Euclidean structure in finite dimensional normed spaces

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1 Introduction

In this article we discuss results which stand between geometry, convex geometry, and functional analysis. We consider the family of n-dimensional normed spaces and study the asymptotic behavior of their parameters as the dimension n grows to infinity. Analogously, we study asymptotic phenomena for convex bodies in high dimensional spaces.

This theory grew out of functional analysis. In fact, it may be viewed as the most recent one among many examples of directions in mathematics which were born inside this field during the twentieth century. Functional analysis was developed during the period between the World Wars by the Polish school of mathematics, an outstanding school with broad interests and connections. The influence of the ideas of functional analysis on mathematical physics, on differential equations, but also on classical analysis, was enormous. The great achievements and successful applications to other fields led to the creation of new directions (among them, algebraic analysis, non-commutative geometry and the modern theory of partial differential equations) which in a short time became autonomous and independent fields of mathematics.

Thus, in the last decades of the twentieth century, geometric functional analysis and even more narrowly the study of the geometry of Banach spaces became the main line of research in what remained as "proper" functional analysis. The two central themes of this theory were infinite dimensional convex bodies and the linear structure of infinite dimensional normed spaces. Several questions in the direction of a structure theory for Banach spaces were asked and stayed open for many years. Some of them can be found in Banach's book. Their common feature was a search for simple building blocks inside an arbitrary Banach space. For example: does every Banach space contain an infinite unconditional basic sequence? Is every Banach space decomposable as a topological sum of two infinite dimensional subspaces? Is it true that every Banach space is isomorphic to its closed hyperplanes? Does every Banach space contain a subspace isomorphic to some ℓ_p or c_0?

This last question was answered in the negative by Tsirelson (1974) who gave an example of a reflexive space not containing any ℓ_p. Before Tsirelson's example, it had been realized by the second named author that the notion of the spectrum of a uniformly continuous function on the unit sphere of a normed space was related to this question and that the problem of distortion was a central geometric question for approaching the linear structure of the space. Although Tsirelson's example was a major breakthrough and introduced a completely new construction of norm, the search for simple linear structure continued to be the aim of most of the efforts in the geometry of Banach spaces. We now know that infinite dimensional Banach spaces have much more complicated structure than what was assumed (or hoped). All the questions above were answered in the negative in the middle of the 90's, starting with the works of Gowers and Maurey, Gowers, Odell and Schlumprecht. Actually, the line of thought related to Tsirelson's example and the concepts of spectrum and distortion were the most crucial for the recent developments.

The systematic quantitative study of n-dimensional spaces with n tending to infinity started in the 60's, as an alternative approach to several unsolved problems of geometric functional analysis. This study led to a new and deep theory with many surprising consequences in both analysis and geometry. When viewed as part of functional analysis, this theory is often called local theory (or asymptotic theory of finite dimensional normed spaces). However, it adopted a significant part of classical convexity theory and used many of its methods and techniques. Classical geometric inequalities such as the Brunn-Minkowski inequality, isoperimetric inequalities and many others were extensively used and established themselves as important technical tools in the development of local theory. Conversely, the study of geometric problems from a functional analysis point of view enriched classical convexity with a new approach and a variety of problems: The "isometric" problems which were typical in convex geometry were replaced by "isomorphic" ones, with the emphasis on the role of the dimension. This change led to a new intuition and revealed new concepts, the concentration phenomenon being one of them, with many applications in convexity and discrete mathematics. This natural melting of the two theories should perhaps correctly be called asymptotic (or convex) geometric analysis.

This paper presents only some aspects of this asymptotic theory. We leave aside type-cotype theory and other connections with probability theory, factorization results, covering and entropy (besides a few results we are going to use), connections with infinite dimension theory, random normed spaces, and so on. Other articles in this collection will cover these topics and complement these omissions. On the other hand, we feel it is necessary to give some background on convex geometry: This is done in Sections 2 and 3.

The theory as we build it below "rotates" around different Euclidean structures associated with a given norm or convex body. This is in fact a reflection of different traces of hidden symmetries every high dimensional body possesses. To recover these symmetries is one of the goals of the theory. A new point which appears in this article is that all these Euclidean structures that are in use in local theory have precise geometric descriptions in terms of classical convexity theory: they may be viewed as "isotropic" ones.

Traditional local theory concentrates its attention on the study of the structure of the subspaces and quotient spaces of the original space (the "local structure" of the space). The connection with classical convexity goes through the translation of these results to a "global" language, that is, to equivalent statements pertaining to the structure of the whole body or space. Such a comparison of "local" and "global" results is very useful, opens a new dimension in our study and will lead our presentation throughout the paper.

We refer the reader to the books of Schneider [177] and of Burago and Zalgaller [35] for the classical convexity theory. Books mainly devoted to the local theory are the ones by: Milman and Schechtman [149], Pisier [162], Tomczak-Jaegermann [195].

2 Classical inequalities and isotropic positions

2.1 Notation

2.1.1. We study finite-dimensional real normed spaces $X = (\mathbb{R}^n, \|\cdot\|)$. The unit ball K_X of such a space is an origin-symmetric convex body in $\mathbb{R}^n$ which we agree to call a *body*. There is a one to one correspondence between norms and bodies in $\mathbb{R}^n$: If K is a body, then $\|x\| = \min\{\lambda > 0 : x \in \lambda K\}$ is a norm defining a space X_K with K as its unit ball. In this way bodies arise naturally in functional analysis and will be our main object of study.

If K and T are bodies in $\mathbb{R}^n$ we can define a multiplicative distance $d(K,T)$ by

$$d(K,T) = \inf\{ab : a, b > 0, K \subseteq bT, T \subseteq aK\}.$$

The natural distance between the n-dimensional spaces X_K and X_T is the *Banach-Mazur distance*. Since we want to identify isometric spaces, we allow a linear transformation and set

$$d(X_K, X_T) = \inf\{d(K, uT) : u \in GL_n\}.$$

In other words, $d(X_K, X_T)$ is the smallest positive number d for which we can find $u \in GL_n$ such that $K \subseteq uT \subseteq dK$. We clearly have $d(X_K, X_T) \geq 1$ with equality if and only if X_K and X_T are isometric. Note the multiplicative triangle inequality $d(X,Z) \leq d(X,Y)d(Y,Z)$ which holds true for every triple of n-dimensional spaces.

2.1.2. We assume that $\mathbb{R}^n$ is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$ and denote the corresponding Euclidean norm by $|\cdot|$. D_n is the Euclidean unit ball and S^{n-1} is the unit sphere. We also write $|\cdot|$ for the volume (Lebesgue measure) in $\mathbb{R}^n$, and μ for the Haar probability measure on the orthogonal group $O(n)$.

Let $G_{n,k}$ denote the Grassmannian of all k-dimensional subspaces of $\mathbb{R}^n$. Then, $O(n)$ equips $G_{n,k}$ with a Haar probability measure $\nu_{n,k}$ satisfying

$$\nu_{n,k}(A) = \mu\{u \in O(n) : uE_k \in A\}$$

for every Borel subset A of $G_{n,k}$ and every fixed element E_k of $G_{n,k}$. The rotationally invariant probability measure on S^{n-1} will be denoted by σ.

2.1.3. Duality plays an important role in the theory. If K is a body in $\mathbb{R}^n$, its *polar body* is defined by

$$K^\circ = \{y \in \mathbb{R}^n : |\langle x, y\rangle| \leq 1 \text{ for all } x \in K\}.$$

That is, $\|y\|_{K^\circ} = \max_{x\in K} |\langle x, y\rangle|$. Note that $X_{K^\circ} = X_K^*$: K° is the unit ball of the dual space of X. It is easy to check that $d(X, Y) = d(X^*, Y^*)$.

2.2 Classical Inequalities

(a) **The Brunn-Minkowski inequality.** Let K and T be two convex bodies in $\mathbb{R}^n$. If $K + T$ denotes the *Minkowski sum* $\{x + y : x \in K, y \in T\}$ of K and T, the Brunn-Minkowski inequality states that

$$|K + T|^{1/n} \geq |K|^{1/n} + |T|^{1/n}, \tag{1}$$

with equality if and only if K and T are homothetical. Actually, the same inequality holds for arbitrary non empty compact subsets of $\mathbb{R}^n$.

One can rewrite (1) in the following form: For every $\lambda \in (0, 1)$,

$$|\lambda K + (1 - \lambda)T|^{1/n} \geq \lambda|K|^{1/n} + (1 - \lambda)|T|^{1/n}. \tag{2}$$

Then, the arithmetic-geometric means inequality gives a dimension free version:

$$|\lambda K + (1 - \lambda)T| \geq |K|^\lambda |T|^{1-\lambda}. \tag{3}$$

There are several proofs of the Brunn-Minkowski inequality, all of them related to important ideas. We shall sketch only two lines of proof.

The first (historically as well) proof is based on the Brunn concavity principle:

Let K be a convex body in $\mathbb{R}^n$ and F be a k-dimensional subspace. Then, the function $f : F^\perp \to \mathbb{R}$ defined by $f(x) = |K \cap (F + x)|^{1/k}$ is concave on its support.

The proof is by symmetrization. Recall that the *Steiner symmetrization* of K in the direction of $\theta \in S^{n-1}$ is the convex body $S_\theta(K)$ consisting of all points of the form $x + \lambda\theta$, where x is in the projection $P_\theta(K)$ of K onto $\theta^\perp$ and $|\lambda| \leq \frac{1}{2} \times \text{length}(x + \mathbb{R}\theta) \cap K$. Steiner symmetrization preserves convexity: in fact, this is the Brunn concavity principle for $k = 1$. The proof is elementary and essentially two dimensional. A well known fact which goes back to Steiner and Schwarz but was later rigorously proved in [45] (see [35]) is that for every convex body K one can find a sequence of successive Steiner symmetrizations in directions $\theta \in F$ so that the resulting convex body $\tilde{K}$ has the following property: $\tilde{K} \cap (F + x)$ is a ball with radius $r(x)$, and $|\tilde{K} \cap (F + x)| = |K \cap (F + x)|$ for every $x \in F^\perp$. Convexity of $\tilde{K}$ implies that r is concave on its support, and this shows that f is also concave. □

The Brunn concavity principle implies the Brunn-Minkowski inequality. If K, T are convex bodies in $\mathbb{R}^n$, we define $K_1 = K \times \{0\}$, $T_1 = T \times \{1\}$ in $\mathbb{R}^{n+1}$ and consider their convex hull L. If $L(t) = \{x \in \mathbb{R}^n : (x, t) \in L\}$, $t \in \mathbb{R}$, we easily check that

$L(0)=K, L(1)=T$, and $L(1/2)=\frac{K+T}{2}$. Then, the Brunn concavity principle for $F=\mathbb{R}^n$ shows that

$$\Big|\frac{K+T}{2}\Big|^{1/n} \geq \frac{1}{2}|K|^{1/n}+\frac{1}{2}|T|^{1/n}. \quad \square \tag{4}$$

A second proof of the Brunn-Minkowski inequality may be given via the *Knöthe map*: Assume that K and T are open convex bodies. Then, there exists a one to one and onto map $\phi : K \to T$ with the following properties:

(i) ϕ is triangular: the i-th coordinate function of ϕ depends only on $x_1,\dots,x_i$. That is,

$$\phi(x_1,\dots,x_n)=(\phi_1(x_1),\phi_2(x_1,x_2),\dots,\phi_n(x_1,\dots,x_n)). \tag{5}$$

(ii) The partial derivatives $\frac{\partial\phi_i}{\partial x_i}$ are nonnegative on K, and the determinant of the Jacobian of ϕ is constant. More precisely, for every $x\in K$

$$(\det J_\phi)(x)=\prod_{i=1}^n \frac{\partial\phi_i}{\partial x_i}(x)=\frac{|T|}{|K|}. \tag{6}$$

The map ϕ is called the Knöthe map from K onto T. Its existence was established in [102] (see also [149, Appendix I]). Observe that each choice of coordinate system in $\mathbb{R}^n$ produces a different Knöthe map from K onto T.

