Counting Plane Graphs: Flippability and its Applications*

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Abstract

We generalize the notions of flippable and simultaneously-flippable edges in a triangulation of a set S of points in the plane, into so called *pseudo simultaneously-flippable edges*. Such edges are related to the notion of convex decompositions spanned by S.

We derive a worst-case tight lower bound on the number of pseudo simultaneously-flippable edges in any triangulation, and use this bound to obtain upper bounds for the maximal number of several types of plane graphs that can be embedded (with crossing-free straight edges) on a fixed set of N points in the plane. More specifically, denoting by $\mathsf{tr}(N) < 30^N$ the maximum possible number of triangulations of a set of N points in the plane, we show that every set of N points in the plane can have at most $6.9283^N \cdot \mathsf{tr}(N) < 207.85^N$ plane graphs, $O(4.8895^N) \cdot \mathsf{tr}(N) = O(146.69^N)$ spanning trees, and $O(5.4723^N) \cdot \mathsf{tr}(N) = O(164.17^N)$ forests (that is, cycle-free graphs). We also obtain upper bounds for the number of crossing-free straight-edge graphs with at most cN edges and for the number of such graphs with at least cN edges.

1 Introduction

In this paper we obtain improved upper bounds for the maximal number of various types of straight-edge crossing-free graphs embedded on a fixed set of points in the plane. The techniques used in this paper rely on edge-flips in triangulations.

A *planar graph* is a graph that can be embedded in the plane in such a way that its vertices are embedded as points and its edges are embedded as Jordan arcs that connect the respective pairs of points and can meet only at a common endpoint. A *straight-edge plane graph* is an embedding of a planar graph in the plane such that its edges are embedded as non-crossing straight line segments. In this paper, we only consider straight-edge plane graphs. Moreover, we only consider embeddings where the points are in general position, that is, where no three points are collinear. (For upper bounds on the number of graphs,

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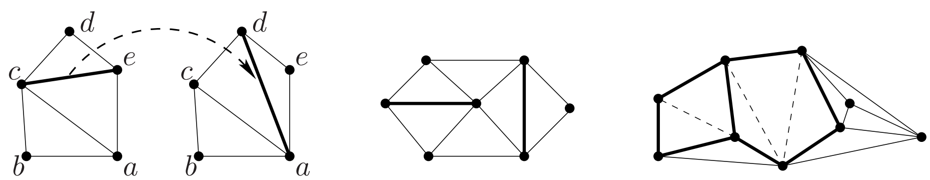
this involves no loss of generality, because the number of graphs can only grow when a degenerate point set is slightly perturbed into general position.)

We note that, by Fáry's theorem, every planar graph G can be embedded as a straight-edge crossing-free graph in the plane (see [7]), but, in general, the set of points realizing G in such an embedding depends on G. (For example, the planar graph K_4 can be embedded with straight edges if and only if the vertex set is not in convex position.) In our setting, we only consider planar graphs that can be realized as straight-edge crossing-free graphs on a fixed but arbitrarily labeled set S. Analysis of the number of plane embeddings of planar graphs in which the set of vertices is not restricted or when the vertices are not labeled can be found, for example, in [13, 18, 26].

A *triangulation* of a set S of N points in the plane is a maximal straight-edge crossing-free graph on S (that is, no additional straight edges can be inserted without crossing some of the existing edges). Triangulations are an important geometric construct which is used in many algorithmic applications, and are also an interesting object of study in discrete and combinatorial geometry (comprehensive surveys can be found in [5, 14]).

A triangulation of S forms a decomposition of the convex hull of S into pairwise interior-disjoint triangles, whose vertices are the points of S, and with no point of S isolated. If the convex hull of S has h vertices, so $n = N - h$ points of S are interior to the hull, then, by Euler's formula, every triangulation of S has $3n + 2h - 3$ edges (h of which are hull edges, common to all triangulations, and $3n + h - 3$ interior), and $2n + h - 2$ bounded triangular faces. We will use throughout the paper the notations h and n for the number of hull vertices and interior points, respectively.



(a) (b) (c)

Figure 1: (a) The edge ce can be flipped to the edge ad. (b) The two bold edges are simultaneously flippable. (c) Interior-disjoint convex quadrilateral and convex pentagon in a triangulation.

We say that an interior edge in a triangulation of such a set S (i.e., not an edge of the convex hull of S) is *flippable*, if its two incident triangles form a convex quadrilateral. A flippable edge can be *flipped*, that is, removed from the graph of the triangulation and replaced by the other diagonal of the corresponding quadrilateral, thereby obtaining a new triangulation of S. Such an operation is depicted in Figure 1(a), where the edge ce can be flipped to the edge ad. Already in 1936, Wagner [28] has shown that any *unlabeled non-embedded* triangulation T can be transformed into any other triangulation T' (with the same number of vertices) through a series of edge-flips (here one uses a more abstract notion of an edge flip). When we deal with a pair of triangulations over a specific common (labeled) set S of points in the plane, there always exists such a sequence of $O(|S|^2)$ flips, and this bound is tight in the worst case (e.g., see [2, 16]). Moreover, there are algorithms that perform such sequences of flips to obtain some "optimal" triangulation (typically, the Delaunay triangulation; see [9] for example), which, as a by-product, provide an edge-flip sequence between any specified pair of triangulations.

So how many flippable edges can a single triangulation have? Given a triangulation T, we denote by $\mathsf{flip}(T)$ the number of flippable edges in T. Hurtado, Noy, and Urrutia [16] proved the following lower bound.

Lemma 1.1 *For any triangulation T over a set of N points in the plane,*

$$\mathsf{flip}(T) \geq N/2 - 2.$$

Moreover, there are triangulaitons (of specific point sets of arbitrarily large size) for which this bound is tight.

We refer to this lemma as *the flippability lemma*.

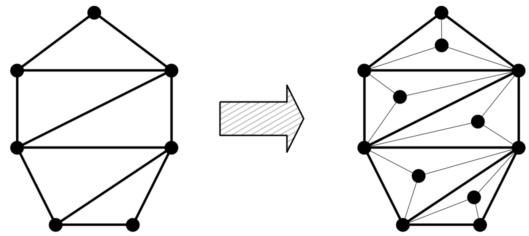


Figure 2: Constructing a triangulation with $N/2 - 2$ flippable edges.

To obtain a triangulation with exactly $N/2 - 2$ flippable edges, start with a convex polygon over $N/2 + 1$ vertices, triangulate it in some arbitrary manner, insert a new point into each of the $N/2 - 1$ resulting bounded triangles, and connect each new point p to the three hull vertices that form the triangle containing p. Such a construction is depicted in Figure 2. The resulting graph is a triangulation with N vertices and exactly $N/2 - 2$ flippable edges, namely the chords of the initial triangulation.

Next, we say that two flippable edges e and e' of a triangulation T are *simultaneously flippable* if no triangle of T is incident to both edges; equivalently, the quadrilaterals corresponding to e and e' are interior-disjoint. See Figure 1(b) for an illustration. Notice that flipping an edge e cannot affect the flippability of any edge simultaneously flippable with e. Given a triangulation T, let $\mathsf{flip}_s(T)$ denote the size of the largest subset of edges of T, such that every pair of edges in the subset are simultaneously flippable. We refer to the following lemma, taken (and slightly adapted) from Galtier et al. [10], as *the simultaneous-flippability lemma*.

