**REPEATED ANGLES IN THREE AND FOUR DIMENSIONS***

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**Abstract.** We show that the maximum number of occurrences of a given angle in a set of $n$ points in $\mathbb{R}^3$ is $O(n^{7/3})$, and that a right angle can actually occur $\Omega(n^{7/3})$ times. We then show that the maximum number of occurrences of any angle different from $\pi/2$ in a set of $n$ points in $\mathbb{R}^4$ is $O(n^{5/2}\beta(n))$, where $\beta(n) = 2^{O(\alpha(n)^2)}$ and $\alpha(n)$ is the inverse Ackermann function.

**Key words.** Combinatorial geometry, Geometric incidences, Repeated angles

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**1. Introduction.** In this paper we consider the following problem: Given a set $P$ of $n$ points in $\mathbb{R}^d$ and some fixed $0 < \alpha < \pi$, how many times can the angle $\alpha$ occur among triplets of point of $P$? That is, how many triplets $p, q, r \in P$ are there such that $\angle pqr = \alpha$? (We identify the triplet $(p, q, r)$ with $(r, q, p)$, and count them as only one angle.) The trivial upper bound is $O(n^3)$, which is the number of triplets, and a simple construction gives a lower bound of $\Omega(n^2)$ repeated angles.

In the plane, Pach and Sharir [4] have shown that the number of occurrences of a fixed angle among $n$ points is $O(n^2 \log n)$, and that this lower bound can be achieved for every angle $\alpha = \arctan \frac{a\sqrt{m}}{b}$, where $a, b$ and $m$ are positive integers.

In $\mathbb{R}^3$, the best known upper bound, $O(n^{8/3})$, is due to Conway *et al.* [2]; see also [1, Section 6.2]. We improve this bound to $O(n^{7/3})$ and show that this bound is tight in case $\alpha = \pi/2$.

In $\mathbb{R}^4$, there is a construction of $n$ points that determine $\Theta(n^3)$ *right angles* [1, Section 6.2, Problems 7 and 8], but for other angles $\alpha \neq \pi/2$, there is a subcubic bound of $O(n^{3-\frac{1}{25}})$, due to Purdy [6]; see [1, Section 6.2]. We improve this bound by showing that the maximum number of repeated angles $\alpha \notin \left\{0, \frac{\pi}{2}, \pi\right\}$ in a set of $n$ points in $\mathbb{R}^4$ is $O(n^{5/2}\beta(n))$, where $\beta(n) = 2^{O(\alpha(n)^2)}$ and $\alpha(n)$ is the inverse Ackermann function.

So far, the only lower bound in $\mathbb{R}^3$ and $\mathbb{R}^4$ that we have for $\alpha \notin \left\{0, \frac{\pi}{2}, \pi\right\}$ is the trivial bound $\Omega(n^2)$, and the planar bound $\Omega(n^2 \log n)$ for the above mentioned special values of $\alpha$.

As it turns out, the main difficulty in upper bounding the number of repeated angles lies in the possibility that the same angle instance is counted many times. Specifically, if $p \in P$ is incident to two rays that form an angle $\alpha$, and if there are $t$ points of $P$ on each ray, then the same angle occurs $t^2$ times among $t^2$ triplets of points of $P$. In particular, if $t = \Omega(n)$ we obtain the trivial lower bound $\Omega(n^2)$ mentioned above.

We overcome this difficulty by using the tradeoff, due to Szemerédi and Trotter [7], between the number of rays containing many points and the number of points on each ray. The more points per ray, the fewer rays. More precisely,

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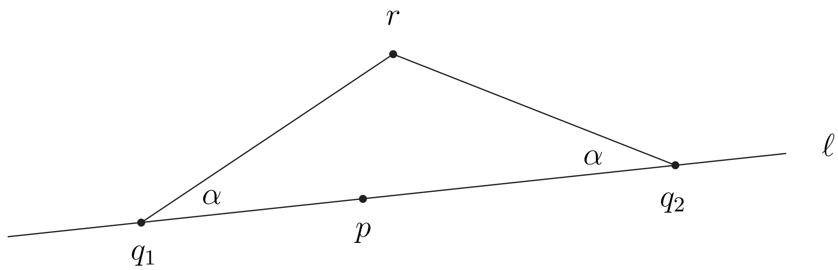


