Computational Game Theory	**Spring Semester, 2003/04**

Lecture 5: March 30

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5.1 Introduction

In this lecture we explore issues concerning the computability of Nash Equilibrium in any general game. First we prove any game has at least one Nash Equilibrium using Brouwer's Fixed Point Theorem (The proof is presented at section 5.6). Then we discuss how to compute a Nash Equilibrium of a game and a Nash Equilibrium approximation called ϵ-Nash. We conclude by proving that the computation of most information concerning Nash Equilibrium is NP-hard.

5.2 Proof of existence of Stochastic Nash Equilibrium in any game

5.2.1 Proof Outline

We prove that any game has a Nash Equilibrium, though not necessarily a deterministic one. Using Brouwer's Fixed Point Theorem, we prove that in any game if we map every game state to another state, such that at least one player is better off, then there is some state which is mapped into itself (a fixed point). In other words, this game state cannot be improved by any player by changing his strategy, and thus is, by definition, a Nash Equilibrium.

5.2.2 Notations

- There are n players: $N = \{1, ..., n\}$
- Each player can choose from a set of m pure strategies, thus the possible Strategies for each Player i are: $A_i = \{a_{i1}, ..a_{im}\}$
- u_i - utility function of player i
- p_i - distribution over A_i
- $\wp = \prod p_i$ - product of players' distributions
- $(q, \wp_{-i})$ - taking $\wp$ with p_i replaced by q
- $u_i(\wp) = E_{\vec{u}\sim\wp}[u_i(\vec{a})]$ - expected utility of player i

- Δ_i - all possible distributions for player i (infinite set - since each distribution contains m values in the range $[0, 1]$)

- $\Lambda = \prod \Delta_i$ - product of all players' possible distributions

5.2.3 Definitions

- Nash Equilibrium - $\wp^*$ is a Nash Equilibrium if:

$$\wp^* \in \Lambda$$

$$\forall i \in N : q_i \in \Delta_i$$

$$u_i(q_i, \wp^*_{-i}) \leq u_i(\wp^*)$$

- Revenue of player i from deterministic action $a_{i,j}$:

$$Revenue_{i,j}(\wp) = u_i(a_{i,j}, \wp_{-i})$$

- Profit of player i from deterministic action $a_{i,j}$:

$$Profit_{i,j}(\wp) = (Revenue_{i,j}(\wp) - u_i(\wp))$$

- Gain of player i from deterministic action $a_{i,j}$:

$$Gain_{i,j} = max(Profit_{i,j}(\wp), 0)$$

  $Gain_{i,j}$ is non zero only if player i has positive profit from playing strategy $a_{i,j}$ instead of his strategy in $\wp$. Thus in Nash Equilibrium ($\wp = \wp^*$) all $Gain_{i,j}$'s are zero.

- We define a mapping:

$$y : \Lambda \rightarrow \Lambda$$

$$y_{i,j}(\wp) = \frac{p_{i,j} + Gain_{i,j}}{1 + \sum_{k=1}^{m} Gain_{i,k}}$$

  - $y_{i,*}$ is a distribution over A_i (where $*$ means for any j)
  - $y_{i,j}$ is continuous (since $Revenue_{i,j}$ is continuous $\rightarrow$ $Profit_{i,j}$ is continuous $\rightarrow$ $Gain_{i,j}$ is continuous $\rightarrow$ $y_{i,j}$ is continuos)
  - Notice that the denominator of the expression on the right is there in order to normalize the expression to be a value in $\wp$.

5.2.4 Proof

By Brouwer's Theorem (see section 5.6) for a continuous function $f : S \to S$ such that S is a convex and compact set, there exists $s \in S$ such that $s = f(s)$. s is called a fixed point of f. Thus in our case, for y, there exists $\wp^*$:

$$y_{i,j}(\wp^*) = p_{i,j} = \frac{p_{i,j} + Gain_{i,j}}{1 + \sum_{k=1}^{m} Gain_{i,k}}$$

We argue that this point is a Nash Equilibrium.