It is clear that $(I+\phi)(K)\subseteq K+T$, therefore we can estimate $|K+T|$ using the arithmetic-geometric means inequality as follows:

$$\begin{aligned}|K+T| &\geq \int_{(I+\phi)(K)} dx = \int_K |\det J_{I+\phi}(x)|dx = \int_K \prod_{i=1}^n \left(1+\frac{\partial\phi_i}{\partial x_i}\right)dx \\ &\geq \int_K (1+\det J_\phi^{1/n})^n dx = |K|\left(1+\frac{|T|^{1/n}}{|K|^{1/n}}\right)^n = \left(|K|^{1/n}+|T|^{1/n}\right)^n.\end{aligned} \tag{7}$$

This proves the Brunn-Minkowski inequality. $\square$

Alternatively, instead of the Knöthe map one may use the *Brenier map* $\psi : K\to T$, where K and T are open convex bodies. This is also a one to one, onto and "ratio of volumes" preserving map (i.e. its Jacobian has constant determinant), with the following property: There is a convex function $f\in C^2(K)$ defined on K such that $\psi=\nabla f$. A remarkable property of the Brenier map is that it is uniquely determined. Existence and uniqueness of the Brenier map were proved in [26] (see also [125] for a different proof and important generalizations).

It is clear that the Jacobian $J_\psi = \mathrm{Hess} f$ is a symmetric positive definite matrix. Again we have $(I+\psi)(K)\subseteq K+T$, hence

$$|K+T| \geq \int_K |\det J_{I+\psi}(x)|dx = \int_K \det\left(I+\mathrm{Hess}f\right)dx = \int_K \prod_{i=1}^n (1+\lambda_i(x))dx, \tag{8}$$

where $\lambda_i(x)$ are the non negative eigenvalues of Hess$f$. Moreover, by the ratio of volumes preserving property of ψ, we have $\prod_{i=1}^n \lambda_i(x) = |T|/|K|$ for every $x \in K$. Therefore, the arithmetic-geometric means inequality gives

$$|K+T| \geq \int_K \left(1 + [\prod_{i=1}^n \lambda_i(x)]^{1/n}\right)^n dx = \left(|K|^{1/n} + |T|^{1/n}\right)^n. \quad \square \tag{9}$$

This proof has the advantage of providing a description for the equality cases: either K or T must be a point, or K must be homothetical to T.

Let us describe here the generalization of Brenier's work due to McCann: Let μ, ν be probability measures on $\mathbb{R}^n$ such that μ is absolutely continuous with respect to Lebesgue measure. Then, there exists a convex function f such that $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is defined μ-almost everywhere, and $\nu(A) = \mu((\nabla f)^{-1}(A)))$ for every Borel subset A of $\mathbb{R}^n$ (∇f pushes forward μ to ν). If both μ, ν are absolutely continuous with respect to Lebesgue measure, then the Brenier map ∇f has an inverse $(\nabla f)^{-1}$ which is defined ν-almost everywhere and is also a Brenier map, pushing forward ν to μ. A regularity result of Caffarelli [44] (see [11]) states that if T is a convex bounded open set, f is a probability density on $\mathbb{R}^n$, and g is a probability density on T such that

(i) f is locally bounded and bounded away from zero on compact sets, and

(ii) there exist $c_1, c_2 > 0$ such that $c_1 \leq g(y) \leq c_2$ for every $y \in T$,

then, the Brenier map $\nabla f : (\mathbb{R}^n, f dx) \to (\mathbb{R}^n, g dx)$ is continuous and belongs locally to the Hölder class C^α for some $\alpha > 0$. The following recent result [11] makes use of all this information:

Fact 1: *Let K_1 and K_2 be open convex bounded subsets of $\mathbb{R}^n$ with $|K_1| = |K_2| = 1$. There exists a C^1-diffeomorphism $\psi : K_1 \to K_2$ which is volume preserving and satisfies*

$$K_1 + \lambda K_2 = \{x + \lambda\psi(x) : x \in K_1\} \quad , \quad \lambda > 0. \tag{10}$$

Proof: Let ρ be any smooth strictly positive density on $\mathbb{R}^n$. Consider the Brenier maps

$$\psi_i = \nabla f_i : (\mathbb{R}^n, \rho dx) \to (K_i, dx) \quad , \quad i = 1, 2. \tag{11}$$

Caffarelli's result shows that they are C^1-smooth. We now apply the following theorem of Gromov [72] (for a proof, see also [11]):

Fact 2: (i) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-smooth convex function with strictly positive Hessian. Then, the image of the gradient map* Im∇f *is an open convex set.*

(ii) *If f_1, f_2 are two such functions, then*

$$\mathrm{Im}(\nabla f_1 + \nabla f_2) = \mathrm{Im}(\nabla f_1) + \mathrm{Im}(\nabla f_2). \quad \square$$

It follows that, for every $\lambda > 0$,

$$K_1 + \lambda K_2 = \{\nabla f_1(x) + \lambda \nabla f_2(x) : x \in \mathbb{R}^n\}. \tag{12}$$

Then, one can check that the map $\psi = \psi_2 \circ (\psi_1)^{-1} : K_1 \to K_2$ satisfies all the conditions of Fact 1. $\square$

The existence of a volume preserving $\psi : K_1 \to K_2$ such that $(I + \psi)(K_1) = K_1 + K_2$ covers a "weak point" of the Knöthe map and should have important applications to convex geometry. We discuss some of them in Section 3.2.

(b) **Consequences of the Brunn-Minkowski inequality**

(b_1) *The isoperimetric inequality for convex bodies.* The surface area $\partial(K)$ of a convex body K is defined by

$$\partial(K) = \lim_{\varepsilon\to 0^+} \frac{|K + \varepsilon D_n| - |K|}{\varepsilon}. \tag{13}$$

It is a well-known fact that among all convex bodies of a given volume the ball has minimal surface area. This is an immediate consequence of the Brunn-Minkowski inequality: If K is a convex body in $\mathbb{R}^n$ with $|K| = |rD_n|$, then for every $\varepsilon > 0$

$$|K + \varepsilon D_n|^{1/n} \geq |K|^{1/n} + \varepsilon |D_n|^{1/n} = (r + \varepsilon)|D_n|^{1/n}. \tag{14}$$

It follows that the surface area $\partial(K)$ of K satisfies

$$\partial(K) = \lim_{\varepsilon\to 0^+} \frac{|K + \varepsilon D_n| - |K|}{\varepsilon} \geq \lim_{\varepsilon\to 0^+} \frac{(r+\varepsilon)^n - r^n}{\varepsilon}|D_n| = n|D_n|^{\frac{1}{n}}|K|^{\frac{n-1}{n}} \tag{15}$$

with equality if $K = rD_n$. The question of uniqueness in the equality case is more delicate.

(b_2) *The spherical isoperimetric inequality.* Consider the unit sphere S^{n-1} with the geodesic distance ρ and the rotationally invariant probability measure σ. For every Borel subset A of S^{n-1} and for every $\varepsilon > 0$, we define the ε-extension of A:

$$A_\varepsilon = \{x \in S^{n-1} : \rho(x, A) \leq \varepsilon\}. \tag{16}$$

Then, the isoperimetric inequality for the sphere is the following statement:

Among all Borel subsets A of S^{n-1} with given measure $\alpha \in (0,1)$, a spherical cap $B(x,r)$ of radius $r > 0$ such that $\sigma(B(x,r)) = \alpha$ has minimal ε-extension for every $\varepsilon > 0$.

This means that if $A \subseteq S^{n-1}$ and $\sigma(A) = \sigma(B(x_0, r))$ for some $x_0 \in S^{n-1}$ and $r > 0$, then

$$\sigma(A_\varepsilon) \geq \sigma(B(x_0, r + \varepsilon)) \tag{17}$$

for every $\varepsilon > 0$. Since the σ-measure of a cap is easily computable, one can give a lower bound for the measure of the ε-extension of any subset of the sphere. We are mainly interested in the case $\sigma(A) = \frac{1}{2}$, and a straightforward computation (see [61]) shows the following:

Theorem 2.2.1. *If A is a Borel subset of S^{n+1} and $\sigma(A) = 1/2$, then*

$$\sigma(A_\varepsilon) \geq 1 - \sqrt{\pi/8}\exp(-\varepsilon^2 n/2) \tag{18}$$

for every $\varepsilon > 0$. $\square$

[The constant $\sqrt{\pi/8}$ may be replaced by a sequence of constants a_n tending to $\frac{1}{2}$ as $n \to \infty$.]

The spherical isoperimetric inequality can be proved by spherical symmetrization techniques (see [176] or [61]). However, it was recently observed [10] that one can give a very simple proof of an estimate analogous to (18) using the Brunn-Minkowski inequality. The key point is the following lemma:

Lemma. *Consider the probability measure $\mu(A) = |A|/|D_n|$ on the Euclidean unit ball D_n. If A, B are subsets of D_n with $\mu(A) \geq \alpha$, $\mu(B) \geq \alpha$, and if $\rho(A, B) = \inf\{|a - b| : a \in A, b \in B\} = \rho > 0$, then*

$$\alpha \leq \exp(-\rho^2 n/8).$$

[In other words, if two disjoint subsets of D_n have positive distance ρ, then at least one of them must have small volume (depending on ρ) when the dimension n is high.]

Proof: We may assume that A and B are closed. By the Brunn-Minkowski inequality, $\mu(\frac{A+B}{2}) \geq \alpha$. On the other hand, the parallelogram law shows that if $a \in A, b \in B$ then

$$|a + b|^2 = 2|a|^2 + 2|b|^2 - |a - b|^2 \leq 4 - \rho^2.$$

It follows that $\frac{A+B}{2} \subseteq (1 - \frac{\rho^2}{4})^{1/2} D_n$, hence

$$\mu\left(\frac{A+B}{2}\right) \leq \left(1 - \frac{\rho^2}{4}\right)^{n/2} \leq \exp(-\rho^2 n/8). \quad \square$$

Proof of Theorem 2.2.1 (with weaker constants). Assume that $A \subseteq S^{n-1}$ with $\sigma(A) = 1/2$. Let $\varepsilon > 0$ and define $B = (A_\varepsilon)^c \subseteq S^{n-1}$. We fix $\lambda \in (0,1)$ and consider the subsets $\tilde{A} = \bigcup\{tA : \lambda \leq t \leq 1\}$ and $\tilde{B} = \bigcup\{tB : \lambda \leq t \leq 1\}$ of D_n. These are disjoint with distance $\simeq \lambda\varepsilon$. The Lemma shows that $\mu(\tilde{B}) \leq \exp(-c\lambda^2\varepsilon^2 n/8)$, and since $\mu(\tilde{B}) = (1 - \lambda^n)\sigma(B)$ we obtain

$$\sigma(A_\varepsilon) \geq 1 - \frac{1}{1 - \lambda^n}\exp(-c\lambda^2\varepsilon^2 n/8). \tag{19}$$

We conclude the proof by choosing suitable $\lambda \in (0,1)$. $\square$

(b_3) *C. Borell's Lemma and Khinchine type inequalities.* Let μ be a Borel probability measure on $\mathbb{R}^n$. We say that μ is *log-concave* if whenever A, B are Borel subsets of $\mathbb{R}^n$ and $\lambda \in (0,1)$ we have

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}. \tag{20}$$

The following lemma of C. Borell [21] holds for all log-concave probability measures:

Lemma. *Let μ be a log-concave Borel probability measure on $\mathbb{R}^n$, and A be a symmetric convex subset of $\mathbb{R}^n$. If $\mu(A) = \theta > 1/2$, then for every $t \geq 1$ we have*

$$\mu\left((tA)^c\right) \leq \theta \left(\frac{1-\theta}{\theta}\right)^{\frac{t+1}{2}}. \tag{21}$$

Proof: Immediate by the log-concavity of μ, after one observes that

$$\mathbb{R}^n \setminus A \supseteq \frac{2}{t+1}(\mathbb{R}^n \setminus tA) + \frac{t-1}{t+1}A. \quad \square \tag{22}$$

Let K be a convex body in $\mathbb{R}^n$. By the Brunn-Minkowski inequality we see that the measure μ_K defined by $\mu_K(A) = |A \cap K|/|K|$ is a log-concave probability measure. In this context, Borell's lemma tells us that if A is convex symmetric and if $A \cap K$ contains more than half of the volume of K, then the proportion of K which stays outside tA decreases exponentially in t as $t \to +\infty$ in a rate independent of the convex body K and the dimension n.

