, $j \in A_i$, $p \in \triangle(A)$ we define

$$g_{ij}(p) = \max\{u_i(p_{-i},\, a_{ij}) - u_i(p),\, 0\}$$

to be the gain for player i from switching to the deterministic action a_{ij}, when p is the joint strategy (if this switch is indeed profitable). We can now define a continuous map between mixed strategies $y : \triangle(A) \to \triangle(A)$ by

$$y_{ij}(p) = \frac{p_{ij} + g_{ij}(p)}{1 + \sum_{j=1}^{m} g_{ij}(p)}.$$

Observe that:

- For every player i and action a_{ij}, the mapping $y_{ij}(p)$ is continuous (w.r.t. p). This is due to the fact that $u_i(p)$ is obviously continuous, making $g_{ij}(p)$ and consequently $y_{ij}(p)$ continuous.
- For every player i, the vector $(y_{ij}(p))_{j=1}^{m}$ is a distribution, i.e. it is in $\triangle(A_i)$. This is due to the fact that the denominator of $y_{ij}(p)$ is a normalizing constant for any given i.

Therefore y fulfills the conditions of Brouwer's Lemma. Using the lemma, we conclude that there is a fixed point p for y. This point satisfies

$$p_{ij} = \frac{p_{ij} + g_{ij}(p)}{1 + \sum_{j=1}^{m} g_{ij}(p)}.$$

This is possible only in one of the following cases. Either $g_{ij}(p) = 0$ for every i and j, in which case we have an equilibrium (since no one can profit from changing his strategy). Otherwise, assume there is a player i s.t. $\sum_{j=1}^{m} g_{ij}(p) > 0$. Then,

$$p_{ij}\left(1 + \sum_{j=1}^{m} g_{ij}(p)\right) = p_{ij} + g_{ij}(p)$$

or

$$p_{ij}\left(\sum_{j=1}^{m} g_{ij}(p)\right) = g_{ij}(p).$$

This means that $g_{ij}(p) = 0$ iff $p_{ij} = 0$, and therefore $p_{ij} > 0 \Rightarrow g_{ij}(p) > 0$. However, this is **impossible** by the definition of $g_{ij}(p)$: $g_{ij}(p) \neq 0 \implies u_i(p_i, a_{ij}) > u_i(p)$ for every j in p_i's support. Taking the mean of these inequalities we get $\sum_j p_{ij} u_i(p_i, a_{ij}) > \sum_j p_{ij} u_i(p)$. But both sides are equal since $\sum_j p_{ij} u_i(p) = u_i(p) \sum_j p_{ij} = u_i(p)$ and by definition $\sum_j p_{ij} u_i(p_i, a_{ij}) = u_i(p)$ so we get the contradiction $u_i(p) < u_i(p)$. Therefore, it cannot be that player i can profit from every pure action in $\vec{p_i}$'s support (with respect to the mean).

We are therefore left with the former possibility, i.e. $g_{ij}(p) = 0$ for all i and j, implying a NE. Q.E.D.

6.4 The Complexity of Finding Nash Equilibrium

6.4.1 Defining the problem

Given a 2-players game if we will ask only if an equilibrium exists then the answer will always be true. We can of course create various decision problems by adding demands to the equilibrium:

- At least 2 equilibrium.
- An equilibrium in which player 1 has utility at least α.
- An equilibrium in which the size of the support is at least k.
- An equilibrium which includes action a_{ij}.

* All the above can be shown to be $NP-Complete$ hard problem.

The complexity class for problems for which the existence of the solution is guaranteed and the challenge is finding this solution is called PPP. Before giving a formal definition to PPP we will see another problem which belongs to this class.
Example: we are given as input 10 2-digits numbers: $S = \{42, 5, 90, 98, 99, 7, 64, 70, 78, 51\}$.
Do we have 2 subgroups s.t their sums are equal: $S_1, S_2 \subseteq S \quad \sum_{i \in S_1} i = \sum_{j \in S_2} j$?
Solution: there are $2^{10} = 1024$ subgroups. The sum of 10 2-digit numbers is at most 1000, so by pigeon hole principle two subgroups with an equal sum must exists!

