Advanced Topics in Computational Geometry

Prof. Micha Sharir

Final Exam, July 30, 2012

Answer four of the following problems. All have the same weight—25%.
You may use any material.
Good Luck!!

PROBLEM 1

Let P be a set of n points in the plane (in general position). We have shown in class that P admits a spanning tree T so that each line crosses at most $O(\sqrt{n})$ edges of T. Suppose that such a tree T is available.

(a) Show that T can be converted to a spanning path E of P with the same asymptotic bound $O(\sqrt{n})$ on the number of edges crossed by a line.

(b) Give an output-sensitive algorithm that, given a query line ℓ, finds all the k edges of E crossed by ℓ, in time close to k. (**Hint:** Store the edges of E in a binary tree in their order along E and maintain the convex hull of the edges stored below each node of the tree. Then search the tree with ℓ.)

(c) Turn the algorithm in (b) into a data structure for answering halfplane range counting queries on P (where each query specifies a line ℓ and asks for the number of points of P below ℓ), using near-linear storage and with query time close to $O(\sqrt{n})$.

PROBLEM 2

(a) Let $P = \{p_1(t), \ldots, p_n(t)\}$ be a set of n points moving in the plane. Assume that for each $i = 1, \ldots, n$, each coordinate of $p_i(t)$ is given as a polynomial in t of degree at most k, where k is a constant. Give an algorithm that runs in close to linear time for computing the smallest disk that is centered at the origin and contains P. That is, we need to find the time t at which the smallest disk centered at the origin and enclosing P is really the smallest.

(b) Same setup, but now we want to find the smallest disk that is centered on the x-axis and contains P. (Explain the ideas behind the algorithm, and give only some of the algorithmic details.) How efficient is the algorithm in this case? (Here the goal is mainly to show understanding of the structure. The algorithmic details are less important, but will count as bonus if given (briefly).)

PROBLEM 3

(a) Let S be a set of n segments in the plane, in general position, with c distinct orientations, where c is a constant. What is the overall complexity of the faces of $\mathcal{A}(S)$ that contain endpoints of the segments? (**Hint:** Consider each orientation separately, and use the combination lemma.)

(b) What is the maximum complexity of any *other* face of $\mathcal{A}(S)$? What can you say about the overall complexity of any m faces of $\mathcal{A}(S)$ (in terms of m and n)?

PROBLEM 4

Let $\mathcal{Q}$ be a set of n axis-parallel squares in the xy-plane. Lift each square to a random height in the z-direction (e.g., enumerate the squares as $Q_1, Q_2, \ldots, Q_n$, choose a random permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of $(1, 2, \ldots, n)$, and assign to square Q_i the height (z-coordinate) π_i.

We say that a vertex v of $\mathcal{A}(\mathcal{Q})$, incident to the boundaries of two squares Q_i, Q_j, *survives* after the lifting if the z-vertical line passing through v meets the two lifted squares at two points w_i, w_j, so that the vertical segment $w_i w_j$ meets no other lifted square.

Show that the expected number of surviving vertices is $O(n \log n)$. (**Hint:** Express the probability of a vertex to survive in terms of the number of squares that contain it, and use Clarkson-Shor.)

PROBLEM 5

(a) Let P be a set of n points in the plane and let o be a fixed point. Show that the number of triangles Δopq, with $p, q \in P$, of area 1 is $O(n^{4/3})$. (**Hint:** For a fixed p, determine the locus of points q that satisfy the constraint with p, and reduce the problem to an incidence problem.)

(b) Show that the number of triangles whose three vertices are points of P and whose area is 1 is $O(n^{7/3})$.

PROBLEM 6

Let P be a set of n points in the plane, and let R be a random sample of r points of P. Show that, with high probability, the following properties hold.

(a) Every angle (infinite wedge) that does not contain a point of R contains at most $O\left(\frac{n}{r} \log r\right)$ points of P.

(b) For every angle W we have $\left| \frac{|R \cap W|}{|R|} - \frac{|P \cap W|}{|P|} \right| = O\left(\sqrt{\frac{\log r}{r}}\right)$.