This observation has important applications to the study of linear functions $f(x) = \langle x, y\rangle$, $y \in \mathbb{R}^n$, defined on convex bodies. Let us denote by $\|f\|_p$ the L_p norm with respect to the probability measure μ_K. Then, for every linear function $f : K \to \mathbb{R}$ we have

$$\|f\|_q \leq \|f\|_p \leq c_p \|f\|_q \quad , \quad 0 < q < p \tag{23}$$

where c_p are universal constants depending only on p. The left hand side inequality is just Hölder's inequality, while the right hand side (in the case $1 \leq q < p$) is a consequence of Borell's lemma (see [83]). One writes

$$\frac{1}{|K|}\int_K |f(x)|^p dx = \int_0^{+\infty} pt^{p-1}\mu_K\left(\{x \in K : |f(x)| \geq t\}\right) dt \tag{24}$$

and estimates $\mu_K(\{x \in K : |f(x)| \geq t\})$ for large values of t using Borell's lemma with say $A = \{x \in \mathbb{R}^n : |f(x)| \leq 3\|f\|_q\}$. The dependence of c_p on p is linear as $p \to \infty$. This is equivalent to the fact that the $L^{\psi_1}(K)$ norm of f

$$\|f\|_{L^{\psi_1}(K)} = \inf\Big\{\lambda > 0 : \frac{1}{|K|}\int_K \exp(|f(x)|/\lambda) \leq 2\Big\} \tag{25}$$

is equivalent to $\|f\|_1$. The question to determine the cases where $c(p) \simeq \sqrt{p}$ as $p \to \infty$ in (23) is very important for the theory. This is certainly true for some bodies (e.g. the cube), but the example of the cross-polytope shows that it is not always so.

Inverse Hölder inequalities of this type are very similar in nature to the classical Khinchine inequality (and are sometimes called *Khinchine type inequalities*). In fact, the argument described above may be used to give proofs of the Kahane-Khinchine inequality (see [149, Appendix III]).

Khinchine type inequalities do not hold only for linear functions. For example, Bourgain [24] has shown that if $f : K \to \mathbb{R}$ is a polynomial of degree m, then

$$\|f\|_p \leq c(p,m)\|f\|_2 \tag{26}$$

for every $p > 2$, where $c(p,m)$ depends only on p and the degree m of f (For this purpose, the Brunn-Minkowski inequality was not enough, and a suitable direct use of the Knöthe map was necessary). It was also recently proved [107] that (23) holds true for any norm f on $\mathbb{R}^n$. Finally the interval of values of p and q in (23) can be extended to $(-1,+\infty)$ (see [145] for linear functions, [76] for norms).

2.3 Extremal problems and isotropic positions

In the study of finite dimensional normed spaces one often faces the problem of choosing a suitable Euclidean structure related to the question in hand. In geometric language, we are given the body K in $\mathbb{R}^n$ and want to find a specific Euclidean norm in $\mathbb{R}^n$ which is naturally connected with our question about K. An equivalent (and sometimes more convenient) approach is the following: we fix the Euclidean structure in $\mathbb{R}^n$, and given K we ask for a suitable "position" uK of K, where u is a linear isomorphism of $\mathbb{R}^n$. That is, instead of keeping the body fixed and choosing the "right ellipsoid" we fix the Euclidean norm and choose the "right position" of the body.

Most of the times the starting point is a question of the following type: we are given a functional f on convex bodies and a convex body K and we ask for the maximum or minimum of $f(uK)$ over all volume preserving transformations u. We shall describe some very important positions of K which solve such extremal problems. What is interesting is that there is a simple variational method which leads to a description of the solution, and that in most cases the resulting position of K is *isotropic*. Moreover, isotropic conditions are closely related to the Brascamp-Lieb inequality [34] and its reverse [19], a fact that was discovered and used by K. Ball in the case of John's representation of the identity. For more information on this very important connection, see the article [18] in this collection.

(*a*) *John's position.* A classical result of F. John [94] states that $d(X,\ell_2^n) \leq \sqrt{n}$ for every n-dimensional normed space X. This estimate is a by-product of the study of the following extremal problem:

Let K be a body in $\mathbb{R}^n$. Maximize $|\mathrm{det}u|$ over all $u : \ell_2^n \to X_K$ with $\|u\| = 1$.

If u_0 is a solution of this problem, then $u_0 D_n$ is the ellipsoid of maximal volume which is inscribed in K. Existence and uniqueness of such an ellipsoid are easy to check. An equivalent formulation of the problem is the following:

Let K be a body in $\mathbb{R}^n$. Minimize $\|u : \ell_2^n \to X_K\|$ over all volume preserving transformations u.

We assume that the identity map I is a solution of this problem, and normalize so that

$$\|I : \ell_2^n \to X_K\| = 1 = \min\{\|u : \ell_2^n \to X_K\| : |\mathrm{det}u| = 1\}. \tag{1}$$

This means that the Euclidean unit ball D_n is the maximal volume ellipsoid of K. We shall use a simple variational argument [82] to give necessary conditions on K:

Theorem 2.3.1. *Let D_n be the maximal volume ellipsoid of K. Then, for every $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ we can find a contact point x of K and D_n (i.e. $|x| = \|x\| = 1$) such that*

$$\langle x, Tx\rangle \geq \frac{\mathrm{tr}T}{n}. \tag{2}$$

Proof: We may assume that K is smooth enough. Let $S \in L(\mathbb{R}^n, \mathbb{R}^n)$. We first claim that

$$\|Sx\| \geq \frac{\mathrm{tr}S}{n} \tag{3}$$

for some contact point x of K and D_n. Let $\varepsilon > 0$ be small enough. From (1) we have

$$\|I + \varepsilon S\| \geq [\det(I + \varepsilon S)]^{1/n} = 1 + \varepsilon\frac{\mathrm{tr}S}{n} + O(\varepsilon^2). \tag{4}$$

Let $x_\varepsilon \in S^{n-1}$ be such that $\|x_\varepsilon + \varepsilon Sx_\varepsilon\| = \|I + \varepsilon S\|$. Since $D_n \subseteq K$, we have $\|x_\varepsilon\| \leq 1$. Then, the triangle inequality for $\|\cdot\|$ shows that

$$\|Sx_\varepsilon\| \geq \frac{\mathrm{tr}S}{n} + O(\varepsilon). \tag{5}$$

We can find $x \in S^{n-1}$ and a sequence $\varepsilon_m \to 0$ such that $x_{\varepsilon_m} \to x$. By (5) we obviously have $\|Sx\| \geq \frac{\mathrm{tr}S}{n}$. Also, $\|x\| = \lim \|x_{\varepsilon_m} + \varepsilon_m Sx_{\varepsilon_m}\| = \|I\| = 1$. This proves (3).

Now, let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and write $S = I + \varepsilon T$, $\varepsilon > 0$. We can find x_ε such that $\|x_\varepsilon\| = |x_\varepsilon| = 1$ and

$$\|x_\varepsilon + \varepsilon Tx_\varepsilon\| \geq \frac{\mathrm{tr}(I + \varepsilon T)}{n} = 1 + \varepsilon\frac{\mathrm{tr}T}{n}. \tag{6}$$

Since $\|x_\varepsilon + \varepsilon Tx_\varepsilon\| = 1 + \varepsilon\langle\nabla\|x_\varepsilon\|, Tx_\varepsilon\rangle + O(\varepsilon^2)$, we obtain $\langle\nabla\|x_\varepsilon\|, Tx_\varepsilon\rangle \geq \frac{\mathrm{tr}T}{n} + O(\varepsilon)$. Choosing again $\varepsilon_m \to 0$ such that $x_{\varepsilon_m} \to x \in S^{n-1}$, we readily see that x is a contact point of K and D_n, and

$$\langle\nabla\|x\|, Tx\rangle \geq \frac{\mathrm{tr}T}{n}. \tag{7}$$

But, $\nabla\|x\|$ is the point on the boundary of K° at which the outer unit normal is parallel to x (see [177, pp. 44]). Since x is a contact point of K and D_n, we must have $\nabla\|x\| = x$. This proves the theorem. $\square$

As a consequence of Theorem 2.3.1 we get John's upper bound for $d(X, \ell_2^n)$:

Theorem 2.3.2. *Let X be an n-dimensional normed space. Then,*

$$d(X, \ell_2^n) \leq \sqrt{n}.$$

Proof: By the definition of the Banach-Mazur distance we may clearly assume that the unit ball K of X satisfies the assumptions of Theorem 2.3.1. In particular, $\|x\| \leq |x|$ for every $x \in \mathbb{R}^n$.

Let $x \in \mathbb{R}^n$ and consider the map $Ty = \langle y, x\rangle x$. Theorem 2.3.1 gives us a contact point z of K and D_n such that

$$\langle z, Tz\rangle \geq \frac{\mathrm{tr}T}{n} = \frac{|x|^2}{n}. \tag{8}$$

On the other hand,

$$\langle z, Tz\rangle = \langle z, x\rangle^2 \leq \|z\|_*^2\|x\|^2 = \|x\|^2, \tag{9}$$

since one can check that $\|z\|_* = 1$. Therefore, $\|x\| \leq |x| \leq \sqrt{n}\|x\|$. This shows that $D_n \subseteq K \subseteq \sqrt{n}D_n$. □

Remark. The estimate given by John's theorem is sharp. If $X = \ell_1^n$ or ℓ_∞^n, one can check that $d(X, \ell_2^n) = \sqrt{n}$.

Theorem 2.3.1 gives very precise information on the distribution of contact points of K and D_n on the sphere S^{n-1}, which can be put in a quantitative form:

Theorem 2.3.3. (Dvoretzky-Rogers Lemma [53]). *Let D_n be the maximal volume ellipsoid of K. Then, there exist pairwise orthogonal vectors $y_1, \ldots, y_n$ in $\mathbb{R}^n$ such that*

$$\left(\frac{n-i+1}{n}\right)^{1/2} \leq \|y_i\| \leq |y_i| = 1 \ , \ i = 1, \ldots, n. \tag{10}$$

Proof: We define the y_i's inductively. The first vector y_1 can be any contact point of K and D_n. Assume that $y_1, \ldots, y_{i-1}$ have been defined. Let $F_i = \mathrm{span}\{y_1, \ldots, y_{i-1}\}$. Then, $\mathrm{tr}(P_{F_i^\perp}) = n-i+1$ and using Theorem 2.3.1 we can find a contact point x_i for which

$$|P_{F_i^\perp}x_i|^2 = \langle x_i, P_{F_i^\perp}x_i\rangle \geq \frac{n-i+1}{n}. \tag{11}$$

We set $y_i = P_{F_i^\perp}x_i / |P_{F_i^\perp}x_i|$. Then,

$$1 = |y_i| \geq \|y_i\| = \|y_i\| \cdot \|x_i\|_* \geq \langle x_i, y_i\rangle = |P_{F_i^\perp}x_i| \geq \left(\frac{n-i+1}{n}\right)^{1/2}. \quad \square \tag{12}$$

Finally, a separation argument and Theorem 2.3.1 give us *John's representation of the identity*:

Theorem 2.3.4. *Let D_n be the maximal volume ellipsoid of K. There exist contact points $x_1, \ldots, x_m$ of K and D_n, and positive real numbers $\lambda_1, \ldots, \lambda_m$ such that*

$$I = \sum_{i=1}^m \lambda_i x_i \otimes x_i.$$

Proof: Consider the convex hull $\mathcal{C}$ of all operators $x \otimes x$, where x is a contact point of K and D_n. We need to prove that $I/n \in \mathcal{C}$. If this is not the case, there exists $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$\langle T, I/n\rangle > \langle x \otimes x, T\rangle \tag{13}$$

for every contact point x. But, $\langle T, I/n\rangle = \frac{\mathrm{tr}T}{n}$ and $\langle x \otimes x, T\rangle = \langle x, Tx\rangle$. This would contradict Theorem 2.3.1. □

Definition. A Borel measure μ on S^{n-1} is called *isotropic* if

$$\int_{S^{n-1}} \langle x, \theta\rangle^2 d\mu(x) = \frac{\mu(S^{n-1})}{n} \tag{14}$$

for every $\theta \in S^{n-1}$.