- If $\sum_{k=1}^{m} Gain_{i,k} = 0$ then we are done since all $Gain_{i,k}$ are zero and thus there is no room for improvement which is by definition a Nash Equilibrium.
- Otherwise $\sum_{k=1}^{m} Gain_{i,k} > 0$:

$$p_{i,j}(1 + \sum_{k=1}^{m} Gain_{i,k}) = p_{i,j} + Gain_{i,j} \Longrightarrow$$

$$p_{i,j} \sum_{k=1}^{m} Gain_{i,k} = Gain_{i,j}$$

  This implies one of two cases:

  1. $$p_{i,j} = 0 \Longleftrightarrow Gain_{i,j} = 0$$

     Then we are in a Nash Equilibrium and the fixed point is 0.
  2. $$p_{i,j} > 0 \Longleftrightarrow Gain_{i,j} > 0$$

     This implies that for every strategy that has a positive probability $a_{i,k}$, playing it purely will net a higher utility. $p_{i,j}$ is a distribution, thus, if we take $p_{i,j}$ over all strategies we will net a higher utility. But this implies that we can improve $p_{i,j}$ by selecting $p_{i,j}$, which is a contradiction

5.3 Computing of Nash equilibria

5.3.1 General sum games

We begin from the example of the general sum game. Let A,B be payoff matrices of player 1 and 2 accordingly.

$$A = \begin{pmatrix} 1 & 5 & 7 & 3 \\ 2 & 3 & 4 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$

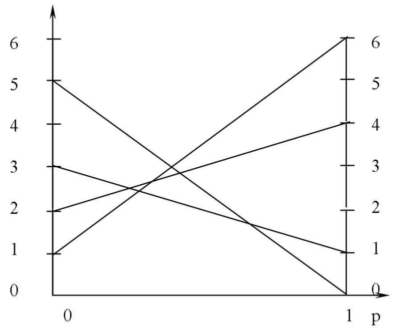


Figure 5.1: The viewpoint of player 2.

Player 1 has only two strategies: (x, 1-x).
Player 2 has vector of strategies. Let number all strategies of player 2. $j \in [1, 4]$ - the index of player 2's strategies. The utility of playing strategy j is $b_{1j}x + b_{2j}(1 - x)$.

$p \in [0, 1]$ is the probability that player 1 plays bottom strategy. (1-p) that he plays top strategy. We can receive Nash equilibrium in points those player 2 is indifferent between strategies 3 and 4. Let see all cases.

- The side (3,3). For this side the player 1 is indifferent, since for all values p lengthwise the side that is equilibrium.
- Extreme points. Let see the case of extreme points p=0, p=1. In these points Equilibrium can be received if the action p is better than $\overline{x}$ - reverse action.
- Pass points. Check the pass points from one to other side. There are two such points. The first one is (3,3)-(1,2) pass point. In this point player 2 prefer other strategy. The second one is (1,2)-(7,4). There is Nash equilibrium in this point. Let compute it.

$$1 \cdot p + 7 \cdot (1 - p) = 2p + 4(1 - p)$$

$$(7 - 4)(1 - p) = (2 - 1)p$$

$$3 = \frac{7 - 4}{2 - 1} = \frac{p}{1 - p}$$

$$p = \frac{3}{4}$$

$$1 - p = \frac{1}{4}$$

In the same way we can compute the case when player 1 is indifferent between top and bottom strategies. For player 2 those are strategies number three and fourth.

5.3.2 An algorithm that uses the support vectors

Consider a general two-players game
A- payoff matrix of player 1.
B- payoff matrix of player 2.
As stipulated by Nash's theorem a Nash Equilibrium (p,q) is exists.
Suppose we know the support sets. p has support $S_p = \{i : p_i \neq 0\}$ and q has support $S_q = \{j : q_j \neq 0\}$. How can we compute the Nash equilibrium? Let's determine requirements for the best response of these players.
For player 1:

$$\forall i \in S_p, \forall l \in A$$
$$e_i A q^T \geq e_l A q^T.$$

Where e_i is unit vector for i=1.
If $i, l \in S_p$ then $e_i A q^T = e_l A q^T$.
For player 2:

$$\forall j \in S_q, \forall l \in B$$
$$pBe_j^T \geq pBe_l^T$$

$$\begin{array}{lllll} & \Sigma q_j = 1 & \Sigma p_i = 1 & \\ j \in S_q & q_j > 0 & p_i > 0 & i \in S_p \\ j \notin S_q & q_j = 0 & p_i = 0 & i \notin S_p \end{array}$$