FIG. 2.1. *For a given point $r$ and line $\ell$, only two points $q \in \ell$ can be the apex of an angle $\angle pqr = \alpha$, with $p \in \ell$ too.*

THEOREM 1.1 (Szemerédi and Trotter [7]). *Let $P$ be a set of $n$ points in the plane. Then the number of lines containing at least $t$ points of $P$, is $O(n^2/t^3 + n/t)$, and the number of incidences between these lines and the points of $P$ is $O(n^2/t^2 + n)$.* We note that the Szemerédi-Trotter theorem is stated for points and lines in the plane, but it can easily be extended to higher dimensions by projecting the given points and lines onto some generic plane. See [5, 7] for details.

## 2. Repeated Angles in $\mathbb{R}^3$: Upper Bound.

THEOREM 2.1. *Let $P \subset \mathbb{R}^3$ be a set of $n$ points and let $0 < \alpha < \pi$ be fixed. Then the number of triplets $(p, q, r) \in P^3$ of distinct points satisfying $\angle pqr = \alpha$ is $O(n^{7/3})$.*

*Proof.* Denote the number of such triplets by $A(P)$. Let $L$ be the set of lines spanned by $P$. Partition $L$ into $m = \lceil \log n \rceil$ classes $L_1, L_2, \ldots, L_m$, so that the class $L_i$ includes all the lines of $L$ that contain at least $2^i$ and at most $2^{i+1} - 1$ points of $P$, for $i = 1, \ldots, m$. We use in the proof the threshold value $k = \lceil \frac{1}{3} \log n \rceil$.

We say that an angle $\angle pqr$ is *supported* by the lines $\ell_1$ and $\ell_2$ if $\ell_1 = \overline{pq}$ and $\ell_2 = \overline{qr}$. For $1 \le j \le i \le m$, let $A_{i,j}$ denote the number of angles supported by one line from $L_i$ and another line from $L_j$, that is,

$$A_{i,j} = \left| \left\{ (p, q, r) \in P^3 \;\middle|\; \angle pqr = \alpha, \overline{pq} \in L_i \text{ and } \overline{qr} \in L_j \right\} \right|,$$

and let $A_i = \sum_{j=1}^{i} A_{i,j}$ (recall that we identify triplets $(p, q, r)$ with their reverses $(r, q, p)$). We have $A(P) \le \sum_{i=1}^{m} A_i = \sum_{i=1}^{k} A_i + \sum_{i=k+1}^{m} A_i$ . We shall bound separately the terms $A' = \sum_{i=1}^{k} A_i$ and $A'' = \sum_{i=k+1}^{m} A_i$.

To bound $A''$, we use the following easy but crucial observation: For each point $r \in P$ and a line $\ell \in L$, there are at most two points $q \in \ell \cap P$ such that $r, q$ and some third point, $p$, on $\ell$ form an angle $\alpha$; see Figure 2.1. For each angle $\angle pqr = \alpha$ that is counted in $A''$, with $\overline{pq} \in L_i$, for some $i > k$, and $\overline{qr} \in L_j$, for some $j \le i$, we charge the triplet $(p, q, r)$ to the pair $(r, \ell)$ where $\ell = \overline{pq}$. The preceding observation implies that if $\ell$ contains $t$ points of $P$, where $2^i \le t < 2^{i+1}$, then the number of triplets that charge $(r, \ell)$ is at most $2t$, that is, at most twice the number of points of $P$ on $\ell$. This, in turn, implies that for a fixed $r \in P$ and for all lines $\ell$ containing $2^{k+1}$ points or more, the number of triplets that charge the pairs $(r, \ell)$ is at most twice the number of incidences between the points of $P$ and these lines. Hence, using Theorem 1.1, we have

$$A'' = O(n(n^2 2^{-2k} + n)) = O(n^3 2^{-2k} + n^2) = O(n^{7/3}).$$

We next bound $A'$. Let $j \le i \le k$ be fixed. We bound the number of angles in $A_{i,j}$ in the following different way. Put $s = 2^i$ and $t = 2^j$. For a point $p \in P$, let $\xi_p$ (resp., $\eta_p$) denote the number of rays emanating from $p$ and contained in lines of $L_i$ (resp., $L_j$). $\sum_{p\in P} \xi_p$ is twice the number of incidences between the points of $P$ and the lines of $L_i$, since each such incidence, $(p, \ell) \in P \times L_i$, contributes exactly 2 to this sum by generating two opposite rays, that are counted in $\xi_p$. As noted above, applying Theorem 1.1 to the lines of $L_i$, each containing $\Theta(s)$ points, implies that