John's representation of the identity implies that

$$\sum_{i=1}^{m} \lambda_i \langle x_i, \theta\rangle^2 = 1$$

for every $\theta \in S^{n-1}$. This means that if we consider the measure ν on S^{n-1} which gives mass λ_i at the point x_i, $i = 1, \dots, m$, then ν is isotropic. In this sense, John's position is an isotropic position. One can actually prove that the existence of an isotropic measure supported by the contact points of K and D_n characterizes John's position in the following sense (see [16], [82]):

"Assume that D_n is contained in the body K. Then, D_n is the maximal volume ellipsoid of K if and only if there exists an isotropic measure ν supported by the contact points of K and D_n."

Note. The argument given for the proof of Theorem 2.3.1 can be applied in a more general context: If K and L are (not necessarily symmetric) convex bodies in $\mathbb{R}^n$, we say that L *is of maximal volume in* K if $L \subseteq K$ and, for every $w \in \mathbb{R}^n$ and $T \in SL_n$, the affine image $w + T(L)$ of L is not contained in the interior of K. Then, one has a description of this maximal volume position, which generalizes John's representation of the identity:

Theorem 2.3.5. *Let L be of maximal volume in K. For every $z \in \mathrm{int}(L)$, we can find contact points $v_1, \dots, v_m$ of $K - z$ and $L - z$, contact points $u_1, \dots, u_m$ of $(K - z)^\circ$ and $(L - z)^\circ$, and positive reals $\lambda_1, \dots, \lambda_m$, such that $\sum \lambda_j u_j = o$, $\langle u_j, v_j\rangle = 1$, and*

$$I = \sum_{j=1}^{m} \lambda_j u_j \otimes v_j. \quad \square$$

This was observed by Milman in the symmetric case with $z = 0$ (see [195, Theorem 14.5]). For the extension to the non-symmetric case see [88], where it is also shown that under mild conditions on K and L there exists an optimal choice of the "center" z so that, setting $z = 0$, we simultaneously have $\sum \lambda_j u_j = \sum \lambda_j v_j = 0$ in the statement above.

(*b*) *Isotropic position – Hyperplane conjecture.* A notion coming from classical mechanics is that of the *Binet ellipsoid* of a body K (actually, of any compact set with positive Lebesgue measure). The norm of this ellipsoid $E_B(K)$ is defined by

$$\|x\|^2_{E_B(K)} = \frac{1}{|K|}\int_K |\langle x, y\rangle|^2 dy. \tag{15}$$

The *Legendre ellipsoid* $E_L(K)$ of K is defined by

$$\int_{E_L(K)} \langle x, y\rangle^2 dy = \int_K \langle x, y\rangle^2 dy \tag{16}$$

for every $x \in \mathbb{R}^n$, and satisfies (see [146])

$$E_B(K) = (n+2)^{1/2}|E_L(K)|^{-1}(E_L(K))^\circ. \tag{17}$$

That is, $E_L(K)$ has the same moments of inertia as K with respect to the axes. A body K is said to be in *isotropic position* if $|K| = 1$ and its Legendre ellipsoid $E_L(K)$ (equivalently, its Binet ellipsoid $E_B(K)$) is homothetical to D_n. This means that there exists a constant L_K such that

$$\int_K \langle y, \theta\rangle^2 dy = L_K^2 \tag{18}$$

for every $\theta \in S^{n-1}$ (K has the same moment of inertia in every direction θ). It is not hard to see that every body K has a position uK which is isotropic. Moreover, this position is uniquely determined up to an orthogonal transformation. Therefore, L_K is an affine invariant which is called the *isotropic constant* of K.

An alternative way to see this isotropic position in the spirit of our present discussion is to consider the following minimization problem:

Let K be a body in $\mathbb{R}^n$. Minimize $\int_{uK} |x|^2 dx$ over all volume preserving transformations u.

Then, we have the following theorem [146]:

Theorem 2.3.6. *Let K be a body in $\mathbb{R}^n$ with $|K| = 1$. The identity map minimizes $\int_{uK} |x|^2 dx$ over all volume preserving transformations u if and only if K is isotropic. Moreover, this isotropic position is unique up to orthogonal transformations.*

Proof: We shall use the same variational argument as for John's position. Let $T \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $\varepsilon > 0$ be small enough. Then, $u = (I + \varepsilon T)/[\det(I + \varepsilon T)]^{1/n}$ is volume preserving, and since $\int_{uK} |x|^2 dx \geq \int_K |x|^2 dx$ we get

$$\int_K |x + \varepsilon Tx|^2 dx \geq [\det(I + \varepsilon T)]^{\frac{2}{n}} \int_K |x|^2 dx. \tag{19}$$

But, $|x + \varepsilon Tx|^2 = |x|^2 + 2\varepsilon\langle x, Tx\rangle + O(\varepsilon^2)$ and $[\det(I + \varepsilon T)]^{\frac{2}{n}} = 1 + 2\varepsilon\frac{\mathrm{tr}T}{n} + O(\varepsilon^2)$. Therefore, (19) implies

$$\int_K \langle x, Tx\rangle dx \geq \frac{\mathrm{tr}T}{n}\int_K |x|^2 dx. \tag{20}$$

By symmetry we see that

$$\int_K \langle x, Tx\rangle dx = \frac{\mathrm{tr}T}{n}\int_K |x|^2 dx \tag{21}$$

for every $T \in L(\mathbb{R}^n, \mathbb{R}^n)$. This is equivalent to

$$\int_K \langle x, \theta\rangle^2 dx = \frac{1}{n}\int_K |x|^2 dx \quad , \quad \theta \in S^{n-1}. \tag{22}$$

Conversely, if K is isotropic and if T is any volume preserving transformation, then
(23)

$$\int_{TK} |x|^2 dx = \int_K |Tx|^2 dx = \int_K \langle x, T^*Tx\rangle dx = \frac{\mathrm{tr}(T^*T)}{n}\int_K |x|^2 dx \geq \int_K |x|^2 dx,$$

which shows that K solves our minimization problem. We can have equality in (23) if and only if $T \in O(n)$. $\square$

It is easily proved that $L_K \geq L_{D_n} \geq c > 0$ for every body K in $\mathbb{R}^n$, where $c > 0$ is an absolute constant. An important open question having its origin in [22] is the following:

Problem. *Does there exist an absolute constant $C > 0$ such that $L_K \leq C$ for every body K?*

A simple argument based on John's theorem shows that $L_K \leq c\sqrt{n}$ for every body K. Uniform boundedness of L_K is known for some classes of bodies: unit balls of spaces with a 1-unconditional basis, zonoids and their polars, etc. For partial answers to the question, see [13], [47], [48], [95], [96], [106], [146]. The best known general upper estimate is due to Bourgain [24]: $L_K \leq c\sqrt[4]{n}\log n$ for every body K in $\mathbb{R}^n$. In the Appendix we give a brief presentation of Bourgain's result.

The problem we have just stated has many equivalent reformulations, which are deeply connected with problems from classical convexity. For a detailed discussion, see [146]. An interesting property of the isotropic position is that if K is isotropic then all central sections $K\cap\theta^\perp$, $\theta \in S^{n-1}$ are equivalent up to an absolute constant. This comes from the fact that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \simeq \frac{1}{|K\cap\theta^\perp|^2} \quad , \quad \theta \in S^{n-1} \tag{24}$$

a consequence of the log-concavity of μ_K. This was first observed in [91]. Then, uniform boundedness of L_K is equivalent to the statement that an isotropic body has *all* its $(n-1)$-dimensional central sections bounded below by an absolute constant. This is equivalent to the

HYPERPLANE CONJECTURE: Is it true that a body K of volume 1 must have an $(n-1)$-dimensional central section with volume bounded below by an absolute constant?

(*c*) *Minimal surface position.* Let K be a convex body in $\mathbb{R}^n$ with normalized volume $|K| = 1$. We now consider the following minimization problem:

Find the minimum of $\partial(uK)$ over all volume preserving transformations u.

This minimum is attained for some u_0 and will be denoted by ∂_K (the *minimal surface invariant* of K). We say that K has minimal surface if $\partial(K) = \partial_K |K|^{\frac{n-1}{n}}$.

Recall that the area measure σ_K of K is defined on S^{n-1} and corresponds to the usual surface measure on K via the Gauss map: For every Borel $A \subseteq S^{n-1}$, we have

$$\sigma_K(A) = \nu\left(\{x \in \mathrm{bd}(K) : \text{ the outer normal to } K \text{ at } x \text{ is in } A\}\right), \tag{25}$$

where ν is the $(n-1)$-dimensional surface measure on K. We obviously have $\partial(K) = \sigma_K(S^{n-1})$.

A characterization of the minimal surface position through the area measure was given by Petty [157]:

Theorem 2.3.7. *Let K be a convex body in $\mathbb{R}^n$ with $|K| = 1$. Then, $\partial(K) = \partial_K$ if and only if σ_K is isotropic. Moreover, this minimal surface position is unique up to orthogonal transformations.*

The proof makes use of the same variational argument. The basic observation is that if u is any volume preserving transformation, then

$$\partial((u^{-1})^* K) = \int_{S^{n-1}} |ux| \sigma_K(dx). \tag{26}$$

K. Ball [15] has proved that the minimal surface invariant ∂_K is maximal when K is a cube in the symmetric case, and when K is a simplex in the general case. It follows that $\partial_K \leq 2n$ for every body K in $\mathbb{R}^n$. For applications of the minimal surface position to the study of hyperplane projections of convex bodies, see [85] (also, [14] for an approach through the notion of volume ratio).

(*d*) *Minimal mean width position.* Let K be a convex body in $\mathbb{R}^n$. The *mean width* of K is defined by

$$w(K) = 2 \int_{S^{n-1}} h_K(u) \sigma(du), \tag{27}$$

where $h_K(x) = \max_{y \in K} \langle x, y \rangle$ is the *support function* of K. We say that K has *minimal mean width* if $w(TK) \geq w(K)$ for every volume preserving linear transformation T of $\mathbb{R}^n$. Our standard variational argument gives the following characterization of the minimal mean width position:

Proposition 2.3.8. *A smooth body K in $\mathbb{R}^n$ has minimal mean width if and only if*

$$\int_{S^{n-1}} \langle \nabla h_K(u), Tu \rangle \sigma(du) = \frac{\mathrm{tr} T}{n} \frac{w(K)}{2} \tag{28}$$

for every linear transformation T. Moreover, this minimal mean width position is uniquely determined up to orthogonal transformations. $\square$

Consider the measure w_K on S^{n-1} with density h_K with respect to σ. If we define

$$I_K(\theta) = \int_{S^{n-1}} \langle \nabla h_K(u), \theta \rangle \langle u, \theta \rangle \sigma(du) \quad , \quad \theta \in S^{n-1}, \tag{29}$$

an application of Green's formula shows that

$$\frac{w(K)}{2} + I_K(\theta) = (n+1) \int_{S^{n-1}} h_K(u) \langle u, \theta \rangle^2 \sigma(du). \tag{30}$$

Combining this identity with Proposition 2.3.8, we obtain an isotropic characterization of the minimal mean width position (see [82]):

Theorem 2.3.9. *A convex body K in $\mathbb{R}^n$ has minimal mean width if and only if w_K is isotropic. Moreover, the position is uniquely determined up to orthogonal transformations.* $\square$

Note. It is natural to ask for an upper bound for the minimal width parameter, if we restrict ourselves to bodies of fixed volume. It is known that every body K has a linear image $\tilde{K}$ with $|\tilde{K}| = |D_n|$ such that

$$w(\tilde{K}) \leq c \log(2d(X_K, \ell_2^n)) \leq c \log(2n), \tag{31}$$

where $c > 0$ is an absolute constant. This statement follows from an inequality of Pisier [159] after work of Lewis [109], Figiel and Tomczak-Jaegermann [60], and plays a central role in the theory. We shall use the minimal mean width position and come back to the estimate (31) in Section 5.