To find NE we only need to solve the following system of constraints:

$$\begin{array}{lllll} j \in S_q & pBe_j^T = v & e_i A q^T = u & i \in S_p \\ k \notin S_q & q_k = 0 & p_k = 0 & k \notin S_p \\ & \Sigma q_j = 1 & \Sigma p_i = 1 & \end{array}$$

There are 2(N+1) equations and 2(N+1) variables.
Unique solution requires non-degenerate system of constraints. Otherwise there is an infinite number of Nash equilibria.
Algorithm: For all possible subsets of supports:

- Check if the corresponding linear programming has feasible solution (using e.g. simplex);
- If so, STOP : the feasible solution is Nash equilibrium.

Question: How many possible subsets supports are there to try?
Answer: At most $2^n \cdot 2^n = 4^n$. The algorithm finds all the Nash equilibria. So, unfortunately, the algorithm requires worst-case exponential time.

The following is an example of the game with an exponential number of Nash equilibria.

$$u_2(i,j) = u_1(i,j) = \begin{cases} 1 & \text{if i=j} \\ 0 & \text{otherwise} \end{cases}$$

For each support, a uniform distribution for both players is Nash Equilibrium.

5.4 Approximate Nash Equilibrium

5.4.1 Algorithms for approximating equilibria

Definition $\wp^*$ is ϵ-Nash if for every player p* and every (mixed) strategy $a_j \in A_i$,

$$u_i(a_j, p^*_-i) \leq u_i(a_j, p^*_-i) + \epsilon$$

If $\epsilon = 0$ then this is a general Nash equilibrium.

Theorem 5.1 *For any two-player game G, there exists an ϵ-Nash, whose support's size is $\frac{12}{\epsilon^2} logn$.*

Proof: The proof is based on the probabilistic method. For the present we assume that all utilities are between 0 to 1. By Chernoff bound, for every i.i.d.$x_1, x_2, ...x_n$ s.t. $E[x_i] = p$. We have

$$Pr[|\overline{x} - p| \geq \epsilon] \leq e^{-2\epsilon^2 l},$$

where

$$\overline{x} = \frac{1}{n} \sum x_i.$$

Let (p,q) be a Nash equilibrium. By sampling each distribution l times we get the distribution ($\hat{p}$,$\hat{q}$). Q.E.D.

We now prove that ($\hat{p}$,$\hat{q}$) is an ϵ-Nash.

Lemma 5.2

$$\mid \hat{p}A\hat{q}^T - pAq^T \mid \leq \epsilon$$

$$\mid \hat{p}B\hat{q}^T - pBq^T \mid \leq \epsilon$$

Proof: We prove the inequality for A, a similar proof holds for B. Using the Chernoff bound again we get:

$$Pr[\exists i \mid e_i A\hat{q}^T - e_i Aq^T \mid \geq \epsilon]$$

$$\leq NPr[\mid e_i A\hat{q}^T - e_i Aq^T \mid \geq \epsilon]$$

$$\leq Ne^{-2\epsilon^2 l}$$

Where N is the number of actions.
We define the random variable:

$$Q^i_j = \{(l_i A)_j = a_{ij} w.p \cdot q_j\}$$

$$E[Q_j^i] = e_i A q^T$$

$$\frac{1}{l} \sum Q^i = e_i A \hat{q}^T$$

The bound of the error probability is $Ne^{-2\epsilon^2 l}$. Perform this action for $(B, p)(B, q)(A, p)(A, q)$. Hence, with probability $4Ne^{-2\epsilon^2 l}$ this takes place for $\hat{q}$ and $\hat{p}$. Show that we receive the ϵ-Nash equilibrium.