$$\sum_{p\in P} \xi_p = O\left(\frac{n^2}{s^2} + n\right) = O\left(\frac{n^2}{s^2}\right), \tag{2.1}$$

where the last equality follows from the fact that $s = O(n^{1/3})$. Similarly, we have

$$\sum_{p\in P} \eta_p = O\left(\frac{n^2}{t^2}\right). \tag{2.2}$$

Let $\sigma_p$ denote the unit sphere centered at $p$. Map each ray emanating from $p$ and contained in a line of $L_i$ or $L_j$ to its intersection point with $\sigma_p$. We thus obtain two sets $C_p$ and $D_p$ of $\xi_p$ and $\eta_p$ points, respectively, on the sphere $\sigma_p$, and we want to count the number of pairs in $C_p \times D_p$ at spherical distance exactly $\alpha$. Each such pair corresponds to a pair of rays that emanate from $p$ and subtend the angle $\alpha$, so that one ray contains $O(s)$ points of $P$ and the other contains $O(t)$ points. Hence each such pair generates $O(st)$ occurrences of the angle $\alpha$ among point triplets of $P$. The number of such pairs in $C_p \times D_p$ is equal to the number of incidences between $\xi_p$ points and $\eta_p$ congruent circles on $\sigma_p$, and is thus bounded by $O((\xi_p\eta_p)^{2/3} + \xi_p + \eta_p)$ (see, e.g., [3]). Multiplying this by $O(st)$, and summing over all points $p$, we get

$$\begin{aligned} A_{i,j} &\le st \sum_{p\in P} O((\xi_p\eta_p)^{2/3} + \xi_p + \eta_p) \\ &= O\left(st \sum_{p\in P} (\xi_p\eta_p)^{2/3} + st \sum_{p\in P} \xi_p + st \sum_{p\in P} \eta_p\right). \end{aligned}$$

Using (2.1) and (2.2), the last two terms can be bounded by

$$O\left(st\left(\frac{n^2}{s^2} + \frac{n^2}{t^2}\right)\right) = O\left(\frac{n^2 t}{s} + \frac{n^2 s}{t}\right) = O\left(\frac{n^2 s}{t}\right),$$

since we have assumed that $s \ge t$. It remains to bound the first term. We observe that $\eta_p = O(n/t)$ for each $p \in P$, because all the rays emanating from $p$ are pairwise disjoint (excluding the common point $p$). Combining this with Hölder's inequality and with the estimates (2.1) and (2.2), we thus have

$$\begin{aligned} \sum_{p\in P} (\xi_p\eta_p)^{2/3} &= (O(n/t))^{1/3} \sum_{p\in P} \xi_p^{2/3}\eta_p^{1/3} \\ &= O\left(n^{1/3}t^{-1/3} \left(\sum_{p\in P} \xi_p\right)^{2/3} \left(\sum_{p\in P} \eta_p\right)^{1/3}\right) \\ &= O\left(n^{1/3}t^{-1/3} \left(\frac{n^2}{s^2}\right)^{2/3} \left(\frac{n^2}{t^2}\right)^{1/3}\right) \\ &= O\left(n^{7/3}s^{-4/3}t^{-1}\right). \end{aligned}$$

This yields

$$A_{i,j} = O\left(n^{7/3}s^{-1/3} + \frac{n^2 s}{t}\right) = O\left(n^{7/3}2^{-i/3} + n^2 2^{i-j}\right). \tag{2.3}$$

We then sum this bound over all $A_{i,j}$'s that contribute to $A'$, to obtain

$$\begin{aligned}
A' &= \sum_{i=1}^{k}\sum_{j=1}^{i} A_{i,j} \\
&= \sum_{i=1}^{k}\sum_{j=1}^{i} O\left(n^{7/3}2^{-i/3} + n^2 2^{i-j}\right) \\
&= O\left(n^{7/3}\sum_{i=1}^{k}\sum_{j=1}^{i} 2^{-i/3} + n^2 \sum_{i=1}^{k}\sum_{j=1}^{i} 2^{i-j}\right) \\
&= O\left(n^{7/3}\sum_{i=1}^{k} i2^{-i/3} + n^2 \sum_{i=1}^{k} 2^{i}\right) \\
&= O\left(n^{7/3} + n^2 2^k\right) \\
&= O\left(n^{7/3}\right).
\end{aligned}$$

