A Tutorial on Computational Game Theory

From: Michael Kearns
Computer and Information Science
University of Pennsylvania
mkearns@cis.upenn.edu

Overview

- Examples of Strategic Conflict as Matrix Games
- Basics Definitions of (Matrix) Game Theory
- Notions of Equilibrium: Overview
- Definition and Existence of Nash Equilibria
- Computing Nash Equilibria for Matrix Games

Example: Prisoner's Dilemma

- Two suspects in a crime are interrogated in separate rooms
- Each has two choices: confess or deny
- With no confessions, enough evidence to convict on lesser charge; one confession enough to establish guilt
- Police offer plea bargains for confessing
- Encode strategic conflict as a payoff matrix:

payoffs	confess	deny
confess	$-3,-3$	$0,-4$
deny	$-4,0$	$-1,-1$

- What should happen?

Example: Hawks and Doves

- Two players compete for a valuable resource
- Each has a confrontational strategy ("hawk") and a conciliatory strategy ("dove")
- Value of resource is V; cost of losing a confrontation is C
- Suppose $C > V$ (think nuclear first strike)
- Encode strategic conflict as a payoff matrix:

payoffs	hawk	dove
hawk	$(V-C)/2$,$(V-C)/2$	V,0
dove	0,V	$V/2$,$V/2$

- What should happen?

A (Weak) Metaphor

- Actions of the players can be viewed as (disrete) variables
- Under any reasonable notion of “rationality”, the payoff matrix imposes constraints on the joint behavior of these two variables
- Instead of being probabilistic, these constraints are strategic
- Instead of computing conditional distributions given the other actions, players optimize their payoff
- Players are selfish and play their best response

Basics of Game Theory

- Set of players $i = 1, \ldots, n$ (assume $n = 2$ for now)
- Each player has a set of m basic actions or pure strategies (such as “hawk” or ”dove”)
- Notation: a_i will denote the pure strategy chosen by player i
- Joint action: $\vec{a}$
- Payoff to player i given by matrix or table $M_i(\vec{a})$
- Goal of players: maximize their own payoff

Notions of Equilibria: Overview (1)

- An equilibrium among the players is a strategic standoff
- No player can improve on their current strategy
- But under what model of communication, coordination, and collusion among the players?
- All standard equilbrium notions are descriptive rather than prescriptive

Notions of Equilibria: Overview (2)

- No communication or bargaining:
  Nash Equilibria
- Communication via correlation or shared randomness:
  Correlated Equilibria
- Full communication and coalitions:
  (Assorted) Cooperative Equilibria
- Equilibrium under evolutionary dynamics:
  Evolutionary Stable Strategy
- This seminar focus is only on Nash Equilibria

Mixed Strategies

- Need to introduce mixed strategies
- Each player i has an independent distribution p_i over their pure strategies ($p_i \in [0, 1]$ in 2-action case)
- Use $\vec{p} = (p_1, \ldots, p_n)$ to denote the product distribution induced over joint action $\vec{a}$
- Use $\vec{a} \sim \vec{p}$ to indicate $\vec{a}$ distributed according to $\vec{p}$
- Expected return to player i: $\mathbf{E}_{\vec{a} \sim \vec{p}}[M_i(\vec{a})]$
- (What about more general distributions over $\vec{a}$?)

Nash Equilibria

- A set of mixed strategies $(\vec{p}_1, \ldots, \vec{p}_n)$ such that no player has a unilateral incentive to deviate
- All players know all payoff matrices
- Informal: no communication, deals or collusion allowed — everyone for themselves
- Let $\vec{p}[i : p_i']$ denote $\vec{p}$ with p_i replaced by p_i'
- Formally: $\vec{p}$ is a Nash equilibrium (NE) if for every player i, and every mixed strategy p_i', $\mathbf{E}_{\vec{a} \sim \vec{p}}[M_i(\vec{a})] \geq \mathbf{E}_{\vec{a} \sim \vec{p}[i:p_i']}[M_i(\vec{a})]$
- Nash 1951: NE always exist in mixed strategies
- Players can announce their strategies