$$| \; pAq^T - \hat{p}A\hat{q}^T \; | \leq$$

$$| \; pAq^T - pA\hat{q}^T \; | + | \; pA\hat{q}^T - \hat{p}A\hat{q}^T \; | \leq 2\epsilon$$

Since (p,q) is Nash equilibrium, $e_i A q^T \leq pAq^T$. From the approximation we get $e_i A\hat{q}^T \leq pAq^T + \epsilon \leq \hat{p}A\hat{q} + 3\epsilon$, with success probability of $2Ne^{-2\epsilon^2 l}$. We choose $\epsilon' = \frac{\epsilon}{3}$ for $l \geq \frac{9}{2} \frac{1}{\epsilon'^2} \ln 2N$. The success probability is positive. This would mean that there are exist $\hat{q}$ and $\hat{q}$ satisfy all conditions. Hence, $(\hat{q}, \hat{q})$ is ϵ-Nash equilibrium. $\Omega(\frac{1}{\epsilon^2} \ln N)$ Q.E.D.

Algorithm: For all groups whose size is at most $\frac{1}{\epsilon^2} \ln N$

- compute ϵ-Nash equlibrium for all pairs of groups.

5.5 Hardness results for Nash Equilibrium

In this section we prove that many problems related to Nash Equilibrium are NP-hard to compute.

5.5.1 Proof Outline

We construct a 2 Player game G. Our decidability problem is the following: Is there an expected Nash equilibrium in G, where both players have an expected utility of 1. We use a reduction from the well known NP-hard problem SAT to this decidability problem. Afterwards we elaborate on this result to illustrate that many Nash Equilibrium problems about 2 player games are NP-hard to compute.

5.5.2 Definitions

- Let θ be a boolean CNF (Conjunctive Normal Form) formula.
- Let V be the set of variables $v_i \in \theta$. $|V| = n$.
- Let L be the set of all possible literals l_i composed from the variables in V. Thus for every $x_i \in V$ there exists $x_i, \bar{x}_i \in L$ and $|L| = 2n$.
- Let C be the set of clauses whose conjunction is θ.
- Let $getVar(\cdot)$ be a function that returns for any literal $l \in L$ the variable appearing in the literal, i.e. $getVar(x_i) = getVar(\bar{x}_i) = x_i$.

- Let f be an arbitrary symbol.
- Let $G(\theta)$ be a 2-Player game. In this game each player selects one of the following: a variable, a literal, a clause from θ or the symbol f. This constitutes his sole action in the game. Game G's outcomes are defined in table 5.1

Thus $\Sigma_1 = \Sigma_2 = V \cup L \cup C \cup \{f\}$

	$\ell_2 \in L$	$v_2 \in V$	$c_2 \in C$	f
$\ell_1 \in L$	$(1,1)$ $\ell_1 \neq \bar{\ell_2}$ $(-2,-2)$ $\ell_1 = \bar{\ell_2}$	$(-2,-n+2)$ $v(\ell_1) = v_2$ $(-2,+2)$ $v(\ell_1) \neq v_2$	$(-2,+2)$ $\ell_1 \notin c_2$ $(-2,-n+2)$ $\ell_1 \in c_2$	$(-2,1)$ $(-2,1)$
$v_1 \in V$	$(+2,-2)$ $v(\ell_2) \neq v_1$ $(-n+2,-2)$ $v(\ell_2) = v_1$	$(-2,-2)$	$(-2,-2)$	$(-2,1)$
$c_1 \in C$	$(+2,-2)$ $\ell_2 \notin c_1$ $(-n+2,-2)$ $\ell_2 \in c_1$	$(-2,-2)$	$(-2,-2)$	$(-2,1)$
f	$(1,-2)$	$(1,-2)$	$(1,-2)$	$(0,0)$