3 Background from classical convexity

3.1 Steiner's formula and Urysohn's inequality

3.1.1. Let $\mathcal{K}_n$ denote the set of all non-empty, compact convex subsets of $\mathbb{R}^n$. We may view $\mathcal{K}_n$ as a convex cone under Minkowski addition and multiplication by nonnegative real numbers. Minkowski's theorem (and the definition of the *mixed volumes*) asserts that if $K_1, \ldots, K_m \in \mathcal{K}_n$, $m \in \mathbb{N}$, then the volume of $t_1K_1 + \ldots + t_mK_m$ is a homogeneous polynomial of degree n in $t_i \geq 0$ (see [35], [177]). That is,

$$|t_1K_1 + \ldots + t_mK_m| = \sum_{1 \leq i_1, \ldots, i_n \leq m} V(K_{i_1}, \ldots, K_{i_n}) t_{i_1} \ldots t_{i_n},$$

where the coefficients $V(K_{i_1}, \ldots, K_{i_n})$ are chosen to be invariant under permutations of their arguments. The coefficient $V(K_1, \ldots, K_n)$ is called the mixed volume of $K_1, \ldots, K_n$.

Steiner's formula, which was already considered in 1840, may be seen as a special case of Minkowski's theorem. The volume of $K+tD_n$, $t>0$, can be expanded as a polynomial in t:

$$|K+tD_n|=\sum_{i=0}^{n}\binom{n}{i}W_i(K)t^i, \tag{1}$$

where $W_i(K)=V(K;n-i,D_n;i)$ is the *i-th Quermassintegral* of K. It is easy to see that the surface area of K is given by

$$\partial(K)=nW_1(K). \tag{2}$$

Kubota's integral formula

$$W_i(K)=\frac{|D_n|}{|D_{n-i}|_{n-i}}\int_{G_{n,n-i}}|P_\xi K|_{n-i}d\nu_{n,n-i}(\xi) \tag{3}$$

applied for $i=n-1$ shows that

$$W_{n-1}(K)=\frac{|D_n|}{2}w(K). \tag{4}$$

3.1.2. The *Alexandrov-Fenchel inequalities* constitute a far reaching generalization of the Brunn-Minkowski inequality and its consequences:

If $K,L,K_3,\dots,K_n\in\mathcal{K}_n$, then

$$V(K,L,K_3,\dots,K_n)^2\geq V(K,K,K_3,\dots,K_n)V(L,L,K_3,\dots,K_n). \tag{5}$$

The proof is due to Alexandrov [1], [2] (Fenchel sketched an alternative proof, see [58]). From (5) one can recover the Brunn-Minkowski inequality as well as the following generalization for the quermassintegrals:

$$W_i(K+L)^{\frac{1}{n-i}}\geq W_i(K)^{\frac{1}{n-i}}+W_i(L)^{\frac{1}{n-i}}\quad,\quad i=0,\dots,n-1 \tag{6}$$

for any pair of convex bodies in $\mathbb{R}^n$.

If we take $L=tD_n$, $t>0$, then Steiner's formula and the Brunn-Minkowski inequality give

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}\frac{W_i(K)}{|D_n|}t^i=\frac{|K+tD_n|}{|D_n|}&\geq\left(\left(\frac{|K|}{|D_n|}\right)^{1/n}+t\right)^n\\ &=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{|K|}{|D_n|}\right)^{\frac{n-i}{n}}t^i\end{aligned} \tag{7}$$

for every $t>0$. Since the first and the last term are equal on both sides of this inequality, we must have

$$\frac{W_1(K)}{|D_n|}\geq\left(\frac{|K|}{|D_n|}\right)^{\frac{n-1}{n}} \tag{8}$$

which is the isoperimetric inequality for convex bodies, and

$$w(K) = 2\frac{W_{n-1}(K)}{|D_n|} \geq 2\left(\frac{|K|}{|D_n|}\right)^{\frac{1}{n}}, \tag{9}$$

which is *Urysohn's inequality.* Both inequalities are special cases of the set of *Alexandrov inequalities*

$$\left(\frac{W_i(K)}{|D_n|}\right)^{\frac{1}{n-i}} \geq \left(\frac{W_j(K)}{|D_n|}\right)^{\frac{1}{n-j}} \quad , \quad n > i > j \geq 0. \tag{10}$$

3.1.3. Let K be a body in $\mathbb{R}^n$. We define

$$M^*(K) = \int_{S^{n-1}} \|x\|_* \sigma(dx) = \frac{w(K)}{2}. \tag{11}$$

The *Blaschke-Santaló inequality* asserts that the volume product $|K||K^\circ|$ is maximized over all symmetric convex bodies in $\mathbb{R}^n$ exactly when K is an ellipsoid:

$$|K||K^\circ| \leq |D_n|^2. \tag{12}$$

A proof of this fact via Steiner symmetrization was given in [12] (see also [129], [130] where the non-symmetric case is treated). Hölder's inequality and polar integration show that

$$\frac{1}{M^*(K)} \leq \left(\int_{S^{n-1}} \|x\|_*^{-n}\right)^{1/n} = \left(\frac{|K^\circ|}{|D_n|}\right)^{1/n}. \tag{13}$$

Combining with (12) and applying (13) for K instead of K°, we obtain

$$\frac{1}{M(K)} \leq \left(\frac{|K|}{|D_n|}\right)^{1/n} \leq M^*(K), \tag{14}$$

that is, Urysohn's inequality.

3.1.4. A third proof of Urysohn's inequality can be given as follows: Let $u_i \in O(n)$, $i = 1, \dots, m$ and $\alpha_i > 0$ with $\sum_{i=1}^m \alpha_i = 1$. It is easily checked that $M^*\left(\sum_{i=1}^m \alpha_i u_i(K)\right) = M^*(K)$. It follows that

$$M^*\left(\int_{O(n)} u(K) d\mu(u)\right) = M^*(K). \tag{15}$$

But, $T = \int_{O(n)} u(K) d\mu(u)$ is a ball of radius $(|T|/|D_n|)^{1/n}$, and the Brunn-Minkowski inequality implies that $|T| \geq |K|$. Therefore,

$$M^*(K) = \left(\frac{|T|}{|D_n|}\right)^{1/n} \geq \left(\frac{|K|}{|D_n|}\right)^{1/n}. \tag{16}$$

3.1.5. For any $(n-1)$-tuple $\mathcal{C} = K_1, \dots, K_{n-1} \in \mathcal{K}_n$, the Riesz representation theorem shows the existence of a Borel measure $S(\mathcal{C}, \cdot)$ on the unit sphere S^{n-1} such that

$$V(L, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{S^{n-1}} h_L(u) dS(\mathcal{C}, u) \tag{17}$$

for every $L \in \mathcal{K}_n$. If $K \in \mathcal{K}_n$, the j-th *area measure* of K is defined by $S_j(K, \cdot) = S(K; j, D_n; n-j-1, \cdot)$, $j = 0, 1, \dots, n-1$. It follows that the quermassintegrals $W_i(K)$ can be written in the form

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h_K(u) dS_{n-i-1}(K, u) \quad , \quad i = 0, 1, \dots, n-1 \tag{18}$$

or, alternatively,

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} dS_{n-i}(K, u) \quad , \quad i = 1, \dots, n. \tag{19}$$

If we assume that h_K is twice continuously differentiable, then $S_j(K, \cdot)$ has a continuous density $s_j(K, u)$, the j-th elementary symmetric function of the eigenvalues of the Hessian of h_K at u.

In the spirit of 2.3, we say that a body K *minimizes* W_i if $W_i(K) \leq W_i(TK)$ for every volume preserving linear transformation T of $\mathbb{R}^n$. The cases $i = 1$ and $i = n-1$ correspond to the minimal surface area and minimal mean width respectively. For every $i = 1, \dots, n-1$ one can prove that, if K minimizes W_i then $S_{n-i}(K, \cdot)$ is isotropic (see [82], where other necessary isotropic conditions are also given).

3.2 Geometric inequalities of "hyperbolic" type

The Alexandrov-Fenchel inequalities are the most advanced representatives of a series of very important inequalities. They should perhaps be called "hyperbolic" inequalities in contrast to the more often used in analysis "elliptic" inequalities: Cauchy-Schwarz, Hölder, and their consequences (various triangle inequalities). A consequence of "hyperbolic" inequalities is *concavity* of some important quantities.

3.2.1. Let us start this short review by recalling some old and classical, but not well remembered, inequalities due to Newton. Let $x_1, \dots, x_n$ be real numbers. We define the elementary symmetric functions $e_0(x_1, \dots, x_n) = 1$, and

$$e_i(x_1, \dots, x_n) = \sum_{1 \leq j_1 < \dots < j_i \leq n} x_{j_1} x_{j_2} \dots x_{j_i} \quad , \quad 1 \leq i \leq n. \tag{1}$$

In particular, $e_1(x_1, \dots, x_n) = \sum_{i=1}^n x_i$, $e_n(x_1, \dots, x_n) = \prod_{i=1}^n x_i$. We then consider the normalized functions

$$E_i(x_1, \dots, x_n) = \frac{1}{\binom{n}{i}} e_i(x_1, \dots, x_n). \tag{2}$$

Newton proved that, for $k = 1, \dots, n-1$,

$$E_k^2(x_1, \dots, x_n) \geq E_{k-1}(x_1, \dots, x_n) E_{k+1}(x_1, \dots, x_n), \tag{3}$$

with equality if and only if all the x_i's are equal. An immediate corollary of (3), observed by Newton's student Maclaurin, is the string of inequalities

$$E_1(x_1, \dots, x_n) \geq E_2^{1/2}(x_1, \dots, x_n) \geq \dots \geq E_n^{1/n}(x_1, \dots, x_n), \tag{4}$$

which holds true for any n-tuple $(x_1, \dots, x_n)$ of positive reals. Note the similarity between (3), (4) and the Alexandrov-Fenchel and Alexandrov inequalities 3.1.2(5) and (10) respectively.

To prove (3) we consider the polynomial

$$P(x) = \prod_{i=1}^{n}(x - x_i) = \sum_{j=0}^{n}(-1)^j \binom{n}{j} E_j(x_1, \dots, x_n) x^{n-j}, \tag{5}$$

or in homogeneous form,

$$Q(t, \tau) = \tau^n P(\frac{t}{\tau}) = \sum_{j=0}^{n}(-1)^j \binom{n}{j} E_j(x_1, \dots, x_n) t^{n-j} \tau^j. \tag{6}$$

Since P has only real roots, the same is true for the derivatives of P (with respect to t or τ) of any order. If we differentiate (6) $(n-k-1)$-times with respect to t and then $(k-1)$-times with respect to τ, we obtain the polynomial

$$\frac{n!}{2} E_{k-1}(x_1, \dots, x_n) t^2 - n! E_k(x_1, \dots, x_n) t\tau + \frac{n!}{2} E_{k+1}(x_1, \dots, x_n) \tau^2, \tag{7}$$

which has two real roots for fixed $\tau = 1$. This is exactly Newton's inequality (3).