Lemma 1.2 *For any triangulation T over a set of N points in the plane,*

$$\mathsf{flip}_s(T) \geq \frac{\mathsf{flip}(T)}{3}.$$

In particular, $\mathsf{flip}_s(T) \geq N/6 - 2/3$.

Galtier et al. [10] also show that the bound $\mathsf{flip}_s(T) \geq N/6 - 2/3$ is not far from being tight in the worst case, by presenting a specific triangulation in which at most $(N - 4)/5$ edges are simultaneously flippable.

Pseudo simultaneously-flippable edges. In the definition above, a set of simultaneously flippable edges can be considered as the set of diagonals of a collection of interior-disjoint convex quadrilaterals (in a triangulation T). One may consider a more liberal definition of simultaneously flippable edges, by taking, within a fixed triangulation T, the diagonals of a set of interior-disjoint convex polygons, each with at least four edges (so that the boundary edges of these polygons belong to T). Consider such a collection of convex polygons $Q_1, \ldots, Q_m$, where Q_i has $k_i \geq 4$ edges, for $i = 1, \ldots, m$. We can then retriangulate each Q_i independently, to obtain many different triangulations. Specifically, each Q_i can be triangulated in C_{k_i-2} ways, where C_j is the j-th Catalan number (see, e.g., [24, Section 5.3]). Hence, we can get $M = \prod_{i=1}^{m} C_{k_i-2}$ different triangulations in this way. In particular, if a graph $G \subseteq T$ (namely, all the edges of G are edges of T) does not contain any diagonal of any Q_i (it may contain boundary edges though) then G is a subgraph of (at least) M distinct triangulations. An example is depicted in Figure 1(c), where by "flipping" (or rather, redrawing) the diagonals of the highlighted quadrilateral and pentagon, we can get $C_2 \cdot C_3 = 2 \cdot 5 = 10$ different triangulations. We refer to such edges, namely, the set of diagonals of such a collection of convex polygons, as *pseudo simultaneously-flippable edges* (*ps-flippable edges* for short). We emphasize that all three notions of flippability are defined within a fixed triangulation T of S (although each of them gives a recipe for producing many other triangulations).

Edge Type	Upper bound	Worst case lower bound
Flippable	$N/2 - 2$ [16]	$N/2 - 2$ [16]
Simultaneously flippable	$N/6 - 2/3$ [10]	$N/5 - 4/5$ [10]
Ps-flippable	$\max(N/2 - 2, h - 3)$	$\max(N/2 - 2, h - 3)$

Table 1: Bounds for minimum numbers of the various types of flippable edges in a triangulation.

Our Results. In Section 2, we derive a lower bound on the size of the largest set of ps-flippable edges in a triangulation, and show that this bound is tight in the worst case. Specifically, we have the following *ps-flippability lemma*, which is a considerable strengthening of Lemma 1.2:

Lemma 1.3 *Let S be a set of N points in the plane, and let T be a triangulation of S. Then T contains a set of at least* $\max(N/2-2, h-3)$ *ps-flippable edges. This bound is tight in the worst case.*

The proof of this lemma is given in Section 2. Table 1 summarizes the known bounds for minimum numbers of the various types of flippable edges in a triangulation.

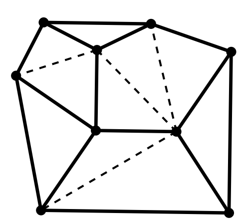


Figure 3: A convex decomposition of S. When completing it into a triangulation, the added (dashed) diagonals form a set of ps-flippable edges.

We also relate ps-flippable edges to the subject of *convex decompositions* of S. These are crossing-free straight-edge graphs on S such that (i) they include all the hull edges, (ii) each of their bounded faces is a convex polygon, and (iii) no point of S is isolated. See Figure 3 for an illustration.

In Section 3, we use the bound on the size of the largest set of ps-flippable edges to derive several upper bounds on the numbers of plane graphs of various kinds embedded

Graph type	Lower bound	Prev. upper bound	New upper bound	In the form $a^N \cdot \mathsf{tr}(N)$
Plane Graphs	$\Omega(41.18^N)$ [1]	$O(239.4^N)$ [19, 22]	$\mathbf{207.85^N}$	$\mathbf{6.9283^N \cdot tr(N)}$
Spanning Trees	$\Omega(11.97^N)$ [6]	$O(158.6^N)$ [4, 22]	$\mathbf{O(146.69^N)}$	$\mathbf{O(4.8895^N) \cdot tr(N)}$
Forests	$\Omega(12.23^N)$ [6]	$O(194.7^N)$ [4, 22]	$\mathbf{O(164.17^N)}$	$\mathbf{O(5.4723^N) \cdot tr(N)}$

Table 2: Upper and lower bounds for the number of several types of straight-edge crossing-free graphs on a set of N points in the plane. The new bounds (two right columns) are obtained by using sets of ps-flippable edges.

as straight-edge crossing-free graphs on a fixed set S. For a set S of points in the plane, we denote by $\mathcal{T}(S)$ the set of all triangulations of S, and put $\mathsf{tr}(S) := |\mathcal{T}(S)|$. Similarly, we denote by $\mathcal{P}(S)$ the set of all (crossing-free straight-edge) plane graphs on S, and put $\mathsf{pg}(S) := |\mathcal{P}(S)|$. We also let $\mathsf{tr}(N)$ and $\mathsf{pg}(N)$ denote, respectively, the maximum values of $\mathsf{tr}(S)$ and of $\mathsf{pg}(S)$, over all sets S of N points in the plane.

Since a triangulation has fewer than $3|S|$ edges, the trivial upper bound $\mathsf{pg}(S) < 8^{|S|} \cdot \mathsf{tr}(S)$ holds for any point set S. Recently, Razen, Snoeyink, and Welzl [19] slightly improved the upper bound on the ratio $\mathsf{pg}(S)/\mathsf{tr}(S)$ from $8^{|S|}$ down to $O\left(7.9792^{|S|}\right)$. We give a more significant improvement with an upper bound of $6.9283^{|S|}$. Combining this bound with the recent bound $\mathsf{tr}(S) < 30^{|S|}$ [22], we get $\mathsf{pg}(N) < 207.85^N$. We provide similar improved ratios and absolute bounds for the numbers of crossing-free straight-edge spanning trees and forests (i.e, cycle-free graphs). Table 2 summarizes these results[1].

We also derive similar ratios (or, equivalently, bounds of the form $a^{|S|} \cdot \mathsf{tr}(S)$) for the number of straight-edge crossing-free graphs with at most $c|S|$ edges and for the number of such graphs with at least $c|S|$ edges, for $0 \le c \le 3$. (Actually, our bounds are established only for certain restricted ranges of c; see Subsection 3.3 for details.)

Notations. Here are some additional notations that we use.

Given a triangulation T of S, we denote by $v_i(T)$ the number of interior vertices of degree i in T, for $i \ge 3$.