Table 5.1: Definiton of Game $G(\theta)$

5.5.3 Reduction Proof

Lemma 5.3 *If θ is satisfiable then there exists a (1,1) Nash equilibrium in G*

Proof: If θ is satisfiable then there are $l_1, l_2, ..., l_n \in L$ (where $getVar(l_i) = x_i$) that when assigned the value $True$ (by setting the value of the underlying variable to the satisfying assignment) satisfy θ. If the other player plays all of these l_i with probability $\frac{1}{n}$ then playing the same strategy as well with the same probability yields a utility of 1. We argue this is a Nash Equilibrium.
We show that neither players can net a higher utility by changing his strategy and thus, by definition, it is a Nash Equilibrium:

- Playing the negation of one of the l_i's gives an expected utility of:

$$\frac{1}{n}(-2) + \frac{n-1}{n}(1) < 1$$

- Playing a $v_i \in V$ yields a utility of:

$$\frac{1}{n}(2-n) + \frac{n-1}{n}(2) = 1$$

  The reason is that there is a probability of $\frac{1}{n}$ that the other player chose an l_i such that $getVar(l_i) = x_i$.

- playing a clause $c \in C$ gives a utility of:

$$\frac{1}{n}(2-n) + \frac{n-1}{n}(2) = 1$$

  This happens since each l_i is a member of one or more of the clauses.

- Finally, by choosing f the utility is also 1.

Q.E.D.

This is the only Nash Equilibrium in the game that has an expected utility of 1 to both players. Actually the game has only one more Nash Equilibrium where both players put probability 1 on f and get $(0, 0)$.

Proof:

1. It is easy to verify there are no equilibriums where one player plays purely f and the other does not.

2. Assume both play a mixed strategy where the probability of playing f is 0. The maximal joint utility $(u_1 + u_2)$ in the game is 2. If any player has expected utility less than 1 than he is better off switching to playing f with probability 1. Thus (1,1) is also the maximal joint utility in any equilibrium. If either player plays V and C with a positive probability it follows that the joint utility is below 2. By the linearity of expectation it follows that at least one player has utility below 1. This player is better off playing purely f and thus V or C are never played in a Nash Equilibrium.

3. Thus we can assume both players put positive probabilities on strategies in $L \cup \{f\}$. If one player puts positive probability on f then the other player is strictly better off playing purely f since, like playing L, it yields 1 when the other player plays L and it performs better than L when the other player plays f. It follows that the only equilibrium where f is ever played is the one when both players play purely f.

4. Now we can assume both player only put positive probabilities on elements of L. Suppose that for some $l \in L$, the probability that player 1 plays either l or $\bar{l}$ is less than $\frac{1}{n}$ then the expected utility of the player 2, playing $v \in V$ such that $v = getVar(l)$, is:
$$u_2 > \frac{1}{n}(2 - n) + \frac{n-1}{n}(2) = 1$$
Hence, this cannot be a Nash Equilibrium. Thus we can assume that for any $l \in L$ the probability that a given player plays l or $\bar{l}$ is precisely $\frac{1}{n}$.

5. If there is an element of L such that player 1 plays with a positive probability and player 2 plays with a positive probability its negation, then both players have expected utilities of less than 1 and thus are better off playing purely f. Thus, in a Nash Equilibrium, if player 1 plays l with some probability, player 2 must play l with probability $\frac{1}{n}$ and thus player 1 must also play l with probability $\frac{1}{n}$. Thus we can assume that for any variable exactly one of its literals is played by both players with a probability of $\frac{1}{n}$. It follows that in any Nash Equilibrium (besides the one where both players play purely f), literals that are played indeed correspond to an assignment to the variables.

Q.E.D.

Lemma 5.4 *If θ is not satisfiable then a Nash Equilibrium with expected utilities of (1,1) doesn't exist in G*

Proof: As we verified above, when θ is satisifiable, the only Nash Equilibrium where the utility is 1 for both players is when both players choose l_i with probability $\frac{1}{n}$. But when θ is not satisfiable, this is not a Nash Equilibrium anymore:

- Let $c \in C$ be a clause that is not satisfied by the assignment, that is, none of its literals are ever played. Thus playing c nets a utility of 2 and each player is better off switching to this strategy.

Q.E.D.