Hence the number of repeated angles in $P$ is at most $A' + A'' = O(n^{7/3})$. □

**3. Repeated Angles in $\mathbb{R}^3$: Lower Bound.** In this section we show that the set $P$ of vertices of the $n^{1/3} \times n^{1/3} \times n^{1/3}$ cubic lattice section determine $\Omega(n^{7/3})$ right angles. The proof outline is as follows. The points of $P$ determine $O(n^{2/3})$ distinct distances. Hence, if we take all the spheres centered at points of $P$ and containing at least one point of $P$, we get at most $O(n^{5/3})$ spheres. For simplicity we consider only the spheres fully contained in the bounding cube of $P$. On each sphere we obtain many right angles as follows. Take a pair of antipodal points $p, r \in P$ and another point $q \in P$ on the sphere. Then $\angle pqr = \pi/2$. On average there are $m = \Omega(n^{1/3})$ points on the sphere. There are $m/2$ choices of an (unordered) antipodal pair $(p, r)$, and $m - 2$ choices of a third point $q$, yielding about $m^2/2 = \Omega(n^{2/3})$ right angles per sphere on average. Multiplying this bound by the number of spheres, $O(n^{5/3})$, we obtain that $P$ determines $\Omega(n^{7/3})$ right angles.

In more detail, we have:

Theorem 3.1. *Let $P = \left\{1, \ldots, \lfloor n^{1/3} \rfloor\right\}^3$. Then the number of triplets $(p, q, r) \in P^3$ such that $\angle pqr = \pi/2$ is $\Omega(n^{7/3})$.*

*Proof.* For simplicity we assume that $n$ is a cubic integer and a multiple of 5, so that all the quantities that appear in the proof are integers. This assumption does not change the order of magnitude of the lower bound.

Let $Q = \left\{\frac{2}{5}n^{1/3} + 1, \ldots, \frac{3}{5}n^{1/3}\right\}^3$ be the middle $\frac{1}{5}n^{1/3} \times \frac{1}{5}n^{1/3} \times \frac{1}{5}n^{1/3}$ portion of $P$. We have $|Q| = \frac{n}{125} = \Theta(n)$. For each pair of points in $Q$, the square of the distance between them is an integer of magnitude at most $\frac{3}{25}n^{2/3}$. Hence there are at most $\frac{3}{25}n^{2/3} = O(n^{2/3})$ distinct distances between the points of $Q$. For every point $o \in Q$ we take the spheres centered at $o$ and containing at least one point $p \in Q$. There are $O(n^{2/3})$ such spheres. We repeat this for all points of $Q$ and let $S$ denote the resulting set of spheres. We have $|S| = O(n^{5/3})$. The choice of $Q$ guarantees that, for every point $p \in P$ on a sphere $\sigma \in S$, the point on $\sigma$ antipodal to $p$ is also in $P$.

For each $\sigma \in S$, let $m_\sigma = |P \cap \sigma|$ denote the number of lattice points on $\sigma$. We observe that $\sum_{\sigma \in S} m_\sigma \geq 2\binom{|Q|}{2} = \Omega(n^2)$, since in this sum we count every pair $p, p' \in Q$ exactly twice — once with $p$ at the center of the sphere and $p'$ on the sphere itself, and once the other way around. Similarly, $\sum_{\sigma \in S} m_\sigma \leq |Q| \cdot |P| = O(n^2)$, so this sum is $\Theta(n^2)$. Let $\sigma \in S$ be one of the spheres and let $p, q, r \in \sigma \cap P$ be three distinct points such that $p$ and $r$ are antipodal points of $\sigma$. Then $\angle pqr = \pi/2$. There are $m_\sigma/2$ choices of an antipodal pair $p, r \in \sigma \cap P$ and $m_\sigma - 2$ choices of a third point $q$, yielding $m_\sigma(m_\sigma - 2)/2$ right angles on $\sigma$. The lower bound on the number of right angles in $P$ is obtained by summing over all the spheres of $S$. Note that each pair of points can be antipodal on at most one sphere, hence every angle is counted only once. This gives a lower bound of