Given two plane graphs G and H over the same point set S, if every edge of G is also an edge of H, we write $G \subseteq H$.

Hull edges and vertices (resp., interior edges and vertices) of a graph embedded on a point set S are those that are part of the convex hull of S (resp., not part of the convex hull).

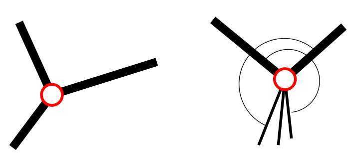


Figure 4: Separable edges.

Separable edges. Let p be an interior vertex in a convex decomposition G of S. Following the notation in [23], we call an edge e incident to p in G *separable at p* if it can be separated

[1]Up-to-date bounds of various sorts can be found in `http://www.cs.tau.ac.il/~sheffera/counting/PlaneGraphs.html` (version of November 2010).

from the other edges incident to p by a line through p (see Figure 4, where the separating lines are not drawn). Equivalently, the two angles between e and its clockwise and counterclockwise neighboring edges (around p) have to sum up to more than π. Following [23], we observe the following easy properties.

(i) If p is an interior vertex of degree 3 in G, its three incident edges are separable at p, for otherwise p would have been a reflex vertex of some face.

(ii) An interior vertex p of degree 4 or higher can have at most two incident edges which are separable at p (and if it has two such edges they must be consecutive in the circular order around p).

Separable edges have a couple of additional properties when G is a triangulation (see [23]), which will not be used in this paper.

2 The number of ps-flippable edges

In this section, we establish the ps-flippability lemma (Lemma 1.3 from the introduction). We restate the lemma for the convenience of the reader.

Lemma 1.3 *Let S be a set of N points in the plane, and let T be a triangulation of S. Then T contains a set of at least* $\max(N/2-2, h-3)$ *ps-flippable edges. This bound is tight in the worst case.*

Proof. Starting with the lower bound, we apply the following iterative process to T. As long as there exists an interior edge whose removal does not create a non-convex face, we pick such an edge and remove it. When we stop, we have a straight-edge plane graph G, all of whose bounded faces are convex; that is, we have a locally minimal convex decomposition of S. Note that all h original hull edges are still in G, and that every interior vertex of G has degree at least 3.

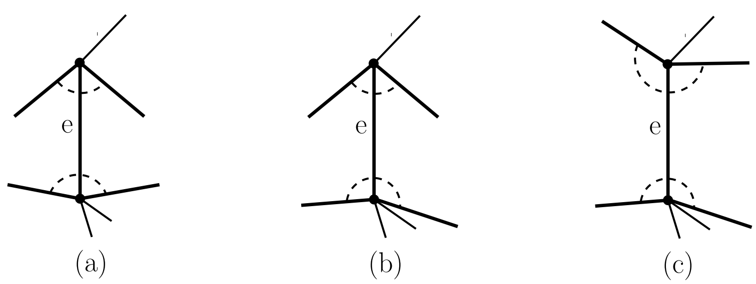


Figure 5: (a) An edge not separable at both of its endpoints can be removed from the graph. (b,c) An edge separable in at least one of its endpoints cannot be removed from the graph.

Note that every edge of G is separable at one or both of its endpoints, for we can remove any other edge and the graph will continue to have only convex faces (see Figure 5). We denote by m the number of edges of G, and by m_{int} the number of its interior edges. Recalling properties (i) and (ii) of separable edges, we have $m = m_{\text{int}} + h$ and

$$m_{\text{int}} \leq 3v_3 + 2v_{4+,2} + v_{4+,1} = a, \tag{1}$$

where v_3 is the number of interior vertices of degree 3 in G, and $v_{4+,i}$ is the number of interior vertices u of degree at least 4 in G with exactly i edges separable at u. Notice that

$$n = v_3 + v_{4+,0} + v_{4+,1} + v_{4+,2}.$$

The estimate in the right-hand side of (1) may be pessimistic, because it doubly counts edges that are separable at both endpoints (such as the one in Figure 5(c)). To address this possible over-estimation, denote by m_{double} the number of edges that are separable at both endpoints, to which we refer as *doubly separable edges*, and rewrite (1) as

$$m_{\text{int}} = 3v_3 + 2v_{4+,2} + v_{4+,1} - m_{\text{double}} = a - m_{\text{double}}. \tag{2}$$

Denoting by m_{faces} the number of bounded faces of G, we have, by Euler's formula, $n + h + (m_{\text{faces}} + 1) = (m_{\text{int}} + h) + 2$ (the expression in the parentheses on the left is the number of faces in G, and the expression in the parentheses on the right is the number of edges), or

$$m_{\text{faces}} = m_{\text{int}} - n + 1. \tag{3}$$

Let f_k, for $k \geq 3$, denote the number of interior faces of degree k in G. By doubly counting the number of edges in G, and then applying (3), we get

$$\sum_{k\geq 3} k f_k = 2m_{\text{int}} + h = 2(m_{\text{faces}} + n - 1) + h = \sum_{k\geq 3} 2f_k + 2n - 2 + h,$$

or

$$\sum_{k\geq 3}(k-2)f_k = 2n + h - 2. \tag{4}$$

The number of edges that were removed from T is $\sum_{k\geq 3}(k-3)f_k$, because a face of G of degree k must have had $k-3$ diagonals that were edges of T. This number is therefore

$$\begin{aligned}\sum_{k\geq 3}(k-3)f_k &= \sum_{k\geq 3}(k-2)f_k - m_{\text{faces}} = 2n + h - 2 - m_{\text{faces}} = \\ &= 2n + h - 2 - (m_{\text{int}} - n + 1) = 3n + h - 3 - a + m_{\text{double}}\end{aligned} \tag{5}$$

(by first applying (4), then (3), and finally (2)).

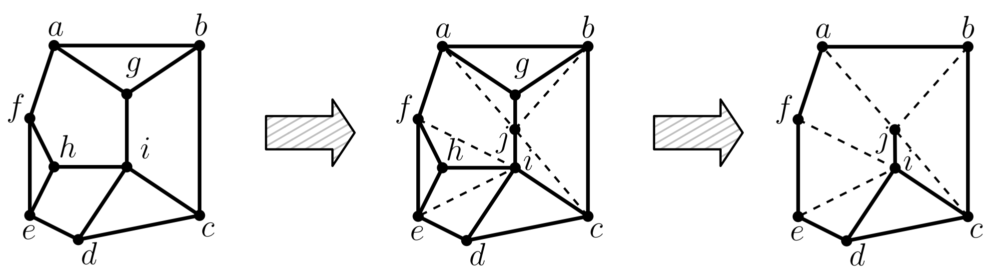


Figure 6: A convex decomposition G of S, its corresponding graph G', and the reduced form of G' after removing vertices of degree 3. The edges that have been added are dashed.