We refer to [167] for a very nice different proof and generalizations.

3.2.2. Let us now turn to a multidimensional, but still numerical, analogue of Newton's inequalities. Consider the space S_n of real symmetric $n \times n$ matrices. We polarize the function $A \to \det A$ to obtain the symmetric multilinear form

$$D(A_1, \dots, A_n) = \frac{1}{n!} \sum_{\varepsilon \in \{0,1\}^n} (-1)^{n + \sum \varepsilon_i} \det\left(\sum \varepsilon_i A_i\right), \tag{8}$$

where $A_i \in S_n$. Then, if $t_1, \dots, t_m > 0$ and $A_1, \dots, A_m \in S_n$, the determinant of $t_1 A_1 + \dots + t_m A_m$ is a homogeneous polynomial of degree n in t_i:

$$\det(t_1 A_1 + \dots + t_m A_m) = \sum_{1 \leq i_1 \leq \dots \leq i_n \leq m} n! D(A_{i_1}, \dots, A_{i_n}) t_{i_1} \dots t_{i_n}. \tag{9}$$

The coefficient $D(A_1, \dots, A_n)$ is called the mixed discriminant of $A_1, \dots, A_n$. The fact that the polynomial $P(t) = \det(A + tI)$ has only real roots for any $A \in S_n$ plays the central role in the proof of a number of very interesting inequalities connecting mixed discriminants, which are quite similar to Newton's inequalities. They were first discovered by Alexandrov [2] in one of his approaches to what is now called Alexandrov-Fenchel inequalities. Today, they are part of a more general theory (see e.g. [93]). We mention some of them: If $A_i, i = 1, \ldots, n$ are positive, then

$$D(A_1, A_2, \ldots, A_n) \geq \prod_{i=1}^{n} [\det A]^{\frac{1}{n}}. \tag{10}$$

Also, the following concavity principle (reverse triangle inequality) is true: The function $[\det A]^{1/n}$ is concave in the positive cone of S_n. This is in fact easy to demonstrate directly. We want to show that, if A_1, A_2 are positive then

$$[\det(A_1 + A_2)]^{\frac{1}{n}} \geq [\det A_1]^{\frac{1}{n}} + [\det A_2]^{\frac{1}{n}}. \tag{11}$$

We may bring two positive matrices to diagonal form without changing their determinants. Then, we should show that for $\lambda_i, \mu_i > 0$,

$$\left(\prod_{i=1}^{n}(\lambda_i + \mu_i)\right)^{1/n} \geq \left(\prod_{i=1}^{n}\lambda_i\right)^{1/n} + \left(\prod_{i=1}^{n}\mu_i\right)^{1/n}, \tag{12}$$

which is a consequence of the arithmetic-geometric means inequality.

3.2.3. We now return to convex sets. The results of 3.2.1 and 3.2.2 have their analogues in this setting, but the parallel results for mixed volumes are much more difficult and look unrelated. Even the fact that the volume of $t_1K_1 + \ldots + t_mK_m$ is a homogeneous polynomial in $t_i \geq 0$ is a non-trivial statement, while the parallel result for determinants follows by definition.

To see the connection between the two theories we follow [11]. Consider n fixed convex open bounded bodies K_i with normalized volume $|K_i| = 1$. As in Section 2.2(a), consider the Brenier maps

$$\psi_i : (\mathbb{R}^n, \gamma_n) \to K_i, \tag{13}$$

where γ_n is the standard Gaussian probability density on $\mathbb{R}^n$. We have $\psi_i = \nabla f_i$, where f_i are convex functions on $\mathbb{R}^n$. By Caffarelli's regularity result, all the ψ_i's are smooth maps. Then, Fact 2 from 2.2(a) shows that the image of $(\mathbb{R}^n, \gamma_n)$ by $\sum t_i\psi_i$ is the interior of $\sum t_iK_i$. Since each ψ_i is a measure preserving map, we have

$$\det\left(\frac{\partial^2 f_i}{\partial x_k \partial x_l}\right)(x) = \gamma_n(x) \quad , \quad i = 1, \ldots, n. \tag{14}$$

It follows that

$$\left|\sum_{i=1}^{n} t_iK_i\right| = \int_{\mathbb{R}^n} \det\left(\sum_{i=1}^{n} t_i(\frac{\partial^2 f_i}{\partial x_k \partial x_l})\right) dx \tag{15}$$

$$= \sum_{i_1,\ldots,i_n=1}^{n} t_{i_1}\ldots t_{i_n} \int_{\mathbb{R}^n} D\left(\frac{\partial^2 f_{i_1}(x)}{\partial x_k \partial x_l}, \ldots, \frac{\partial^2 f_{i_n}(x)}{\partial x_k \partial x_l}\right) dx.$$

In particular, we recover Minkowski's theorem on polynomiality of $|\sum t_i K_i|$, and see the connection between the mixed discriminants $D(\mathrm{Hess} f_{i_1}, \ldots, \mathrm{Hess} f_{i_n})$ and the mixed volumes

$$V(K_{i_1}, \ldots, K_{i_n}) = \int_{\mathbb{R}^n} D(\mathrm{Hess} f_{i_1}(x), \ldots, \mathrm{Hess} f_{i_n}(x)) dx. \tag{16}$$

The Alexandrov-Fenchel inequalities do not follow from the corresponding mixed discriminant inequalities, but the deep connection between the two theories is obvious. Also, some particular cases are indeed simple consequences. For example, in [11] it is proved (as a consequence of (16)) that

$$V(K_1, \ldots, K_n) \geq \prod_{i=1}^{n} |K_i|^{1/n}. \tag{17}$$

3.3 Continuous valuations on compact convex sets

(a) **Polynomial valuations.** We denote by $\mathcal{K}_n$ the set of all non-empty compact convex subsets of $\mathbb{R}^n$ and write L for a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$.

A function $\varphi : \mathcal{K}_n \to L$ is called a *valuation*, if $\varphi(K_1 \cup K_2) + \varphi(K_1 \cap K_2) = \varphi(K_1) + \varphi(K_2)$ whenever $K_1, K_2 \in \mathcal{K}_n$ are such that $K_1 \cup K_2 \in \mathcal{K}_n$. We shall consider only *continuous valuations*: valuations which are continuous with respect to the Hausdorff metric.

The notion of valuation may be viewed as a generalization of the notion of measure defined only on the class of compact convex sets. Mixed volumes provide a first important example of valuations.

A valuation $\varphi : \mathcal{K}_n \to L$ is called *polynomial* of degree at most l if $\varphi(K + x)$ is a polynomial in x of degree at most l for every $K \in \mathcal{K}_n$. The following theorem of Khovanskii and Pukhlikov [105] generalizes Minkowski's theorem on mixed volumes (see also [126], [4]):

Theorem 3.3.1. *Let $\varphi : \mathcal{K}_n \to L$ be a continuous valuation, which is polynomial of degree at most l. Then, if $K_1, \ldots, K_m \in \mathcal{K}_n$, $\varphi(t_1 K_1 + \ldots + t_m K_m)$ is a polynomial in $t_j \geq 0$ of degree at most $n + l$.* □

Let $\tilde{K} = (K_1, \ldots, K_s)$ be an s-tuple of compact convex sets in $\mathbb{R}^n$, and $F : \mathbb{R}^n \to \mathbb{C}$ be a continuous function. Alesker studied the Minkowski operator $M_{\tilde{K}}$ which maps F to $M_{\tilde{K}} F : \mathbb{R}^s_+ \to \mathbb{C}$ with

$$(M_{\tilde{K}} F)(\lambda_1, \ldots, \lambda_s) = \int_{\sum_{i \leq s} \lambda_i K_i} F(x) dx.$$

Let $\mathcal{A}(\mathbb{C}^n)$ be the Frechet space of entire functions of n variables and $C^r(\mathbb{R}^n)$ be the Frechet space of r-times differentiable functions on $\mathbb{R}^n$, with the topology of uniform convergence on compact sets. The following facts are established in [3]:

(i) If $F \in \mathcal{A}(\mathbb{C}^n)$, then $M_{\tilde{K}}F$ has a unique extension to an entire function on $\mathbb{C}^s$, and the operator $M_{\tilde{K}} : \mathcal{A}(\mathbb{C}^n) \to \mathcal{A}(\mathbb{C}^s)$ is continuous. It follows that if F is a polynomial of degree d then $M_{\tilde{K}}F$ is a polynomial of degree at most $d+n$.

(ii) If $F \in C^r(\mathbb{R}^n)$, then $M_{\tilde{K}}F \in C^r(\mathbb{R}^s_+)$, and $M_{\tilde{K}}$ is a continuous operator.

Moreover, continuity of the map $\tilde{K} \mapsto M_{\tilde{K}}$ with respect to the Hausdorff metric is established.

(*b*) **Translation invariant valuations.** A valuation of degree 0 is simply translation invariant. If $\varphi(uK) = \varphi(K)$ for every $K \in \mathcal{K}_n$ and every $u \in SO(n)$, we say that φ is $SO(n)$-invariant. Hadwiger [89] characterized the translation and $SO(n)$ invariant valuations as follows (see also [101] for a simpler proof):

Theorem 3.3.2. *A valuation φ is translation and $SO(n)$-invariant if and only if there exist constants c_i, $i = 0, \dots, n$ such that*

$$\varphi(K) = \sum_{i=0}^{n} c_i W_i(K) \tag{1}$$

for every $K \in \mathcal{K}_n$. $\square$

After Hadwiger's classical result, two natural questions arise: to characterize translation invariant valuations without any assumption on rotations, and to characterize $O(n)$ or $SO(n)$ invariant valuations without any assumption on translations. Both questions are of obvious interest in translative integral geometry and in the asymptotic theory of finite dimensional normed spaces respectively (consider, for example, the valuation $\varphi(K) = \int_K |x|^2 dx$ which was discussed in 2.3(b)).

It is a conjecture of McMullen [127] that every continuous translation invariant valuation can be approximated (in a certain sense) by linear combinations of mixed volumes. This is known to be true in dimension $n \leq 3$. In [126], [127] it is proved that every translation invariant valuation φ can be uniquely expressed as a sum $\varphi = \sum_{i=0}^{n} \varphi_i$, where φ_i are translation invariant continuous valuations satisfying $\varphi_i(tK) = t^i \varphi(K)$ (homogeneous of degree i). Moreover, in the case $L = \mathbb{R}$, homogeneous valuations φ_i as above can be described in some cases: φ_0 is always a constant, φ_n is always a multiple of volume, φ_{n-1} is always of the form

$$\varphi_{n-1}(K) = \int_{S^{n-1}} f(u) dS_{n-1}(K, u), \tag{2}$$

where $f : S^{n-1} \to \mathbb{R}$ is a continuous function (which can be chosen to be orthogonal to every linear functional, and then it is uniquely determined).

Under the additional assumption that φ is *simple* ($\varphi(K) = 0$ if $\dim K < n$), a recent theorem of Schneider [178] completely describes φ:

Theorem 3.3.3. *Every simple, continuous translation invariant valuation $\varphi : \mathcal{K}_n \to \mathbb{R}$ has the form*

$$\varphi(K) = c|K| + \int_{S^{n-1}} f(u) dS_{n-1}(K, u), \tag{3}$$

where $f : S^{n-1} \to \mathbb{R}$ is a continuous odd function. $\square$

Remark: McMullen's conjecture was recently proved by Alesker [5] in dimension $n = 4$.

Added in Proofs: Even more recently, Alesker [6] gave a description of translation invariant valuations on convex sets, which in particular confirms McMullen's conjecture in all dimensions.