5.5.4 Conclusions and Corollaries

Thus it is NP-hard to determine whether a certain equilibrium exists in a game. Also, since $(1,1)$ is the equilibrium that maximizes the "social welfare" (combined utility of both players) it can be viewed as the optimal Nash Equilibrium of the game. Thus finding the optimal Nash Equilibrium of the game is also NP-hard. Using the same game G, depending on how we state the decidability problem, we can deduce that many other Nash Equilibrium related problems are NP-hard. For example the number of different Nash Equilibrium in the game, is there an Nash Equilibrium where all players are guaranteed at least k and so on. Thus information regarding the Nash Equilibrium of games is generally hard to compute.

5.6 Brouwer's fixed point theorem

Theorem 5.5 *(Brouwer) Let $f : S \to S$ be a continuous function from a non-empty, compact, convex set $S \in \Re^n$ into itself, then there is $x^{\cdot} \in S$ such that $x^{\cdot} = f(x^*)$ (i.e. $x^{\cdot}$ is a fixed point of function f).*

5.6.1 Proof Outline

For $\Re^1$ the proof is a simple one is proved directly. For the two dimensional case we prove the theorem on triangles, aided by Sperner's lemma (which we will prove as well). Since we can "cut out" a triangle out of any convex, compact set, the theorem holds for any such set in $\Re^2$. Generalization of the theorem for a triangle in $\Re^n$ follows, but will not be shown here.

5.6.2 In One Dimension

Let $f : [0,1] \longrightarrow [0,1]$ be a continuous function. Then, there exists a fixed point, i.e. there is a x^* in $[0,1]$ such that $f(x^*) = x^*$. There are 2 possibilities:

1. If $f(0) = 0$ or $f(1) = 1$ then we are done.

2. If $f(0) \neq 0$ and $f(1) \neq 1$. Then define:

$$F(x) = f(x) - x$$

In this case:

$$F(0) = f(0) - 0 = f(0) > 0$$
$$F(1) = f(1) - 1 < 0$$

Thus $F : [0,1] \longrightarrow \Re$ where $F(0) \cdot F(1) < 0$. Since $f(\cdot)$ is continuous, $F(\cdot)$ is continuous as well. By the *Intermidate Value Theorem*, there exists $x^* \in [0,1]$ such that $F(x^*) = 0$. By definition of $F(\cdot)$:

$$F(x^*) = f(x^*) - x^*$$

And thus:

$$f(x^*) = x^*$$

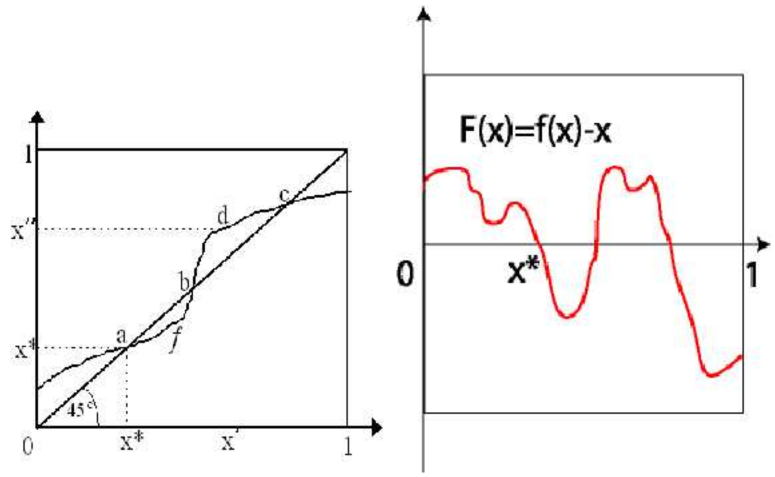


Figure 5.2: A one dimensional fixed point (left) and the function $F(\cdot)$ (right)

5.6.3 In Two Dimensions

Sperner's Lemma

Lemma 5.6 *(Sperner's Lemma) Given a triangle and a triangulation of it into an arbitrary number of "baby triangles".*

1. *Mark the vertices of the original triangle by 0,1,2.*

2. *Mark all other vertices with one of the labels according to the following rule:*

   - *If a vertex lies on the edge of the original triangle, label it by one of the numbers at the end points of that edge.*

- *If a vertex is inside the original triangle label it any way you like.*

There exists a baby triangle that has all three of the labels (one vertex is 0, the second is 1 and the third is 2). In fact there is an odd number of such triangles.