We next derive a lower bound for the right-hand side of (5). For this, we transform G into another graph G' as follows. We first split each doubly separable edge of G at its midpoint, say, and add the splitting point as a new vertex of G (e.g., see the vertex j in Figure 6). We now modify G as follows. We take each vertex u of degree 3 in G and surround it by a triangle, by connecting all pairs of its three neighbors. Notice that some of these neighbors may be new splitting vertices, and that some of the edges of the surrounding triangle may already belong to G. For example, see Figure 6, where the edges ei, fi are added around the vertex h and the edges aj, bj are added around the vertex g. Next we take each interior vertex u with two separable edges at u and complete these two edges into a triangle by connecting their other endpoints, each of which is either an original point or a new splitting point; here too the completing edge may already belong to G. For example, see the edge cj in Figure 6. We then take the resulting graph G' and remove each vertex of degree 3 and its three incident edges; see the reduced version of G' in Figure 6. A crucial and easily verified property of this transformation is that the newly embedded edges do not cross each other, nor do they cross old edges of G.

The number m'_{faces} of bounded faces of the new graph G' is at least $v_3 + v_{4+,2}$, which is the number of triangles that we have created, and the number n' of its interior vertices is $n - v_3 + m_{\text{double}}$. Also, G' still has h hull edges. Using Euler's formula, as in (3) and (4) above, we have $m'_{\text{faces}} \leq 2n' + h - 2$. Combining the above, we get

$$v_3 + v_{4+,2} \leq 2(n - v_3 + m_{\text{double}}) + h - 2,$$

or

$$m_{\text{double}} \geq \frac{3}{2}v_3 + \frac{1}{2}v_{4+,2} - n - \frac{1}{2}h + 1.$$

Hence, the right-hand side of (5) is at least

$$\begin{aligned}
&3n + h - 3 - a + m_{\text{double}} \\
&\quad\geq\quad 3n + h - 3 - (3v_3 + 2v_{4+,2} + v_{4+,1}) + \left(\frac{3}{2}v_3 + \frac{1}{2}v_{4+,2} - n - \frac{1}{2}h + 1\right) \\
&\quad=\quad 2n + \frac{1}{2}h - 2 - \frac{3}{2}v_3 - \frac{3}{2}v_{4+,2} - v_{4+,1} \\
&\quad\geq\quad 2n + \frac{1}{2}h - 2 - \frac{3}{2}\left(v_3 + v_{4+,2} + v_{4+,1} + v_{4+,0}\right) \\
&\quad=\quad \frac{n+h}{2} - 2 = \frac{N}{2} - 2.
\end{aligned}$$

In other words, the number of edges that we have removed is at least $N/2-2$. On the other hand, we always have $m_{\text{double}} \geq 0$ and $a \leq 3n$. Substituting these trivial bounds in (5) we get at least $h - 3$ ps-flippable edges. This completes the proof of the lower bound.

It is easily noticed that only flippable edges of T could have been removed in the initial pruning stage. Hurtado, Noy, and Urrutia [16] present two distinct triangulations that contain exactly $N/2 - 2$ flippable edges (one of those is depicted in Figure 2). These triangulations cannot have a set of more than $N/2 - 2$ ps-flippable edges. Therefore, there are point sets for which our bound is tight in the worst case. Similarly, for point sets in convex position, all $h - 3$ interior edges form a set of ps-flippable edges, showing that the other term in the lower bound is also tight in the worst case. ◻

Remark. The actual lower bound yielded by the first part of the preceding analysis is

$$\frac{1}{2}N + \frac{1}{2}v_{4+,1} + \frac{3}{2}v_{4+,0} - 2.$$

That is, for the bound to be tight, every interior vertex u of degree 4 or higher must have two incident edges separable at u (note that this condition holds vacuously for the triangulation in Figure 2).

Convex decompositions. The preceding analysis is also related to the notion of *convex decompositions*, as defined in the introduction. Urrutia [27] asked what is the minimum number of faces that can always be achieved in a convex decomposition of any set of N points in the plane. Hosono [15] proved that every planar set of N points admits a convex decomposition with at most $\lceil \frac{7}{5}(N+2) \rceil$ (bounded) faces. For every $N \geq 4$, García-Lopez and Nicolás [11] constructed N-element point sets that do not admit a convex decomposition with fewer than $\frac{12}{11}N - 2$ faces. By Euler's formula, if a connected plane graph has N vertices and e edges, then it has $e - N + 2$ faces (including the exterior face). It follows that for convex decompositions, minimizing the number of faces is equivalent to minimizing the number of edges. (For convex decompositions contained in a given triangulation, this is also equivalent to maximizing the number of removed edges.)

Lemma 1.3 directly implies the following corollary. (The bound that it gives is weaker than the bound in [15], but it holds for every triangulation.)

Corollary 2.1 *Let S be a set of N points in the plane, so that its convex hull has h vertices, and let T be a triangulation of S. Then T contains a convex decomposition of S with at most $\frac{3}{2}N - h \leq \frac{3}{2}N - 3$ convex faces and at most $\frac{5}{2}N - h - 1 \leq \frac{5}{2}N - 4$ edges. Moreover, there exist point sets S of arbitrarily large size, and triangulations $T \in \mathcal{T}(S)$ for which these bounds are tight.*

3 Applications of ps-flippable edges

In this section we apply the ps-flippability lemma (Lemma 1.3) to obtain several improved bounds on the number of plane graphs of various kinds on a fixed set of points in the plane.

3.1 The ratio between the number of plane graphs and the number of triangulations

Let S be a set of N points in the plane. Every straight-edge crossing-free graph in $\mathcal{P}(S)$ is contained in at least one triangulation in $\mathcal{T}(S)$. Additionally, since a triangulation has fewer than $3N$ edges, every triangulation $T \in \mathcal{T}(S)$ contains fewer than $2^{3N} = 8^N$ plane graphs. This immediately implies

$$\mathsf{pg}(S) < 8^N \cdot \mathsf{tr}(S).$$

However, this inequality seems rather weak since it potentially counts some plane graphs many times. More formally, given a graph $G \in \mathcal{P}(S)$ contained in x distinct triangulations of S, we say that G has a *support* of x, and write $supp(G) = x$. Thus, every graph $G \in \mathcal{P}(S)$ will be counted $supp(G)$ times in the preceding inequality.

Recently, Razen, Snoeyink, and Welzl [19] managed to break the 8^N barrier by overcoming the above inefficiency. However, they obtained only a slight improvement, with the bound $\mathsf{pg}(S) = O\left(7.9792^N\right) \cdot \mathsf{tr}(S)$. We now present a more significant improvement, using a much simpler technique that relies on the ps-flippability lemma.

