Computational Game Theory **Spring Semester, 2005/6**

Lecture 12:June 13

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12.1 Introduction

In this lecture we consider **Combinatorial Auctions** (abbreviated *CA*), that is, auctions where instead of competing for a single resource we have multiple resources. The resource assignments and bids are defined on subsets of resources and each player has a valuation defined on subsets of the resource set he was assigned. The interesting cases here are when the valuation of a given set of resources differs from the sum of valuations of each resource separately (the whole is *different* from the sum of its parts). This might happen when we have a set of complementary products, i.e., each product alone is useless but the group has a significantly larger value (e.g., left and right shoes). On the other hand we might have a set of substitutional products where the opposite takes place (e.g., tickets to a movie; no use in having two tickets if you are going alone).
In these cases there is an importance for pricing groups of resources rather than single resources separately, i.e. in the absence of complementarity and substitutability (if every participant values a set of goods at the sum of the values of its elements), one should organize the multiple auction as a set of independent simple auctions, but, in the presence of these two attributes, organizing the multiple auction as a set or even a sequence of simple auctions will not lead to optimal results, in such a case we can use **Combinatorial Auctions**.

12.2 Preliminaries

Throughout this lecture, we shall consider single-side combinatorial auctions, that is, auctions with a single seller and multiple buyers.
Any such auction must specify three elements:

- The bidding rules, i.e., what one is allowed to bid for and when.
- The market clearing rules (i.e., when is it decided who bought what and who pays what)
- The information disclosure rules (i.e., what information about the bid state is disclosed to whom and when).

We consider only one-stage, sealed-bid *CAs*: each bidder submits zero or more bids, the auction clears, and the results are announced.
The third element of the specification is thus straightforward: no information is released about other bidders' bids prior to the closing of the auction. The first element of the specification is almost as straightforward: each bidder may submit one or more bids, each of which mentions a subset of the goods and a price. One has to be precise, however, about the semantics of the collection of bids submitted by a single bidder, because, as mentioned, the bid for a group isn't necessarily equal to the sum of bids for its elements.
Only the second element of the above specification, the clearing policy, provides choices for the designer of the *CA*. There are two choices to be made here: which goods each bidder receives, and how much each bidder pays? We address these below. We will focos on auctions where truth telling is a dominant strategy.

Claim 12.1 *If a player's outcome doesn't depend on his action (bid), then truth telling is a dominant strategy*

12.2.1 The model

- $N = \{1..n\}$ set of players.
- $S = \{1..m\}$ set of resources (products).
- Θ - set of players' private information, player i has information $\theta_i \in \Theta_i$ which is the internal state he is currently in.
- D - Mechanisms decision space. Each vector specifies resource allocation amongst the players.
  - Limited Supply - $D = \{< s_1..s_n > \mid (\forall i \neq j \ s_i \cap s_j = \emptyset) \wedge (\bigcup_{1 \leq i \leq n} s_i \subseteq S)\ \}$.
  - Unlimited Supply - $D = 2^S \times \cdots \times 2^S$.
- $V = \{V_1..V_n\}$ - set of preference functions $V_i : D \times \Theta_i \rightarrow R$ which is the value that player i attributes to a subset of S given its internal state θ_i.
- $\vec{t} = \{t_1..t_n\}$ - set of payment functions defined for each player by the mechanism, i.e., $t_i : \Theta \rightarrow R$ where the t_is are dependent on the bids of the other players. We let $u_i(d, \theta_i) = v(d, \theta_i) - t_i(\vec{t_i})$.

Assumption 12.2 ***Monotonicity***: *for every $s_1, s_2 \in S$ such that $s_1 \subseteq s_2$, the value attributed to s_2 will not be smaller to that of s_1. i.e. $s_1 \subseteq s_2 \Rightarrow V_i(s_1) \leq V_i(s_2)$ for any player i.*

Assumption 12.3 ***No Externalities***: *The valuation for a player depends* solely *on the set of products he won.*

12.2.2 Goals

- A possible goal would be be guaranteeing **Efficiency** - finding a *pareto-optimal* allocation, that is, no further trade among the buyers can improve the situation of some trader without hurting any of them. This is typically achieved by using an assignment which brings the sum of benefits to a maximum.
- Maximizing seller's revenue - we will focus on this goal.

12.3 Single Product, Single Player

The valuation for the player is in the range $[1, h]$. Consider the following algorithm for the seller to choose his price:

1. Pick in random $i \in \{0, ..., \lfloor \log_2 h \rfloor\}$ with uniform distribution.
2. Offer the player a price of 2^i.