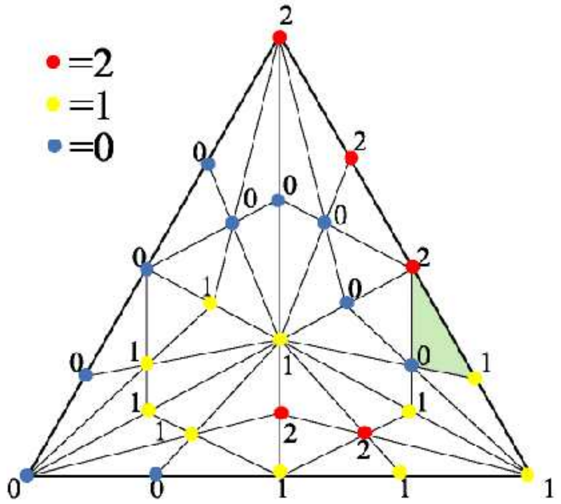


Figure 5.3: Sperner labeled triangle

Proof of Sperner's lemma in one dimension

Given a line segment whose endpoints are labeled 0 and 1 is divided to subsegments, each interior point is labeled is 0 or 1.

Definition A segment is called *completely labeled* if it has a label of 0 at one end and a label of 1 at the other end

- $C =$ number of completely labeled subsegments.
- $Z =$ number of subsegments labeled with 0 and 0.
- $O =$ number of occurrences of 0 at an endpoint of a subsegment - *an odd number* (since for every 0 you label two edges are added and you start off with one such edge)
- $O = 2Z + C$

Thus, C must be odd numbered (and thus $\neq$ 0s)

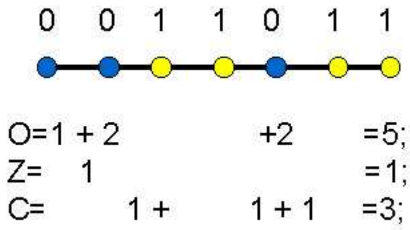


Figure 5.4: Sperner labeled segment

Proof of Sperner's lemma in two dimensions

Definition A triangle is called *completely labeled* if it has a label of 0 at one vertex, a label of 1 at another vertex and a label of 2 at the third vertex

- C = number of baby triangles that are completely labeled
- Z = number of baby triangles with 0 and 1 but not 2 (with 0,1,0 or with 0,1,1)
- O = number of occurrences of 0,1 at an edge of a baby triangle *an odd number*

  1. All occurrences of 0,1 for an interior baby triangle are paired up (since each interior edge is shared by two baby triangles ...)- an even number.
  2. All occurrences of 0,1 for a side baby triangle occur along the base of the original triangle, and thus are an odd number by the one dimension argument.

- $O = 2Z + C$
  Since each triangle in Z has two edges labeled 0,1 it contributes two edges to O. In sum, $2Z$ edges are contributed by all triangles in Z. This suggests the remaining edges in O come from triangles in C. Each triangle in C has one edge labeled 0,1 and thus contributes one edge to O.

Since O is odd and $2Z$ is even, C must be odd numbered (and thus $\neq 0$); which implies there exists at least one completely labeled baby triangle.

Proof of Brouwer's theorem

We can find a triangle in any convex, compact two-dimensional shape. Define the three vertices of the original triangle as A,B and C. Each point p in the triangle will be represented by its barycentric coordinates:

$$p = x_0A + x_1B + x_2C; x_0 + x_1 + x_2 = 1$$

Thus p can be represented as $p = (x_0, x_1, x_2)$.