6.4.2 The Equal Subsets Problem

Definition ES (Equal Subsets) Problem:
input: $A = \{a_1, \ldots, a_n\} \quad a_{i\in} \left[1, \frac{(2^n-1)}{n}\right]$
output: $S_1, S_2 \subseteq A_1$ s.t $\sum_{a_i \in S_1} a_i = \sum_{a_j \in S_2} a_j$.

Notice that A always has a solution because $\sum_{a_i \in A} a_i \leq n \cdot \frac{(2^n-1)}{n} = 2^n - 1$ and there are 2^n subsets of A so - by Pigeonhole principle -there are at least 2 subsets of A with an equal sum.

Let's try to analyze the complexity of the ES:

- The problem is clearly in NP: a non deterministic polynomial algorithm - guess S_1,$S_2 \subseteq A$ and test if $\sum_{a_i \in S_1} a_i = \sum_{a_j \in S_2} a_j$.
- The question if ES is in P is an open question.
- Is ES an $NP-Complete$ problem?

It is unlikely that ES is an $NP-Complete$ problem because this will give the result $NP = Co - NP$ which is commonly considered not to be the case.

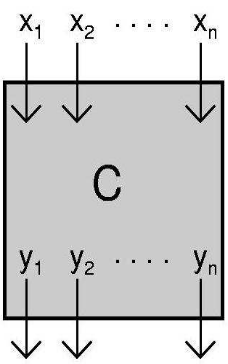
Proof: Suppose it is NP-complete, that SAT≤pEQUAL SUBSETS. Then there is an algorithm for SAT that runs in polynomial time, provided that it has access to a poly-time algorithm for ES. Now suppose that this algorithm is given a non-satisfiable formula γ. Presumably it calls the ES algorithm some number of times, and receives a sequence of solutions to various instances of ES, and eventually the algorithm returns the answer "no, γ is not satisfiable". Now suppose that we replace this hypothetical poly-time algorithm for ES with the natural non-deterministic "guess and test" algorithm, which gives us a non-deterministic polynomial-time algorithm for SAT. Notice that when γ is given to this new algorithm, the "guess and test" subroutine for ES will always succeed to produce the same sequence of solutions as the hypothetical poly-time algorithm becuase the solution always exists. As a result, the entire algorithm can recognise this non-satisfiable formula γ as before. Thus we have a NP algorithm that recognizes unsatisfiable formulae, which gives the consequence NP=co-NP. Q.E.D.

6.4.3 A Module for ES

The proof we gave for the existence of solution for ES is based on the pigeonhole principle. We will now define a complexity class which tries to "catch" that principle.

Basic problem -PC (Pigeonhole Circuit):

- A circuit C



- Find
  - An $x \in [1, 2^n]$ s.t $C(x) = 0$.

  Or

  - $x_1, x_2 \in [1, 2^n]$ s.t $C(x_1) = C(x_2)$.

Observation: There is always a solution (pigeonhole principle). The challenge is to find the solution in polynomial time.

Definition PPP (Polynomial Pigeonhole Principle) complexity class: contains all the problems which have polynomial time reduction to PC