Claim 12.4 *For every $v \in [1, h]$ valuation for the player, the algorithm's expected profit is* $\Omega\left(\frac{v}{\log_2 h}\right)$.

Proof: With probability $\frac{1}{\log_2 h}$ the picked i satisfies $\frac{v}{2} < 2^i \leq v$, in which case the profit will be $\frac{v}{2}$ at the least. Therefore the expectation is at least $\frac{v}{2 \log_2 h}$. □

12.4 Single Product, Multiple Players, Unlimited Supply

We have N players and we would like to maximize the seller's profit.

For each player i, $v_i \in [1, h]$. Define $\mathcal{G}_\epsilon = \{(1+\epsilon)^k : 0 \leq k \leq \log_{1+\epsilon} h\}$, thus $|\mathcal{G}_\epsilon| = O\left(\frac{\log h}{\epsilon}\right)$.

Consider the following RSOO (Random Sampling Optimal Offer) algorithm:

- Get v_i from player i.
- Partition N into N_1 and N_2, *s.t.* $\Pr[i \in N_1] = \Pr[i \in N_2] = 1/2$.
- Pick the optimal price $g_1 \in \mathcal{G}_\epsilon$ for N_1.
- Pick the optimal price $g_2 \in \mathcal{G}_\epsilon$ for N_2.

- Finally, offer the players in N_1 the price of g_2 and the players in N_2 the price of g_1.

Let OPT be the profit gained with the optimal price, i.e.,

$$OPT = \max_{p\in[1,h]} p \cdot |\{i : v_i \geq p\}|$$

Theorem 12.5 *If* $OPT \geq \frac{18h}{\epsilon^2} \ln \frac{2|\mathcal{G}_\epsilon|}{\delta}$ *then with probability* $1-\delta$ *the profit of RSOO is at least* $(1-\epsilon)OPT$.

Definition For a player i with private value v_i, let

$$g_v(i) = \begin{cases} v & v \leq v_i \\ 0 & else \end{cases}$$

Similarly for a set of players S, let

$$g_v(S) = \sum_{i\in S} g_v(i)$$

Lemma 12.6 *For every* g_v *and every* p, *with probability* $1 - 2e^{-\epsilon^2 p/2h}$ *we have*

$$|g_v(N_1) - g_v(N_2)| \leq \epsilon \cdot \max\{g_v(N), p\}$$

Proof: Let

$$Y_i = \begin{cases} 1 & with\ probability\ 1/2 \\ 2 & with\ probability\ 1/2 \end{cases}$$

then

$$g_v(N_1) = f(Y_1, ..., Y_n) = v \cdot |\{i : Y_i = 1 \wedge v_i \geq v\}|$$

Clearly $E[f] = \frac{1}{2} g_v(N)$, therefore according to McDiarmid:

$$\Pr\left[\left|g_v(N_1) - \frac{g_v(N)}{2}\right| \geq \epsilon \cdot \frac{g_v(N)}{2}\right] \leq 2e^{-\frac{\epsilon^2}{2}\cdot\frac{g_v^2(N)}{\sum_{i\in N} g_v^2(i)}} \tag{12.1}$$

Where

$$\sum_{i\in N} g_v^2(i) \leq \max_i\{g_v(i)\} \cdot \sum_{i\in N} g_v(i) = v \cdot g_v(N) \leq h \cdot g_v(N)$$

Substituting that back in (12.1) we get:

$$\Pr\left[\left|g_v(N_1) - \frac{g_v(N)}{2}\right| \geq \epsilon \cdot \frac{g_v(N)}{2}\right] \leq 2e^{-\epsilon^2 g_v(N)/2h}$$

In case $g_v(N) \geq p$, we get $p \leq g_v(N) = g_v(N_1) + g_v(N_2)$ and thus:

$$\Pr\left[|g_v(N_1) - g_v(N_2)| \geq \epsilon \cdot g_v(N)\right] \leq 2e^{-\epsilon^2 p/2h}$$

Which completes the proof in this case. Alternatively, if $g_v(N) < p$:

$$\Pr\left[|g_v(N_1) - g_v(N_2)| \geq \epsilon p\right] \leq e^{-\frac{\epsilon^2 p^2}{2\sum g_v^2(i)}} \leq e^{-\epsilon^2 p/2h}$$

Where the last inequality follows from:

$$\sum g_v^2(i) \leq h g_v(N) \leq hp$$

□

We can now complete the proof of the theorem:

Proof: Let g_1 be the optimal offer for N_1, let g_2 be the optimal offer for N_2 and let g_{OPT} be the optimal offer for N, then clearly,

