Assignment 5
Advanced Topics in Computational Geometry

Due: June 30, 2016, in my mailbox or in Orit's mailbox

Problem 1: Incidences

(a) Prove the Kővári–Sós–Turán theorem: Let $G = (V, E)$ be a bipartite graph with vertex sets P, Q (so $E \subseteq P \times Q$), such that G does not contain $K_{r,s}$ as a subgraph, where r and s are constants. Show that $|E| = O(mn^{1-1/r} + n)$ (and, symmetrically, $|E| = O(nm^{1-1/s} + m)$), where $m = |P|$ and $n = |Q|$.

(**Hints:** (i) Double count the number of tuples $(p_1, p_2, \ldots, p_r, q)$, such that $p_1, p_2, \ldots, p_r$ are distinct vertices in P, $q \in Q$, and all the edges (p_1, q), $(p_2, q), \ldots, (p_r, q)$ are in E. (ii) As a further hint, one of the counts should be $\sum_{q\in Q} \binom{\deg(q)}{r}$. (iii) Note that $|E| = \sum_{q\in Q} \deg(q)$, and use Hölder's inequality to estimate $|E|$.)

(b) For a set P of m points in the plane, and a set C of n curves in the plane, denote by $G(P, C)$ the *incidence graph* of P and C; its edges are all the pairs $(p, c) \in P \times C$ such that p is incident to c. Apply (a) to the incidence graphs for the cases where C is (i) a set of lines; (ii) a set of unit circles; (iii) a set of circles with arbitrary radii. Get three respective weak incidence bounds for $I(P, C)$ in these three cases.

(c) Consider case (iii) from (b) (where C is a set of arbitrary circles). Combine the bound from (b) with the cutting method, to show that $I(P, C) = O(m^{3/5}n^{4/5} + m + n)$. (No need to give full details, but try to discuss issues where the analysis here is somewhat different from the one shown in class for lines.)

Problem 2

Extend the proof technique of the Crossing Lemma to show the following: Let P be a set of n points in the plane in general position, and let D be a set of M disks, each having a pair of points of P as a diameter. If $M \geq 4n$ then there exists a point of P that lies in the interior of $\Omega(M^2/n^2)$ disks of D.

(**Hints:** (a) Show that if $M \geq 3n$ then there exists a disk $d \in D$ and a point $p \in P$ such that p lies in the interior of d. (To show this, use the graph G drawn on the set P as vertices, where for each disk $d \in D$ we draw in G the straight edge which is the diameter of d that connects its two defining points.) (b) Apply the random sampling technique used in the proof of the Lemma to get a good lower bound on the number of such pairs (p, d). (c) Conclude from (b) the existence of a point $p \in P$ that has the desired property.)

Problem 3

For each of the following range spaces $(X, \mathcal{R})$, obtain an upper bound on the number of possible ranges for any subset of m points of X. Whenever possible, compute the VC-dimension too.

(a) X is a set of points in $\mathbb{R}^3$ and each range of $\mathcal{R}$ is the set of points of X inside some axis-parallel box.

(b) X is a set of points in $\mathbb{R}^3$ and each range of $\mathcal{R}$ is the set of points of X inside some ball.

(c) X is a set of lines in $\mathbb{R}^3$ and each range of $\mathcal{R}$ is the set of lines of X that intersect some unit ball. (**Hint:** Move to a dual space where each ball is represented by its center, and each line of X is represented by ...)

Problem 4

Using Problem 3(c), give a simple-minded solution for the following problem. Given a set X of n lines in $\mathbb{R}^3$, and a parameter $\varepsilon > 0$, preprocess them into a data structure that supports efficiently queries of the form: For a query point $q \in \mathbb{R}^3$, estimate the number of lines at distance at most 1 from q, up to an error of ε. (The problem is somewhat vaguely defined, but you should know what to do...)