Label all points of the triangle as follows:

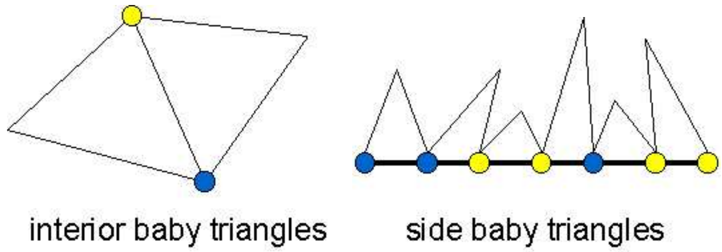


Figure 5.5: Labeling baby triangles in two dimensions

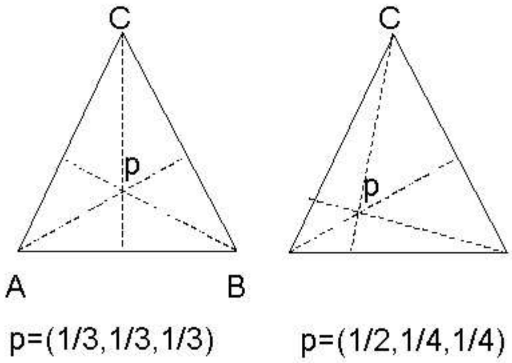
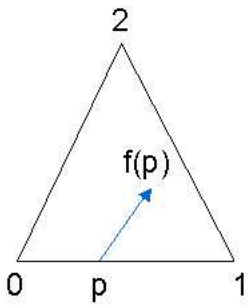


Figure 5.6: Sample barycentric coordinates of point p

- If $p = (x_0, x_1, x_2)$ and $f(p) = (x_0^{'}, x_1^{'}, x_2^{'})$ inspect the coordinates of p and $f(p)$ until you find the first index $i \in \{0, 1, 2\}$ such that $x_i^{'} < x_i$, label p by the label i.
- If it happens that for some point p there is no strict inequality like that, it must be that $x_0^{'} = x_0$, $x_1^{'} = x_1$, $x_2^{'} = x_2$. Thus p is a fixed point and we are done.
- According to this rule, the vertices A,B,C are labeled by 0,1,2.
- Each point of the edge 0,1 is marked by either 0 or 1. Similar statements hold for the other edges.
- Divide the triangles into smaller and smaller triangles with diameters approaching 0. At each step label the triangles by the rule above. Labeling is as in Sperner's Lemma.
- For each subdivision there exists at least one triangle labeled with all three labels.
- As we divide to smaller and smaller triangles, the vertices of the baby triangles that are completely labeled must eventually converge to some point $q = (y_0, y_1, y_2)$ (see figure 5.8). Now we use the labeling scheme we developed earlier and the completely labeled property of q that assures us that, due



the coordinates of p are $(*, *, 0)$ and they transformed by f into $(*, *, *)$; the x_2 increases, thus the label of p cannot be 2, thus it must be either 0 or 1;

Figure 5.7: Example of how a labeling of a point p is determined

to continuity, for q and $f(q)$ we must have:

$$y_0^{'} \leq y_0;\ y_1^{'} \leq y_1;\ y_2^{'} \leq y_2$$

Since the barycentric coordinates of a point add up to 1, these must be equalities:

$$y_0^{'} = y_0;\ y_1^{'} = y_1;\ y_2^{'} = y_2$$

And thus $f(q) = q$ and q is a fixed point. The proof for n-dimensional spaces is similar using an n-dimensional triangle.

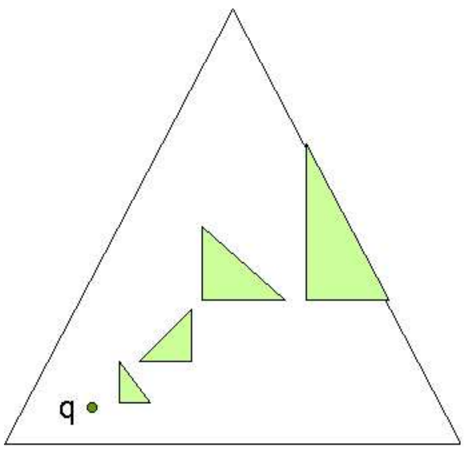


Figure 5.8: Dividing into smaller and smaller triangles - approaching to point q