(c) **Rotation invariant valuations.** Alesker [4] has recently obtained a characterization of $O(n)$ (respectively $SO(n)$) invariant continuous valuations. The first main point is that every such valuation can be approximated uniformly on the compact subsets of $\mathcal{K}_n$ by continuous polynomial $O(n)$ (or $SO(n)$) invariant valuations.

Then, one can describe polynomial rotation invariant valuations in a concrete way. To this end, let us introduce some specific examples of such valuations. We write ν for the $(n-1)$-dimensional surface measure on K and $n(x)$ for the outer normal at $\mathrm{bd}(K)$ (this is uniquely determined ν-almost everywhere). If p, q are non-negative integers, we consider a valuation $\psi_{p,q} : \mathcal{K}_n \to \mathbb{R}$ with

$$\psi_{p,q}(K) = \int_{\mathrm{bd}(K)} \langle x, n(x)\rangle^p |x|^{2q} d\nu(x). \tag{4}$$

All $\psi_{p,q}$ are continuous, polynomial of degree at most $p+2q+n$, and $O(n)$-invariant. Theorem 3.3.1 shows that, for every $K \in \mathcal{K}_n$, $\psi_{p,q}(K + \varepsilon D_n)$ is a polynomial in $\varepsilon \geq 0$, therefore it can be written in the form

$$\psi_{p,q}(K + \varepsilon D_n) = \sum_{i=0}^{p+2q+n} \psi_{p,q}^{(i)}(K)\varepsilon^i. \tag{5}$$

All $\psi_{p,q}^{(i)}$ are continuous, polynomial and $O(n)$-invariant. These particular valuations suffice for a description of all rotation invariant polynomial valuations [4]:

Theorem 3.3.4. *If $n \geq 3$, then every $SO(n)$-invariant continuous polynomial valuation $\varphi : \mathcal{K}_n \to \mathbb{R}$ is a linear combination of the $\psi_{p,q}^{(i)}$.* $\square$

Since $\psi_{p,q}^{(i)}$ are $O(n)$-invariant, Theorem 3.3.4 describes $O(n)$-invariant valuations as well. The case $n = 2$ is also completely described in [4] (and the same statements hold true if $\mathbb{R}$ is replaced by $\mathbb{C}$).

4 Dvoretzky's theorem and concentration of measure

4.1 Introduction

A version of the Dvoretzky-Rogers Lemma [53] asserts that for every body K whose maximal volume ellipsoid is D_n, there exist $k \simeq \sqrt{n}$ and a k-dimensional subspace

E_k of $\mathbb{R}^n$ such that $D_k \subseteq K \cap E_k \subseteq 2Q_k$, where D_k denotes the Euclidean ball in E_k and Q_k the unit cube in E_k (for an appropriately chosen coordinate system). Inspired by this, Grothendieck asked whether Q_k can be replaced by D_k in this statement. He did not specify what the dependence of k on n might be, asking just that k should increase to infinity with n. A short time after, Dvoretzky [51], [52] proved Grothendieck's conjecture:

Theorem 4.1.1. *Let $\varepsilon > 0$ and k be a positive integer. There exists $N = N(k, \varepsilon)$ with the following property: Whenever X is a normed space of dimension $n \geq N$ we can find a k-dimensional subspace E_k of X with $d(E_k, \ell_2^k) \leq 1 + \varepsilon$.*

Geometrically speaking, every high-dimensional body has central sections of high dimension which are almost ellipsoidal. The dependence of $N(k, \varepsilon)$ on k and ε became a very important question, and Dvoretzky's theorem took a much more precise quantitative form:

Theorem 4.1.2. *Let X be an n-dimensional normed space and $\varepsilon > 0$. There exist an integer $k \geq c\varepsilon^2 \log n$ and a k-dimensional subspace E_k of X which satisfies $d(E_k, \ell_2^k) \leq 1 + \varepsilon$.*

This means that Theorem 4.1.1 holds true with $N(k, \varepsilon) = \exp(c\varepsilon^{-2} k)$. Dvoretzky's original proof gave an estimate $N(k, \varepsilon) = \exp(c\varepsilon^{-2} k^2 \log k)$. Later, Milman [131] established the estimate $N(k, \varepsilon) = \exp(c\varepsilon^{-2} |\log \varepsilon| k)$ with a different approach. The logarithmic in ε term was removed by Gordon [68], and then by Schechtman [174]. Other proofs and extensions of Dvoretzky's theorem in different directions were given in [59], [185], [112] (see also the surveys [110], [113], [142]).

The logarithmic dependence of k on n is best possible for small values of ε. One can see this by analyzing the example of ℓ_∞^n. Every k-dimensional central section of Q_n is a polytope with at most $2n$ facets. If we assume that we can find a subspace E_k of ℓ_∞^n with $d(E_k, \ell_2^k) \leq 1 + \varepsilon$, then there exists a polytope P_k in $\mathbb{R}^k$ with $m \leq 2n$ facets satisfying $D_k \subseteq P_k \subseteq (1 + \varepsilon) D_k$. The hyperplanes supporting the facets of P_k create m spherical caps $J_1, \dots, J_m$ on $(1 + \varepsilon) S^{k-1}$ such that $(1 + \varepsilon) S^{k-1} \subseteq \bigcup_{i=1}^m J_i$. On the other hand, since $D_k \subseteq P_k$, if we assume that ε is small, then each J_i has angular radius of the order of $\sqrt{\varepsilon}$. An elementary computation shows that the normalized measure of such a cap does not exceed $(c\varepsilon)^{\frac{k-1}{2}}$. Therefore, we must have $2n \geq (c\varepsilon)^{-\frac{k-1}{2}}$ which shows that

$$k \leq c \log n / \log(1/\varepsilon). \tag{1}$$

The same argument shows that if P is a symmetric polytope and $f(P)$ is the number of its facets, then $k \leq c(\varepsilon) \log f(P)$.

The right dependence of $N(k, \varepsilon)$ on ε for a fixed (even small) positive integer k is not clear. It seems reasonable that ℓ_∞^n is the worst case and that the computation we have just made gives the correct order:

Question 4.1.3. *Can we take $N(k, \varepsilon) = c(k)\varepsilon^{-\frac{k-1}{2}}$ in Theorem 4.1.1?*

Using ideas from the theory of irregularities of distribution, Bourgain and Lindenstrauss [29] have shown that the choice $N(k, \varepsilon) = c(k)\varepsilon^{-\frac{k-1}{2}} |\log \varepsilon|$ is possible for spaces X with a 1-symmetric basis. There are numerous connections of this question with other branches of mathematics (algebraic topology, number theory, harmonic analysis). For instance, an affirmative answer to Question 4.1.3 would be a consequence of the following *hypothesis of Knaster*: Let $f : S^{k-1} \to \mathbb{R}$ be a continuous function and $x_1, \dots, x_k$ be points on S^{k-1}. Does there exist a rotation u such that f is constant on the set $\{ux_i : i \leq k\}$? This hypothesis has been settled only in special cases (see [137] for a discussion of this and other problems related to Question 4.1.3).

Note. Bourgain and Szarek [33] proved a stronger form of the Dvoretzky-Rogers Lemma: If D_n is the ellipsoid of minimal volume containing K, then for every $\delta \in (0,1)$ one can choose $x_1, \dots, x_m$, $m \geq (1-\delta)n$, among the contact points of K and D_n such that for every choice of scalars $(t_i)_{i \leq m}$,

$$f(\delta) \left(\sum_{i=1}^{m} t_i^2 \right)^{1/2} \leq \Big| \sum_{i=1}^{m} t_i x_i \Big| \leq \Big\| \sum_{i=1}^{m} t_i x_i \Big\|_K \leq \sum_{i=1}^{m} |t_i|. \qquad (2)$$

This is a Dvoretzky-Rogers Lemma for arbitrary proportion of the dimension. It can also be stated as a *factorization result*: For any n-dimensional normed space X and any $\delta \in (0,1)$, one can find $m \geq (1-\delta)n$ and two operators $\alpha : \ell_2^m \to X$, $\beta : X \to \ell_\infty^n$ such that the identity $\mathrm{id}_{2,\infty} : \ell_2^m \to \ell_\infty^m$ can be written as $\mathrm{id}_{2,\infty} = \beta \circ \alpha$ and $\|\alpha\| \|\beta\| \leq 1/f(\delta)$. For an extension to the non-symmetric case see [116].

Using this result, Bourgain and Szarek answered in the negative the question of uniqueness, up to a constant, of the centre of the Banach-Mazur compactum, and gave the first non-trivial estimate $o(n)$ for the Banach-Mazur distance from an n-dimensional space X to ℓ_∞^n. It is now known [186], [63] that (2) holds true with $f(\delta) = c\delta$. The question about the best possible exponent of δ in the Dvoretzky-Rogers factorization is also open. By [63], [169] it must lie between 1/2 and 1.

In the Appendix we give a brief account on these and other questions related to the geometry of the Banach-Mazur compactum.

4.2 Concentration of measure on the sphere and a proof of Dvoretzky's theorem

We shall outline the approach of [131] to Dvoretzky's theorem. The method uses the concentration of measure on the sphere and was further developed in [61]. We need to introduce the average parameter

$$M = M(X_K) = \int_{S^{n-1}} \|x\| \, \sigma(dx), \qquad (1)$$

the average on the sphere S^{n-1} of the norm that K induces on $\mathbb{R}^n$.

Remarks on M. (i) It is clear from the definition that M depends not only on the body K but also on the Euclidean structure we have chosen in $\mathbb{R}^n$. If we assume that $\frac{1}{a}|x| \le \|x\| \le b|x|$ and that $a, b > 0$ are the smallest constants for which this is true for all $x \in \mathbb{R}^n$, then we have the trivial bounds $\frac{1}{a} \le M \le b$.

(ii) For every $p > 0$ we define

$$M_p = M_p(X_K) = \left(\int_{S^{n-1}} \|x\|^p \sigma(dx)\right)^{\frac{1}{p}}. \tag{2}$$

In this notation $M = M_1$ and as a consequence of the Kahane-Khinchine inequality one can check that $M_1 \simeq M_2$ independently from the dimension and the norm. It can be actually shown [118] that, for every $1 \le p \le n$,

$$\max\Big\{M_1, c_1\frac{b\sqrt{p}}{\sqrt{n}}\Big\} \le M_p \le \max\Big\{2M_1, c_2\frac{b\sqrt{p}}{\sqrt{n}}\Big\}, \tag{3}$$

where $c_1, c_2 > 0$ are absolute constants.

(iii) Let $g_1, \ldots, g_n$ be independent standard Gaussian random variables on some probability space Ω and $\{e_1', \ldots, e_n'\}$ be any orthonormal basis in $\mathbb{R}^n$. Integration in polar coordinates establishes the identity

$$\left(\int_\Omega \Big\| \sum_{i=1}^n g_i(\omega)e_i' \Big\|^2 d\omega\right)^{1/2} = \sqrt{n}M_2. \tag{4}$$

Using the symmetry of the g_i's and the triangle inequality for $\|\cdot\|$ we get

$$\int_\Omega \Big\| \sum_{i=1}^k g_i(\omega)e_i' \Big\| d\omega \le \int_\Omega \Big\| \sum_{i=1}^n g_i(\omega)e_i' \Big\| d\omega, \tag{5}$$

for every $1 \le k \le n$, and combining with the previous observations we have

$$M(E_k) \le c\sqrt{n/k}M \tag{6}$$

for every k-dimensional subspace E_k of X_K.