NE for Prisoner's Dilemma

- Recall payoff matrix:

payoffs	confess	deny
confess	-3,-3	0,-4
deny	-4,0	-1,-1

- One (pure) NE: (confess,confess)

- Failure to cooperate despite benefits

- Source of great and enduring angst in game theory

NE for Hawks and Doves

- Recall payoff matrix ($V < C$):

payoffs	hawk	dove
hawk	$(V - C)/2$,$(V - C)/2$	V,0
dove	0,V	$V/2$,$V/2$

- Three NE:
  - pure: (hawk,dove)
  - pure: (dove,hawk)
  - mixed: (Pr[hawk] $= V/C$,Pr[hawk] $= V/C$)
- Rock-Paper-Scissors: Only mixed NE

NE Existence Intuition

- Suppose that $\vec{p}$ is not a NE
- For some player i, must be some pure strategy giving higher return against $\vec{p}$ than p_i
- For each such player, shift some of the weight of p_i to this pure strategy
- Leave all other p_j alone
- Formalize as continuous mapping $\vec{p} \to F(\vec{p})$
- Brouwer Fixed Point Theorem: continuous mapping F of a compact set into itself must possess $\vec{p}^*$ such that $F(\vec{p}^*) = \vec{p}^*$
- One-dimensional case easy, high-dimensional difficult

Some NE Facts

- Existence not guaranteed in pure strategies
- May be multiple (exponential) NE
- Suppose (p_1, p_2) and (p'_1, p'_2) are two NE
- Zero-sum: (p_1, p'_2) and (p'_1, p_2) also NE, and give players same payoffs (games have a unique value)
- General sum: (p_1, p'_2) may not be a NE; different NE may give different payoffs
- Which will be chosen?
  - dynamics, additional criteria, structure of interaction?
  - Great concern in Game Theory.

Computing NE

- Inputs:
  - Payoff matrices M_i
  - Note: each has m^n entries (n players, m actions each)
- Output:
  - Any NE?
  - All NE? (output size)
  - Some particular NE?

Complexity Status of Computing a NE

- Zero-sum, 2-player case (input size m^2):
  - Linear Programming
  - Polynomial time solution
- General-sum case, 2 players (input size m^2):
  - Closely related to Linear Complementarity Problems
  - Can be solved with the Lemke-Howson algorithm
  - Exponential worst-case running time
  - Probably not in P, but probably not NP-complete?

Outline of THIS seminar

Outline of the seminar (1)

Coordination Ratio

Comparing the cost of the optimal solution to the worse NE solution.

How bad can NE can be?

- Worst-case Equilibria Koutsoupias and Papadimitriou
  Tight Bounds for Worst-Case Equilibria A. Czumaj and B. Vocking
- Selfish Traffic Allocation for Server Farms A. Czumaj, P. Krysta, and B. Vocking
- How Bad is Selfish Routing? Roughgarden and Tardos

Outline of the seminar (2)

Computing Nash Equilibria

The existing algorithms for computing NE.

- Computation of Equilibria in Finite Games Richard McKelvey and Andrew McLennan. [Chapter 1-3]
  Computing Equilibria for Two-Person Games Bernhard von Stengel [Chapters 1-2]
- Efficient Computation of Equilibria for Two-Person Games Koller, Megiddo, and von Stengel

Outline of the seminar (3)

Load Balancing, Congestion games and potential games

Simple models for computing and converging to deterministic NE.

- Potential Games, D. Monderer and L. Shapley
- Competitive Routing in Multi-User Communication Networks A. Orda R. Rom and N. Shimkin
- Congestion Games with Player-Specific Payoff Functions Igal Milchtaich
- Convergence time to Nash Equilibrium E. Even-Dar, A. Kesselman and Y. Mansour

Outline of the seminar (4)

Nash Convergence

- Graphical Models for Game Theory M. Littman, M. Kearns and S. Singh.
  An Efficient Exact Algorithm for Singly Connected Graphical Games M. Littman, M. Kearns and S. Singh
- Fictitious Play [Chapter 2, The theory of learning in Games, D. Fudenberg and D. Levine]