Theorem 3.1 *For any set S of N points in the plane,*

$$\mathsf{pg}(S) \leq \begin{cases} \frac{(4\sqrt{3})^N}{2^h} \cdot \mathsf{tr}(S) < \frac{6.9283^N}{2^h} \cdot \mathsf{tr}(S), & \text{for } h \leq N/2, \\ 8^N (3/8)^h \cdot \mathsf{tr}(S), & \text{for } h > N/2 \end{cases}$$

Proof. The exact value of $\mathsf{pg}(S)$ is easily seen to be

$$\mathsf{pg}(S) = \sum_{T \in \mathcal{T}(S)} \sum_{\substack{G \in \mathcal{P}(S) \\ G \subseteq T}} \frac{1}{supp(G)}. \tag{6}$$

We obtain an upper bound on this sum as follows. Consider a graph $G \in \mathcal{P}(S)$ and a triangulation $T \in \mathcal{T}(S)$, such that $G \subseteq T$. By Lemma 1.3, there is a set F of $t = \max(N/2-2, h-3)$ ps-flippable edges in T. Let $F_{\bar{G}}$ denote the set of edges that are in F but *not* in G, and put $j = |F_{\bar{G}}|$. Removing the edges of $F_{\bar{G}}$ from T yields a convex decomposition of S which still contains G and whose non-triangular interior faces have a total of j missing diagonals. Suppose that there are m such faces, with $j_1, j_2, \ldots, j_m$ diagonals respectively, where $\sum_{k=1}^{m} j_k = j$. Then these faces can be triangulated in $\prod_{k=1}^{m} C_{j_k+1}$ ways, and each of the resulting triangulations contains G. We always have $C_{i+1} \geq 2^i$, for any $i \geq 1$, as is easily verified by induction on i, and so $supp(G) \geq 2^j$. (Equality occurs when all the non-triangular faces of $T \setminus F_{\bar{G}}$ are quadrilaterals.)

Next, we estimate the number of subgraphs $G \subseteq T$ for which the set $F_{\bar{G}}$ is of size j. Denote by E the set of edges of T that are not in F, and assume that the convex hull of S has h vertices. Since there are $3N-3-h$ edges in any triangulation of S, $|E| \leq 3N-3-h-t$. To obtain a graph G for which $|F_{\bar{G}}| = j$, we choose any subset of edges from E, and any j edges from F (the j edges of F that will not belong to G). Therefore, the number of such subgraphs is at most $2^{3N-h-t-3} \cdot \binom{t}{j}$.

We can thus rewrite (6) to obtain

$$\begin{aligned} \mathsf{pg}(S) &\leq \sum_{T \in \mathcal{T}(S)} \sum_{j=0}^{t} 2^{3N-h-t-3} \cdot \binom{t}{j} \cdot \frac{1}{2^j} = \\ &= \mathsf{tr}(S) \cdot 2^{3N-h-t-3} \sum_{j=0}^{t} \binom{t}{j} \frac{1}{2^j} = \\ &= \mathsf{tr}(S) \cdot 2^{3N-h-t-3} \cdot (3/2)^t. \end{aligned}$$

If $t = N/2-2$, we get $\mathsf{pg}(S) < \mathsf{tr}(S) \cdot \frac{(4\sqrt{3})^N}{2^h} < \frac{6.9283^N}{2^h} \cdot \mathsf{tr}(S)$. If $t = h-3$, we have $\mathsf{pg}(S) \leq \mathsf{tr}(S) \cdot 2^{3N-2h} \cdot (3/2)^h = \mathsf{tr}(S) \cdot 8^N \cdot (3/8)^h$. To complete the proof, we note that $N/2-2 > h-3$ when $h < n+2$, or $h < N/2+1$. □

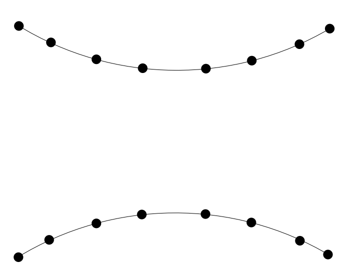


Figure 7: A double chain configuration with 16 vertices.

For a lower bound on $\mathsf{pg}(S)/\mathsf{tr}(S)$, we consider the *double chain* configurations, presented in [12] (and depicted in Figure 7). It is shown in [12] that, when S is a double chain configuration, $\mathsf{tr}(S) = \Theta^*\left(8^N\right)$ and $\mathsf{pg}(S) = \Theta^*\left(39.8^N\right)$ (the upper bound for $\mathsf{pg}(S)$ appears in [1])[2]. Thus, we have $\mathsf{pg}(S) = \Theta^*\left(4.95^N\right) \cdot \mathsf{tr}(S)$ (for this set $h = 4$, so h has no real effect on the bound of Theorem 3.1).

For another lower bound, consider the case where S is in convex position. In this case we have $\mathsf{tr}(S) = C_{N-2} = \Theta^*(4^N)$, and $\mathsf{pg}(S) = \Theta^*(11.65^N)$ (see [8]). Hence, $\mathsf{pg}(S)/\mathsf{tr}(S) = \Theta^*(2.9125^N)$, whereas the upper bound provided by Theorem 3.1 is 3^N in this case[3].

Finally, recall the notations $\mathsf{tr}(N) = \max_{|S|=N} \mathsf{tr}(S)$ and $\mathsf{pg}(N) = \max_{|S|=N} \mathsf{pg}(S)$. Combining the bound $\mathsf{tr}(N) < 30^N$, obtained in [22], with the first bound of Theorem 3.1 implies $\mathsf{pg}(N) < 207.849^N$. The bound improves significantly as h gets larger.

3.2 The number of spanning trees and forests

Spanning Trees. For a set S of N points in the plane, we denote by $\mathcal{ST}(S)$ the set of all straight-edge crossing-free spanning trees of S, and put $\mathsf{st}(S) := |\mathcal{ST}(S)|$. Moreover, we let $\mathsf{st}(N) = \max_{|S|=N} \mathsf{st}(S)$.

Buchin and Schulz [4] have recently shown that every plane graph contains $O\left(5.2852^N\right)$ spanning trees, improving upon the earlier bound of $5.\bar{3}^N$ due to Ribó Mor and Rote [20, 21]. We thus get $\mathsf{st}(S) = O\left(5.2852^N\right) \cdot \mathsf{tr}(S)$ for every set S of N points in the plane. The bound from [4] cannot be improved much further, since there are triangulations with at least 5.0295^N spanning trees [20, 21]. However, the ratio between $\mathsf{st}(S)$ and $\mathsf{tr}(S)$ can be improved beyond that bound, by exploiting and overcoming the same inefficiency as in the case of plane graphs; that is, the fact that some spanning trees may get multiply counted in many triangulations.

We now derive such an improved ratio by using ps-flippable edges. The proof goes along the same lines of the proof of Theorem 3.1.

Theorem 3.2 *For any set S of N points in the plane,*

$$\mathsf{st}(S) = O\left(4.8895^N\right) \cdot \mathsf{tr}(S).$$

Proof. The exact value of $\mathsf{st}(S)$ is

$$\mathsf{st}(S) = \sum_{T \in \mathcal{T}(S)} \sum_{\substack{\tau \in \mathcal{ST}(S) \\ \tau \subset T}} \frac{1}{supp(\tau)}.$$

Consider a spanning tree $\tau \in \mathcal{ST}(S)$ and a triangulation $T \in \mathcal{T}(S)$, such that $\tau \subset T$. As in Theorem 3.1, let F be a set of $N/2 - 2$ ps-flippable edges in T. (Here we do not exploit the alternative bound of $h - 3$ on the size of F.) Also, let $F_{\bar{\tau}}$ denote the set of edges that are in F but *not* in τ, and put $j = |F_{\bar{\tau}}|$. Thus, as argued earlier, $supp(\tau) \geq 2^j$.