$$\frac{1}{2} \sum_{\sigma \in S} m_\sigma(m_\sigma - 2) \geq \frac{1}{2\,|S|} \left( \sum_{\sigma \in S} m_\sigma \right)^2 - \sum_{\sigma \in S} m_\sigma = \frac{1}{2\,|S|} \Theta(n^4) - \Theta(n^2),$$

where we have used the Cauchy-Schwarz inequality. Substituting $|S| = O(n^{5/3})$ in the inequality gives $\Omega(n^{7/3})$ right angles determined by the points of $P$. $\square$

*Remark:*. It is an interesting open problem whether the same lower bound also holds for other angles $\neq \pi/2$.

**4. Repeated Angles in $\mathbb{R}^4$.** Recall that there is a construction of $n$ points in $\mathbb{R}^4$ that determine $\Theta(n^3)$ right angles, but for other angles $\alpha \neq \pi/2$, there is a subcubic upper bound of $O(n^{3-\frac{1}{25}})$, due to Purdy [6]; see [1, Section 6.2]. In this section we improve this upper bound, and derive the following result.

THEOREM 4.1. *Let $P \subset \mathbb{R}^4$ be a set of $n$ points and let $\alpha \notin \{0, \pi/2, \pi\}$ be fixed. Then the number of triplets $(p, q, r) \in P^3$ of distinct points satisfying $\angle pqr = \alpha$ is $O(n^{5/2}\beta(n))$, where $\beta(n) = 2^{c\alpha^2(n)}$ for some constant $c > 0$, and where $\alpha(n)$ is the extremely slowly growing inverse Ackermann function.*

*Proof.* The machinery of Section 2 can be easily extended to four dimensions as follows. We use the same partition of the set $L$ of lines spanned by $P$ into $\lceil \log n \rceil$ classes, where the $i$-th class consists of all lines that contain at least $2^i$ and at most $2^{i+1} - 1$ points of $P$. The values $A(P), A_{i,j}$ and $A_i$ are defined as in the three-dimensional case. Unlike the case of $\mathbb{R}^3$, we use the threshold value $k = \lceil \frac{1}{4} \log n \rceil$, and obtain $A(P) = A' + A''$, where $A' = \sum_{i \leq k} A_i$ and $A'' = \sum_{i > k} A_i$. Bounding $A''$ proceeds exactly as before and yields $A'' = O(n^3 2^{-2k}) = O(n^{5/2})$.

For $A'$, we bound each of the $A_{i,j}$ terms separately. We set $s = 2^i$ and $t = 2^j$. As before, for each $p \in P$, we take the unit 3-sphere $\sigma_p$ centered at $p$, intersect the rays emanating from $p$ and contained in the lines of $L_i$ and $L_j$ with $\sigma_p$, and reduce the problem to that of counting repeated distances on the sphere $\sigma_p$, all equal to the spherical distance $\alpha$. Each such incidence defines a pair of rays at angle $\alpha$ emanating from $p$, which contribute $O(st)$ angles to our count. The number of repeated distances on $\sigma_p$ is equal to the number of incidences between $\xi_p$ points and $\eta_p$ congruent copies of the sphere $\mathbb{S}^2$ scaled according to the spherical distance $\alpha$. If $\alpha \neq \pi/2$, then these copies of $\mathbb{S}^2$ are not great spheres on $\sigma_p$, which easily implies that three distinct copies intersect in at most two points (and not in a common circle, as may happen if they are great spheres).

We can then apply the analysis of [3, Section 6] for the number of incidences between points and unit spheres in $\mathbb{R}^3$. Since our spheres lie on a 3-sphere rather than in Euclidean 3-space, the analysis of [3] requires some easy modifications. For example, we can project $\sigma_p$ onto $\mathbb{R}^3$, using stereographic projection. The $\eta_p$ 2-spheres on $\sigma_p$ are then mapped to 2-spheres in $\mathbb{R}^3$, not necessarily of equal radius. Nevertheless, the analysis in [3] carries over to this situation. The two main properties that the analysis uses are that the incidence graph between the projected points and spheres does not contain $K_{3,3}$, and that the size of the vertical decomposition of an arrangement of $r$ projected spheres is $O(r^3\beta(r))$, and, as is easily verified, both properties hold for the projected spheres and points.