Claim 6.3 *ES is in PPP*

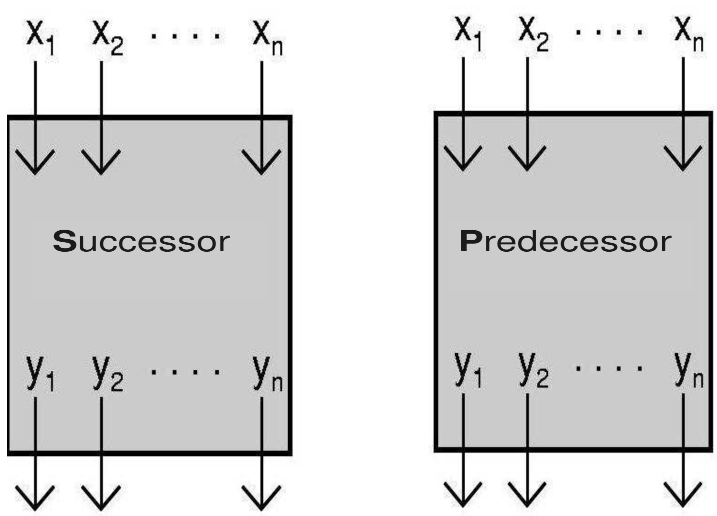
Proof: We will see $ES \leq_p PC$. Given an instance of ES: $A = \{a_1, \ldots, a_n\}$, our circuit will be $C(x) = 1 + \sum a_i x_i$. $C(x) \geq 1$ so it is guranteed that a solution will be only $x_1, x_2 \in [1, 2^n]$ s.t $C(x_1) = \sum a_i x_{1_i} = C(x_2) = \sum a_i x_{2_i}$. Given such x_1, x_2 we can construct a solution for the ES: $S_1 = \{a_i | x_{1_i} = 1\}, S_1 = \{a_i | x_{2_i} = 1\}$. Q.E.D.

Corollary 6.4 *ES is hard to PPP.*

Sadly we don't know of an opposite reduction so it is an open question whether ES is complete for PPP.

6.4.4 $PPAD$

Definition End-Of-Line problem:
Input - 2 circuits:



- The 2 circuits define a directed graph over $\{0,1\}^n$.
- There is a directed edge from x to y if $P(y) = x$ and $S(x) = y$.

output: find a vertex different from $\overrightarrow{0}$ with degree 1.

Observations:

- The input $(0, \ldots, 0)$ has no predecessor $\Rightarrow$ its in-degree is zero - $\overrightarrow{0}$ is a source vertex in the graph.
- For each vertex v $\quad indeg\,(v) \leq 1, \quad outdeg(v) \leq 1$. In such a graph there must be at least one sink.

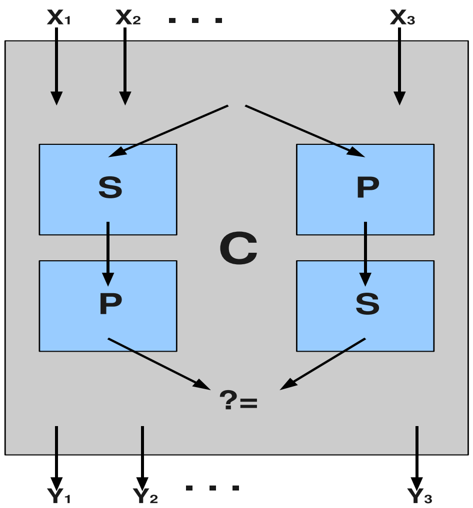
Corollary 6.5 *The resulting input graph has at least one sink which can be reached from* $\overrightarrow{0}$.

Definition $PPAD$ complexity class: contains all the problems which have polynomial time reduction to End-Of-Line

Notice that The solution can be a different source or a sink which is not accessible from $\overrightarrow{0}$. If the demanded was to find a sink which is accessible from $\overrightarrow{0}$ then the complexity class would have been$PSPACE$ - same as finding a final configuration of Turing machine with space m).

Claim 6.6 $PPAD \subseteq PPP$

Proof: We will show a reduction from PC toES ($PC \leq_p ES$):



- If $P\,(S\,(x)) = x$ then x is not a sink ($x \rightarrow y$).
- If $S\,(P\,(x)) = x$ then x is not a source ($y \rightarrow x$).
- 2 inputs will be mapped to the same value iff they are sink or source.