Next, we estimate the number of spanning trees $\tau \subset T$ for which the set $F_{\bar{\tau}}$ is of size j. First, there are $\binom{|F|}{j} < \binom{N/2}{j}$ ways to choose the j edges of F that τ does not use. Then τ uses $N/2 - 2 - j$ edges of F, and its other $N/2 + j + 1$ edges have to be chosen from the complementary set E, as in the preceding proof. Since $|E| < 5N/2 - h$, there are fewer than $\binom{N/2}{j} \cdot \binom{5N/2-h}{N/2+j+1}$ spanning trees $\tau \subset T$ with $|F_{\bar{\tau}}| = j$. However, when j is large, it is better to use the bound $O\left(5.2852^N\right)$ (from [4]) instead.

[2] In the notations $O^*()$, $\Theta^*()$, and $\Omega^*()$, we neglect polynomial factors.

[3] Informally, the discrepancy between the exact bound in [8] and our bound in the convex case comes from the fact that when j is large, the faces of the resulting convex decomposition are likely to have many edges, which makes $\mathrm{supp}(G)$ substantially larger than 2^j.

We thus get, for a threshold parameter $a < 0.5$ that we will set in a moment,

$$\mathsf{st}(S) < \sum_{T \in \mathcal{T}(S)} \left(\sum_{j=0}^{aN} \binom{N/2}{j} \cdot \binom{5N/2 - h}{N/2 + j + 1} \cdot \frac{1}{2^j} + \sum_{j=aN+1}^{N/2} O\left(5.2852^N\right) \cdot \frac{1}{2^j} \right).$$

The terms in the first sum over j increase when $a \le 0.5$, so the sum is at most $N/2$ times its last term. Using Stirling's formula and ignoring the effect of h, we get that for $a \approx 0.1152$, the last term in the first sum is $\Theta^*\left(5.2852^N/2^{aN}\right) = O\left(4.8895^N\right)$. Since this also bounds the second sum, we get

$$\mathsf{st}(S) < \sum_{T \in \mathcal{T}(S)} O\left(4.8895^N\right) = O\left(4.8895^N\right) \cdot \mathsf{tr}(S),$$

as asserted. ◻

Combining the bound just obtained with $\mathsf{tr}(N) < 30^N$ [22] implies

Corollary 3.3 $\mathsf{st}(N) = O\left(146.685^N\right)$.

This improves all previous upper bounds, the smallest of which is $O(158.6^N)$ [4, 22].

It would be interesting to refine the bound in Theorem 3.2 so that it also depends on h, as in Theorem 3.1. An extreme situation is when S is in convex position (in which case $|F| = N-3$). In this case it is known that $\mathsf{tr}(S) = \Theta^*(4^N)$ and $\mathsf{st}(S) = \Theta^*(6.75^N)$ (see [8]), so the exact ratio is only $\mathsf{st}(S)/\mathsf{tr}(S) = \Theta^*(1.6875^N)$. This suggests that when h is large the ratio should be considerably smaller, but we have not pursued this in this paper.

Forests. For a set S of N points in the plane, we denote by $\mathcal{F}(S)$ the set of all straight-edge crossing-free forests (i.e., cycle-free graphs) of S, and put $\mathsf{f}(S) := |\mathcal{F}(S)|$. Moreover, we let $\mathsf{f}(N) = \max_{|S|=N} \mathsf{f}(S)$. Buchin and Schulz [4] have recently shown that every plane graph contains $O\left(6.4884^N\right)$ forests (improving a simple upper bound of $O^*(6.75^N)$ which we will note later in Subsection 3.3). Using this bound, we obtain

Theorem 3.4 *For any set S of N points in the plane,*

$$\mathsf{f}(S) = O\left(5.4723^N\right) \cdot \mathsf{tr}(S).$$

Proof. Immediate by adapting the proof of Theorem 3.2, replacing there $O\left(5.2852^N\right)$ by $O\left(6.4884^N\right)$. We note that there are $O^*\left(\binom{N/2}{j} \cdot \binom{5N/2}{N/2+j}\right)$ ways to choose forests G for which $|F_{\bar{G}}| = j$, which summarizes the number of ways to choose *at most* $N/2+j+1$ edges from E (after having chosen the j unused edges of F). However, we can drop the $O^*(\cdot)$ notation, since we round up the resulting base of the bound. This allows us to proceed as in the previous proof, except that now the optimal value of a is about 0.2457 (this is the value for which $\binom{N/2}{aN} \cdot \binom{5N/2}{N/2+aN} = \Theta^*(6.4884^N)$), from which the asserted bound follows. ◻

As in the previous cases, we can combine this with the bound $\mathsf{tr}(N) < 30^N$ [22] to obtain

Corollary 3.5 $\mathsf{f}(N) = O\left(164.169^N\right)$.

Again, this should be compared with the best previous upper bound $O(194.7^N)$ [4, 22].

Consider once again the case where S consists of N points in convex position. In this case we have $\mathsf{tr}(S) = \Theta^*(4^N)$ and $\mathsf{f}(S) = \Theta^*(8.22^N)$ (see [8]), so the exact ratio is $\mathsf{f}(S)/\mathsf{tr}(S) = \Theta^*(2.055^N)$, again suggesting that the ratio should be smaller when h is large.

3.3 The number of plane graphs with a bounded number of edges

In this subsection we derive an upper bound for the number of straight-edge plane graphs on a set S of N points in the plane, with some constraints on the number of edges. Specifically, we bound the number of plane graphs with at most cN edges and the number of plane graphs with at least cN edges, for any constant parameter $0 < c \leq 3$. (Our actual bounds pose some further restrictions on this range for c; see below for details.)

Plane graphs with at most cN edges. For a set S of N points in the plane and a constant $0 < c \leq 3$, we denote by $\mathcal{P}_c(S)$ the set of all plane graphs of S with at most cN edges, and put $\mathsf{pg_c}(S) := |\mathcal{P}_c(S)|$.

For $c \leq 1.5$, we can obtain a trivial upper bound for $\mathsf{pg_c}(S)$ by bounding the maximal number of such plane graphs that any fixed triangulation of S can contain. Using Stirling's formula, we have

$$\begin{aligned}\mathsf{pg_c}(S) &< \mathsf{tr}(S) \cdot \left(\binom{3N}{0} + \binom{3N}{1} + \cdots + \binom{3N}{cN}\right) \\ &= O^*\left(\mathsf{tr}(S) \cdot \binom{3N}{cN}\right) = O^*\left(\mathsf{tr}(S) \cdot \left(\frac{27}{c^c \cdot (3-c)^{3-c}}\right)^N\right).\end{aligned}$$

For example, the above implies that there are at most $O^*\left(6.75^N\right) \cdot \mathsf{tr}(S)$ plane graphs with at most N edges, over any set S of N points in the plane. In particular, this is also an upper bound on the number of crossing-free forests on S, a bound already observed in [1], or of spanning trees, or of spanning cycles. Of course, better bounds exist for these three special cases, as demonstrated earlier in this paper for the first two bounds.

We now present an improvement of this trivial bound. The proof goes along the same lines of the proof of Theorem 3.1, but is technically somewhat more involved.