- $g_i(N_i) \geq g_{OPT}(N_i)$
- $g_{OPT}(N) \geq g_i(N)$

We pick $p = \frac{18h}{\epsilon^2} \ln \frac{2|\mathcal{G}_\epsilon|}{\delta}$ and apply the lemma for each function in $\mathcal{G}_\epsilon$ with p, $\delta' = \frac{\delta}{|\mathcal{G}_\epsilon|}$, $\epsilon' = \epsilon/2$ and so with probability $1 - \delta$ we get:

$$\forall g_v \in \mathcal{G}_\epsilon \; |g_v(N_1) - g_v(N_2)| \leq \frac{\epsilon}{2} \max\{g_v(N), p\}$$

We assumed $OPT \geq p$ and by definition $g_1(N), g_2(N) \leq OPT$, we get:

$$g_1(N_2) \geq g_1(N_1) - \frac{\epsilon}{2} \max\{g_1(N), p\}$$

$$g_2(N_1) \geq g_2(N_2) - \frac{\epsilon}{2} \max\{g_2(N), p\}$$

and summing up we finally get:

$$RSOO = g_1(N_2) + g_2(N_1) \geq g_1(N_1) + g_2(N_2) - \epsilon \cdot OPT \geq (1 - \epsilon)OPT$$

as required. (We note that $\mathcal{G}_\epsilon$ must be discrete since a union bound was used.)

□

12.5 Combinatorial Auctions

We consider N players, where each player has a valuation for each subset of products. We will consider the following possibilities:

Unit Demand: Each player wants to win only one product, and has a different valuation for each product.

Single Mind: Each player has one subset of products he's interested in.

We let each product i have price p_i, and player j chooses s_j $s.t.$

$$s_j = \arg\max_{s_j \subseteq S} \left(v_j(s_j) - \sum_{i \in s_j} p_i \right)$$

We now wish to estimate the loss of discretization, since the subset a player wins can dramatically change due to a slight change in the prices.
We define

$$H_\epsilon = \{ \frac{h}{(1+\epsilon)^k} : 0 \leq k \leq \log_{1+\epsilon} \frac{mn}{\epsilon} \}$$

where h is the highest price and $\frac{\epsilon h}{mn}$ is the lowest price.

Theorem 12.7 *Let p^* be the vector of prices that maximize the profit, and let $p' \in H_\epsilon$ be the vector of prices that maximize the profit (over H_ϵ), then*

$$Rev(p') \geq (1 - 2\sqrt{\epsilon}) Rev(p^*)$$

Proof: Let $\delta = \sqrt{\epsilon}$, and for p^* we choose for each j a price $p_j \in [(1-\delta)p_j^*, (1-\delta+\delta^2)p_j^*]$ from H_ϵ (such a price exists since $\delta^2 = \epsilon$) or 0.
We define $R_j = p_j^* - p_j$ and get

$$(\delta - \delta^2) p_j^* \leq R_j \leq \max\{\delta p_j^*, \frac{\delta^2 h}{mn}\} \leq \delta p_j^* + \frac{\delta^2 h}{mn}$$

and for a set of prices J and every other price set J':

$$\sum_{j \in J} R_j \geq \delta(1-\delta) Rev(J, p^*)$$

$$\sum_{j \in J'} R_j \leq \delta Rev(J', p^*) + \frac{\delta^2 h}{n}$$

Assuming that player i prefers J^* with prices p^* and prefers J if the prices are p, then:

$$\delta(1-\delta)Rev(J^*, p^*) \le \sum_{j \in J^*} R_j \le \sum_{j \in J} R_j \le \delta Rev(J, p^*) + \frac{\delta^2 h}{n}$$

and therefore:

$$(1-\delta)Rev(J^*, p^*) \le Rev(J, p^*) + \frac{\delta h}{n}$$

and summing over all players we get:

$$(1-\delta)Rev(J^*, p^*) \le Rev(\vec{J}, p^*) + \delta h$$

And if $h \le Rev(J^*, p^*)$ then

$$(1-2\delta)Rev(J^*, p^*) \le Rev(J', p')$$

□

This implies that we can use discretize prices and have a ”small” loss. Now we can use *RSOO* and derive an approximate optimal combinatorial auction.

12.6 References

1. Yosef N., Nabriski I., Sharkov N. ”Topics in Computational Game Theory” Course, Lecture Notes (Scribe) from lecture 11, 2004, Tel Aviv University, Israel.

2. Mansour Y. ”Topics in Computational Game Theory” Course, Notes from lecture 12 presented at 13/6/2006, Tel Aviv University, Israel.