Q.E.D.

Claim 6.7 *Finding Nash Equilibrium is a PPAD-Complete problem*

Proof: To prove it we need to show 2 reductions:

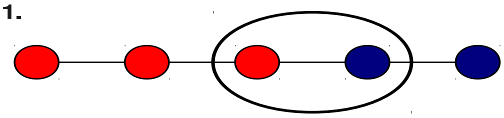
1. From Nash to End-Of-Line - to prove that Nash is in PPAD.
2. From End-Of-Line to Nash - to prove that Nash is complete to PPAD.

To show 1 (Nash $\leq_p$ End-Of-Line) we will use a coloring problem for the reduction:

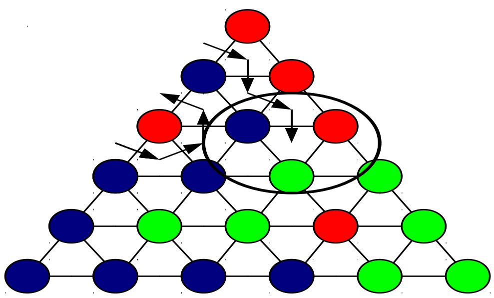
- Input - a "triangle" in the taring$\mathbb{R}$ space with $d + 1$ vertexes.
- Each vertex is colored.
- A vertex on one of the "triangle" edges can only be colored with one of the edge colors.

Lemma 6.8 *Spencer Lemma: There will always exist a triangle with d+1 colors. Examples:*

1.



2.



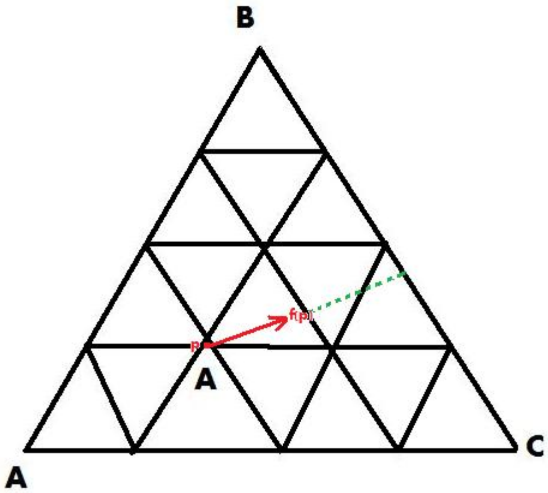
- *1-dimensional (example 1) - must be a segment where we switch from R to B.*
- *2-dimensional (example 2) - we find a segment colored (R,B) on the (R,B) edge of the outer triangle and start searching for an inner triangle with 3 colors. The advancing rule in the search is always to pass through a segment colored with (R,B) and not the one we entered from. The search must end with a 3-colored triangle.*

- *d-dimensional - we find on one of the outer edges a "triangle" colored with d (out of d+1) colors and construct a path where at each step we go through the edge which is d-colored (which guarantee we reach to a "triangle" which has at least d colors). This path must end with a "triangle" which has d+1 colors.*

This is actually a reduction to End-Of-Line (with a none directed graph).

To show 2 (End-Of-Line$\leq_p$Nash) we will prove Brouwer's lemma (for more then 1 dimension):

We can find a triangle in any convex, compact 2-dimensional shape. Label the tree vertexes of the original triangle as A, B and C. We will now divide the triangle to smaller triangles, and label each new vertex by looking at $f(p)$. The label of vertex p will be the vertex of the original triangle from which the vector $f(p)$ is moving away. This labeling is well defined since if we 'continue' the line of $f(p)$ until it reach one of the original triangle sides, we could label p to be the label of the vertex opposing that side:



Sperner's lemma guarantees the existence of a triangle with A-B-C vertexes. We will now divide this triangle to smaller triangles and start the process again

Eventually, we'll find a point p for which $p = f(p)$, and that's our fixed point.

Q.E.D.