Theorem 3.6 *For any set S of N points in the plane and $0 < c \leq 5/4$,*

$$\mathsf{pg_c}(S) = O^*\left(\left(\frac{5^{5/2}}{8(c+t-1/2)^{c+t-1/2}(3-c-t)^{3-c-t}(2t)^t(1/2-t)^{1/2-t}}\right)^N\right) \cdot \mathsf{tr}(S),$$

where

$$t = \frac{1}{2}\left(\sqrt{(7/2)^2 + 3c + c^2} - 5/2 - c\right). \tag{7}$$

Proof. The exact value of $\mathsf{pg_c}(S)$ is

$$\mathsf{pg_c}(S) = \sum_{T \in \mathcal{T}(S)} \sum_{\substack{G \in \mathcal{P}_c(S) \\ G \subseteq T}} \frac{1}{supp(G)},$$

where $supp(G)$, the support of G, is defined as in the case of general plane graphs treated in Subsection 3.1. We obtain an upper bound on this sum as follows. Consider a graph $G \in \mathcal{P}_c(S)$ and a triangulation $T \in \mathcal{T}(S)$, such that $G \subseteq T$. By Lemma 1.3, there is a set F of $N/2 - 2$ ps-flippable edges in T. Let $F_{\bar{G}}$ denote the set of edges that are in F but *not* in G, and put $j = |F_{\bar{G}}|$. As in the preceding proof, we have $supp(G) \geq 2^j$.

Next, we estimate the number of subgraphs $G \subseteq T$ for which the set $F_{\bar{G}}$ is of size j. Denote by E the set of edges of T that are not in F. As argued above, $|E| < 5N/2$. To obtain a graph for which $|F_{\bar{G}}| = j$, we choose any j edges from F (the j edges of F that will not belong to G), and any subset of at most $cN - (N/2 - 2 - j) = (c - 1/2)N + j + 2$ edges from E. If $(c - 1/2)N + j + 2 < 0$, there are no such graphs and we ignore these values of j. The number of ways to pick the edges from E is at most

$$\sum_{i=0}^{(c-1/2)N+j+2} \binom{5N/2}{i} = O(N) \cdot \max_i \binom{5N/2}{i} = O^*\left(\max_i \binom{5N/2}{i}\right). \tag{8}$$

It is easily checked that, for $c \le 5/4$, the maximum term is always the last one, for any value of $j \le N/2-2$ (that is, the maximum possible value of i is at most $5N/4$). Thus, the number of subgraphs G of T for which $G \in \mathcal{P}_c(S)$ and $|F_{\bar{G}}| = j$ is $O^*\left(\binom{5N/2}{\lfloor (c-1/2)N + j \rfloor} \cdot \binom{N/2}{j}\right)$. We use the simpler (and sloppier) bound $O^*\left(\binom{5N/2}{(c-1/2)N + j} \cdot \binom{N/2}{j}\right)$ for convenience; the $O^*(\cdot)$ notation hides the additional factor that might arise. This implies that

$$\begin{aligned}\mathsf{pg_c}(S) &< \sum_{T \in \mathcal{T}(S)} \sum_{j=0}^{N/2} O^*\left(\binom{5N/2}{(c-1/2)N + j} \cdot \binom{N/2}{j}\right) \cdot \frac{1}{2^j} = \\ &= \mathsf{tr}(S) \cdot \sum_{j=0}^{N/2} O^*\left(\binom{5N/2}{(c-1/2)N + j} \cdot \binom{N/2}{j}\right) \cdot \frac{1}{2^j}.\end{aligned} \tag{9}$$

(As already noted, when $c < 1/2$, only the terms for which $(c - 1/2)N + j \ge 0$ are taken into account.)

Once again, it suffices to consider only the largest term of the sum. For this, we consider the quotient of the j-th and $(j-1)$-st terms (ignoring the $O^*(\cdot)$ notation, which will not affect the exponential order of growth of the terms), which is

$$\frac{\binom{N/2}{j}\binom{5N/2}{(c-1/2)N+j}}{2\binom{N/2}{j-1}\binom{5N/2}{(c-1/2)N+j-1}} = \frac{\Big(N/2 - j + 1\Big)\Big(5N/2 - (c-1/2)N - j + 1\Big)}{2j\Big((c-1/2)N + j\Big)}.$$

To simplify this, put $a = N/2$ and $b = (c - 1/2)N$. Moreover, since we are only looking for an asymptotic bound, and are willing to incur small multiplicative errors within the $O^*(\cdot)$ notation, we may ignore the two $+1$ terms in the numerator when N is sufficiently large; we omit the routine algebraic justification of this statement. The above quotient then becomes (approximately) $\dfrac{(a-j)(5a-b-j)}{2j(b+j)}$, which is larger than 1 whenever

$$\begin{aligned} j &< \tfrac{1}{2}(\sqrt{56a^2 + 8ab + b^2} - 6a - b) = \\ &= \tfrac{N}{2}\left(\sqrt{(7/2)^2 + 3c + c^2} - 5/2 - c\right) = tN, \end{aligned}$$

with t given in (7). A simple calculation shows that $0 \le t < 1/2$ and $0 \le c - 1/2 + t \le 5/2$ for $0 \le c \le 3$. In other words (and rather unsurprisingly), the index $j = tN$ attaining the maximum does indeed lie in the range where the two binomial coefficients in the corresponding terms in (9) are both well defined (non-zero).

Now that we have the largest term of the sum in (9), we obtain

$$\mathsf{pg_c}(S) = \mathsf{tr}(S) \cdot O^* \left(\binom{5N/2}{(c-1/2)N + tN} \cdot \binom{N/2}{tN} \cdot \frac{1}{2^{tN}} \right).$$

Using Stirling's approximation, we have

$$\begin{aligned} \mathsf{pg_c}(S) &= \mathsf{tr}(S) \cdot O^* \left(\left(\frac{(5/2)^{5/2}}{(c+t-1/2)^{c+t-1/2}(3-c-t)^{3-c-t}} \cdot \frac{(1/2)^{1/2}}{t^t(1/2-t)^{1/2-t}} \cdot \frac{1}{2^t} \right)^N \right) = \\ &= \mathsf{tr}(S) \cdot O^* \left(\left(\frac{5^{5/2}}{8(c+t-1/2)^{c+t-1/2}(3-c-t)^{3-c-t}(2t)^t(1/2-t)^{1/2-t}} \right)^N \right), \end{aligned}$$

as asserted. ◻

It is not complicated to modify the proof for larger values of c, since the restriction $c \le 5/4$ is required only for assuring that the maximum term of (8) is the last term for each value of j. When $c \ge 7/4$, the maximum term is obtained, for all values of j, when $i = 5N/4$, so the sum in (8) becomes $O^*\left(\binom{5N/2}{5N/4}\right) = O^*\left(2^{5N/2}\right)$. Since this is the same bound used in the analysis of general plane graphs in Subsection 3.1, we obtain the same upper bound on the desired ratio as the one for general plane graphs (see Theorem 3.1).