We conclude that the number of incidences between $\xi_p$ points and $\eta_p$ 2-spheres on $\sigma_p$ is $O((\xi_p\eta_p)^{3/4}\beta(\xi_p,\eta_p)+\xi_p+\eta_p)$, where $\beta(m,n)=2^{c'\alpha^2(m^3/n)}$ for some constant $c'>0$ independent of $m$ and $n$, and where $\alpha(\cdot)$ is the inverse Ackermann function. Put $\beta(n)=\beta(n,1)=2^{c'\alpha^2(n^3)}$. Since $\alpha(\cdot)$ is very slowly growing, we have $\alpha(n^3)=O(\alpha(n))$, and consequently, $\beta(n)\le 2^{c\alpha^2(n)}$ for an appropriate constant $c>0$ depending only on $c'$. Note that $\beta(m,n)$ is ascending in $m$ and descending in $n$, hence $\beta(\xi_p,\eta_p)\le\beta(n,1)$ (unless $\eta_p=0$, but in that case we trivially have 0 angle instances from $A_{i,j}$ at the apex $p$). Plugging this bound into an appropriately modified variant of the analysis of Section 2 gives

$$\begin{aligned}
A_{i,j} &= O\left(st\sum_{p\in P}(\xi_p\eta_p)^{3/4}\beta(\xi_p,\eta_p)+st\sum_{p\in P}\xi_p+st\sum_{p\in P}\eta_p\right)\\
&= O\left(st\sum_{p\in P}(\xi_p\eta_p)^{3/4}\beta(n,1)+st\sum_{p\in P}\xi_p+st\sum_{p\in P}\eta_p\right)\\
&= O\left(st\beta(n)\sum_{p\in P}(\xi_p\eta_p)^{3/4}+\frac{n^2t}{s}+\frac{n^2s}{t}\right).
\end{aligned}$$

As above, we have

$$\begin{aligned}
\sum_{p\in P}(\xi_p\eta_p)^{3/4} &= (O(n/t))^{1/2}\sum_{p\in P}\xi_p^{3/4}\eta_p^{1/4}\\
&= O\left(n^{1/2}t^{-1/2}\left(\sum_{p\in P}\xi_p\right)^{3/4}\left(\sum_{p\in P}\eta_p\right)^{1/4}\right)\\
&= O\left(n^{1/2}t^{-1/2}\left(\frac{n^2}{s^2}\right)^{3/4}\left(\frac{n^2}{t^2}\right)^{1/4}\right)\\
&= O\left(n^{5/2}s^{-3/2}t^{-1}\right),
\end{aligned}$$

and thus

$$A_{i,j}=O\left(n^{5/2}\beta(n)2^{-i/2}+n^2 2^{i-j}\right). \tag{4.1}$$

Finally we sum over all the relevant $A_{i,j}$'s to obtain

$$\begin{aligned} A' &= O\left(n^{5/2}\beta(n)\sum_{i=1}^{k}\sum_{j=1}^{i}2^{-i/2} + n^2\sum_{i=1}^{k}\sum_{j=1}^{i}2^{i-j}\right) \\ &= O\left(n^{5/2}\beta(n) + n^2 2^k\right) \\ &= O\left(n^{5/2}\beta(n)\right). \end{aligned}$$

Hence the number of repeated angles in $\mathbb{R}^4$ is $O(n^{5/2}\beta(n))$. □

*Remarks:.* The lower bound construction for $\mathbb{R}^3$ can be easily extended to $\mathbb{R}^4$ to yield a lower bound of $\Omega(n^{5/2})$ right angles, but this bound is very weak, since, as mentioned, right angles can be repeated $\Theta(n^3)$ times in $\mathbb{R}^4$. An interesting open problem is to match the upper bound of Theorem 4.1 by a lower bound close to $\Omega(n^{5/2})$. As mentioned in the introduction, the only lower bounds that we have so far (for $\alpha \neq \pi/2$) are $\Omega(n^2)$, and $\Omega(n^2 \log n)$ for the special values of $\alpha$ used in [4].

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