When $5/4 < c < 7/4$, the maximum term depends on the value of j. If $cN + j \le 7N/4$, the maximum term is the last one. Otherwise, the maximum term is obtained when $i = 7N/4$. In this case, finding the maximum term of (9) becomes considerably more complicated, and we omit the analysis of this case.

By applying Theorem 3.6 with $c = 1$, we get the bound $\mathsf{f}(S) = O\left(5.4830^N \cdot \mathsf{tr}(S)\right)$. Surprisingly, the ratio in this bound is fairly close to the ratio $O\left(5.4723^N\right)$ that we derived in Subsection 3.2.

Plane graphs with at least cN edges. For a set S of N points in the plane and a constant $0 < c \le 3$, we denote by $\bar{\mathcal{P}}_c(S)$ the set of all plane graphs of S with at least cN edges, and put $\bar{\mathsf{pg}}_\mathsf{c}(S) := \left|\bar{\mathcal{P}}_c(S)\right|$.

Similarly to the case of graphs with at most cN edges, we can obtain a trivial bound by bounding the number of graphs in $\bar{\mathcal{P}}_c(S)$ that are contained in a single triangulation. For $c > 1.5$, this leads to the same bound as before (albeit for a different range of c):

$$\bar{\mathsf{pg}}_\mathsf{c}(S) = O^*\left(\mathsf{tr}(S) \cdot \binom{3N}{(3-c)N}\right) = O^*\left(\mathsf{tr}(S) \cdot \left(\frac{27}{c^c \cdot (3-c)^{3-c}}\right)^N\right).$$

By slightly modifying the proof of Theorem 3.6, we obtain

Theorem 3.7 *For any set S of N points in the plane and $7/4 \le c < 3$,*

$$\bar{\mathsf{pg}}_\mathsf{c}(S) = O^*\left(\left(\frac{5^{5/2}}{8(c+t-1/2)^{c+t-1/2}(3-c-t)^{3-c-t}(2t)^t(1/2-t)^{1/2-t}}\right)^N \cdot \mathsf{tr}(S)\right),$$

where

$$t = \frac{1}{2}\left(\sqrt{(7/2)^2 + 3c + c^2} - 5/2 - c\right).$$

Proof. We start the proof in the same manner as the proof of Theorem 3.6, and define F, $F_{\bar{G}}$, and E accordingly. To estimate the number of subgraphs G of T in $\bar{\mathcal{P}}_c(S)$ for which $|F_{\bar{G}}| = j$, we consider the number of ways to choose j edges from F and at least $cN - (N/2 - 2 - j) = (c - 1/2)N + j + 2$ edges from E. Ignoring the term 2, as above, the number of ways to pick the edges from E is

$$\sum_{i=0}^{(3-c)N-j} \binom{5N/2}{i} = O(N) \cdot \max_i \binom{5N/2}{i}. \tag{10}$$

(The sum actually gives the number of choices of the edges of E that will not be in the graph.) It is easily checked that, for $c \geq 7/4$, the maximum term is always the last one, for any value of $0 \leq j \leq N/2 - 2$ (that is, the largest possible value of i is at most $5N/4$). Therefore, the number of subgraphs of T for which $|F_{\bar{G}}| = j$ is

$$O^*\left(\binom{5N/2}{(3-c)N - j} \cdot \binom{N/2}{j}\right) = O^*\left(\binom{5N/2}{(c-1/2)N + j} \cdot \binom{N/2}{j}\right).$$

This expression is identical to the one in Theorem 3.6, and thus the rest of the proof proceeds essentially unchanged. ◻

As above, when $c \leq 5/4$, the bound on the number of graphs with at least cN edges is asymptotically (in the $O^*(\cdot)$ sense) the same as the bound on the overall number of plane graphs on S. For $5/4 < c < 7/4$, the bound is more complicated and, as above, we do not present it in this paper.

It is interesting to note that the expressions in the bounds in Theorems 3.6 and 3.7 are identical albeit for different (and disjoint) ranges of c.

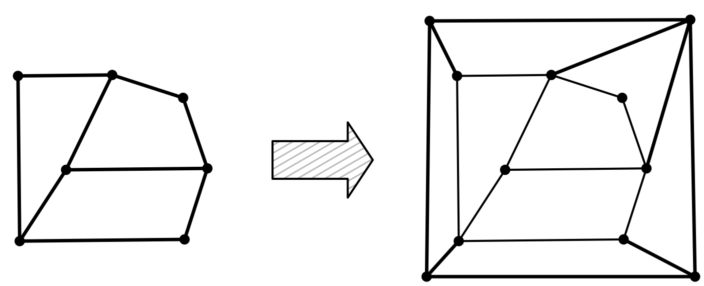


Figure 8: A quadrangulation of S and a quadrangulation of S' that contains it.

As an application, consider the problem of bounding the number of quadrangulations of S, namely straight-edge crossing-free connected graphs on the vertex set S with no isolated vertices, that include all the hull edges of $conv(S)$, and where every bounded face is a quadrilateral. We will assume that h is even, and create a superset $S' \supset S$ as follows. We take a quadrilateral Q that contains S in its interior, and add the vertices of Q to S. It is easy to see that every quadrangulation of S is contained in at least one quadrangulation of S'; see Figure 8 for a "proof by picture". Therefore, it suffices to bound the number of quadrangulations of S'. (The case where h is odd can be handled similarly, by enclosing S in a sufficiently large *triangle* and proceeding as above.)

Using Euler's formula, we notice that a quadrangulation of S' has $N + 1$ quadrilaterals, and $2N + 5$ edges (since $|S'| = N + 4$). Therefore, we can use Theorem 3.7 with $c = 2$, which implies a bound of $O^*(6.1406^N) \cdot \mathsf{tr}(N) = O(184.219^N)$. (There are actually $N + 4$ points and $c = (2N + 5)/(N + 4)$. However, since we are only interested in the exponential part of the bound, the above bound, with the $O^*(\cdot)$ notation, does hold.)

4 Conclusion

In this paper we have introduced the notion of pseudo simultaneously-flippable edges in triangulations, and have shown how to use it to obtain several refined bounds on the number of straight-edge crossing-free graphs on a fixed (labeled) set of N points in the plane. The paper raises several open problems and directions for future research.

First, can one further extend the notion of ps-flippability? For example, one could consider, within a fixed triangulation G, the set of diagonals of a collection of pairwise interior-disjoint simple polygons, not necessarily convex. The number of such diagonals is likely to be larger than the size of the maximal set of ps-flippable edges, but it not clear how large is the number of triangulations that can be obtained by redrawing diagonals.

Notice that we have not considered in this paper crossing-free straight-edge perfect matchings and spanning (Hamiltonian) cycles. These classes of graphs can be studied within the framework introduced by Kasteleyn (see [3, 17]), a direction that we are currently studying.

Finally, counting straight-edge crossing-free graphs *within* triangulations (even by taking their support into account) might not necessarily be the best method for obtaining such bounds. For example, a much better bound on the number of crossing-free straight-edge perfect matchings can be obtained using a direct approach, which does not consider matchings within a triangulation (compare the bound in [25] with the one obtained in [3]). It would be interesting to come up with similar "direct" schemes for other classes of graphs, and this is another direction that we are currently pursuing.

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