- The main step for our proof of Theorem 4.1.2 will be the following [131]:

Theorem 4.2.1. *Let X be an n-dimensional normed space satisfying $\frac{1}{a}|x| \le \|x\| \le b|x|$. For every $\varepsilon \in (0,1)$ there exist $k \ge c\varepsilon^2 n(M/b)^2$ and a k-dimensional subspace E_k of $\mathbb{R}^n$ such that*

$$\frac{1}{1+\varepsilon}L|x| \le \|x\| \le (1+\varepsilon)L|x| \quad , \quad x \in E_k.$$

The constant L appearing in the statement above is the *Lévy mean* (or median) of the function $f(x) = \|x\|$ on S^{n-1}. This is the unique real number $L = L_f$ for which

$$\sigma(\{x : f(x) \ge L\}) \ge \frac{1}{2} \quad \text{and} \quad \sigma(\{x : f(x) \le L\}) \ge \frac{1}{2}.$$

A few observations arise directly from this statement: Assume that $x \in S^{n-1}$ has maximal norm $\|x\| = b$. Consider the one-dimensional subspace E_1 spanned by

x. We have $b = M(E_1) \leq c\sqrt{n}M$, and this shows that $n(M/b)^2 \geq c > 0$ for every norm. This is of course not enough for a proof of Dvoretzky's theorem.

On the other hand, recall that $M \geq 1/a$. By Theorem 4.2.1, every X has a subspace of dimension $k \geq c\varepsilon^2 n/(ab)^2$ on which $\|\cdot\|$ is $(1+\varepsilon)$-equivalent to the Euclidean norm. Since we can choose a linear transformation of K_X so that $ab \leq d(X, \ell_2^n)$, we obtain the following corollary [131]:

Corollary 4.2.2. *For every n-dimensional space X and every $\varepsilon \in (0,1)$ we can find a subspace E_k of X with* $\dim E_k = k \geq c\varepsilon^2 n/d^2(X, \ell_2^n)$ *such that* $d(E_k, \ell_2^k) \leq 1+\varepsilon$. $\square$

This already shows that spaces with small Banach-Mazur distance from ℓ_2^n have Euclidean sections of dimension much larger than $\log n$ (even proportional to n). However, since John's theorem is sharp this observation is not enough for the general case.

• The proof of Theorem 4.2.1 is based on the concentration of measure on the sphere. Recall that as a consequence of the spherical isoperimetric inequality we have the following fact:

If $A \subseteq S^{n-1}$ and $\sigma(A) = \frac{1}{2}$, then $\sigma(A_\varepsilon) \geq 1 - c_1\exp(-c_2\varepsilon^2 n)$.

This inequality explains the term "concentration of measure": However small $\varepsilon > 0$ may be, the measure of the set outside a "strip" of width ε around the boundary of any subset of the sphere of half measure is less than $2c_1\exp(-c_2\varepsilon^2 n)$, which decreases exponentially fast to 0 as the dimension n grows to infinity. This surprising fact was observed and used by P. Lévy [108]:

Let f be a continuous function on the sphere. By $\omega_f(\cdot)$ we denote the modulus of continuity of f:

$$\omega_f(t) = \max\{|f(x) - f(y)| : \rho(x,y) \leq t,\ x, y \in S^{n-1}\}.$$

Consider the Lévy mean L_f of f. It is not hard to see that

$$\{x : f = L_f\}_\varepsilon = (\{x : f \geq L_f\})_\varepsilon \cap (\{x : f \leq L_f\})_\varepsilon.$$

Since $|f(x) - L_f| \leq \omega_f(\varepsilon)$ on $\{x : f = L_f\}_\varepsilon$, the spherical isoperimetric inequality has the following direct consequence:

Fact 1. *For every continuous function $f : S^{n-1} \to \mathbb{R}$ and every $\varepsilon > 0$,*

$$\sigma\left(x \in S^{n-1} : |f(x) - L_f| \geq \omega_f(\varepsilon)\right) \leq c_1\exp(-c_2\varepsilon^2 n). \quad \square \tag{7}$$

If the modulus of continuity of f behaves well, then Fact 1 implies strong concentration of the values of f around its median. Moreover, from a set of big measure on which f is almost constant we can extract a *subspace* of high dimension, on the sphere of which f is almost constant:

Fact 2. *Let $f : S^{n-1} \to \mathbb{R}$ be a continuous function and $\delta, \theta > 0$. There exists a subspace F of $\mathbb{R}^n$ with* $\dim F = k \geq c\delta^2 n/\log(3/\theta)$ *such that*

$$|f(x) - L_f| \leq \omega_f(\delta) + \omega_f(\theta)$$

for every $x \in S(F) := S^{n-1} \cap F$.

Proof: Fix $k < n$ (to be determined) and $F_k \in G_{n,k}$. A standard argument shows that there exists a θ-net $\mathcal{N}$ of $S(F_k)$ with cardinality $|\mathcal{N}| \leq (1+\frac{2}{\theta})^k \leq \exp(k \log(3/\theta))$. If $x \in \mathcal{N}$, then

$$\mu\left(u \in O(n) : |f(ux) - L_f| > \omega_f(\delta)\right) \leq c_1 \exp(-c_2\delta^2 n). \tag{8}$$

Therefore, if $c_1|\mathcal{N}|\exp(-c_2\delta^2 n) < 1$ then most $u \in O(n)$ satisfy

$$|f(ux) - L_f| \leq \omega_f(\delta) \tag{9}$$

for every $x \in \mathcal{N}$. It follows that $|f(x) - L_f| \leq \omega_f(\delta) + \omega_f(\theta)$ for every $x \in S(uF_k)$. A simple computation shows that the necessary condition will be satisfied for some $k \geq c\delta^2 n/\log(3/\theta)$. $\square$

For the proof of Theorem 4.2.1 we are going to apply this fact to the *norm* $f(x) = \|x\|$. In this case, one can say even more (see [149, pp. 12]):

Fact 3. *Let* $X = (\mathbb{R}^n, \|\cdot\|)$ *and assume that* $\|x\| \leq b|x|$. *For every* $\varepsilon \in (0,1)$ *there exists a subspace* E_k *with* $\dim E_k = k \geq \frac{c\varepsilon^2}{\log(1/\varepsilon)} n (\frac{L_f}{b})^2$ *such that*

$$\frac{1}{1+\varepsilon} L_f|x| \leq \|x\| \leq (1+\varepsilon) L_f |x|$$

for every $x \in E_k$. $\square$

The proof of Theorem 4.2.1 is now complete. We just have to observe that if $f(x) = \|x\|$ on S^{n-1}, then $L_f \simeq M$. By Markov's inequality, $\sigma(x : f(x) \geq 2M) \leq \frac{1}{2}$ and this shows that $L_f \leq 2M$. It can be checked that $L_f \geq cM$ as well, where $c > 0$ is an absolute constant [149]. It follows that we can have almost spherical sections of dimension $k \geq \frac{c\varepsilon^2}{\log(1/\varepsilon)} n (\frac{M}{b})^2$ in Theorem 4.2.1. In order to remove the logarithmic in ε term, one needs to put additional effort (see [68], [174]). $\square$

From Theorem 4.2.1 we may deduce Dvoretzky's theorem (Theorem 4.1.2): For every n-dimensional space X and any $\varepsilon \in (0,1)$ there exists a subspace E_k of X with $\dim E_k = k \geq c\varepsilon^2 \log n$, such that $d(E_k, \ell_2^k) \leq 1+\varepsilon$.

Proof: We may assume that D_n is the maximal volume ellipsoid of K_X. Then, $\|x\| \leq |x|$ on $\mathbb{R}^n$ and in view of Theorem 4.2.1 we only need to show that $M^2 \geq c \log n/n$. This is a consequence of the Dvoretzky-Rogers lemma: There exists an orthonormal basis $y_1, \ldots, y_n$ in $\mathbb{R}^n$ with $\|y_i\| \geq (\frac{n-i+1}{n})^{1/2}$. In particular, $\|y_i\| \geq \frac{1}{2}$, $i = 1, \ldots, \frac{n}{4}$.

From the equivalence of M_1 and M_2 we see that

$$\begin{aligned} M &\geq \frac{c}{\sqrt{n}} \int_\Omega \Big\| \sum_{i=1}^n g_i(\omega) y_i \Big\| \, d\omega \geq \frac{c}{\sqrt{n}} \int_\Omega \Big\| \sum_{i=1}^{n/4} g_i(\omega) y_i \Big\| \, d\omega \\ &\geq \frac{c}{\sqrt{n}} \int_\Omega \max_{i \leq n/4} \Big\| g_i(\omega) y_i \Big\| \, d\omega \geq \frac{c'}{\sqrt{n}} \int_\Omega \max_{i \leq n/4} |g_i(\omega)| d\omega \geq \frac{c''\sqrt{\log n}}{\sqrt{n}}, \end{aligned} \tag{10}$$

where we have used the fact (see e.g. [115, pp. 79]) that if $g_1, \ldots, g_m$ are independent standard Gaussian random variables on Ω then $\int_\Omega \max_{i \leq m} |g_i| \simeq \sqrt{\log m}$. $\square$

4.3 Probabilistic and global form of Dvoretzky's Theorem

The proof of Theorem 4.2.1, being probabilistic in nature, gives that a subspace E_k of X with $\dim E_k = [c\varepsilon^2 n(M/b)^2]$ is $(1+\varepsilon)$-Euclidean with high probability. This leads to the definition of the following characteristic of X:

Definition. Let X be an n-dimensional normed space. We set $k(X)$ to be the largest positive integer $k \leq n$ for which

$$\text{(1)} \qquad \mathrm{Prob}\left(E_k \in G_{n,k} : \frac{1}{2}M|x| \leq \|x\| \leq 2M|x|, x \in E_k\right) \geq 1 - \frac{k}{n+k}.$$

In other words, $k(X)$ is the largest possible dimension $k \leq n$ for which the majority of k-dimensional subspaces of X are 4-Euclidean. Note that the presence of M in the definition corresponds to the right normalization, since the average of $M(E_k)$ over $G_{n,k}$ is equal to M for all $1 \leq k \leq n$.

Theorem 4.2.1 implies that $k(X) \geq cn(M/b)^2$. What is surprisingly simple is the observation [151] that an inverse inequality holds true. The estimate in Theorem 4.2.1 is sharp in full generality:

Theorem 4.3.1. $k(X) \leq 4n(M/b)^2$.

Proof: Fix orthogonal subspaces $E^1, \ldots, E^t$ of dimension $k(X)$ such that $\mathbb{R}^n = \sum_{i=1}^t E^i$ (there is no big loss in assuming that $k(X)$ divides n). By the definition of $k(X)$, most orthogonal images of each E^i are 4-Euclidean, so we can find $u \in O(n)$ such that

$$\text{(2)} \qquad \frac{1}{2}M|x| \leq \|x\| \leq 2M|x| \quad , \quad x \in uE^i$$

for every $i = 1, \ldots, t$. Every $x \in \mathbb{R}^n$ can be written in the form $x = \sum_{i=1}^t x_i$, where $x_i \in uE^i$. Since the x_i's are orthogonal, we get

$$\text{(3)} \qquad \|x\| \leq 2M \sum_{i=1}^t |x_i| \leq 2M\sqrt{t}|x|.$$

This means that $b \leq 2M\sqrt{t}$, and since $t = n/k(X)$ we see that $k(X) \leq 4n(M/b)^2$. □

In other words, the following *asymptotic formula* holds true:

Theorem 4.3.2. *Let X be an n-dimensional normed space. Then,*

$$k(X) \simeq n(M/b)^2. \quad \square$$

Dvoretzky's theorem gives information about the central sections of a body, or equivalently, about the local structure of the corresponding normed space. By a *global* result we mean a statement about the full body or space. In order to describe the global version of Dvoretzky's theorem, we need to introduce a new quantity:

Definition. Let $X = (\mathbb{R}^n, \|\cdot\|)$. We define $t(X)$ to be the smallest positive integer t for which there exist $u_1, \ldots, u_t \in O(n)$ such that

$$\frac{1}{2}M|x| \leq \frac{1}{t}\sum_{i=1}^{t} \|u_i x\| \leq 2M|x|$$

for every $x \in \mathbb{R}^n$.

Geometrically speaking, $t(X)$ is the smallest integer t for which there exist rotations $v_1, \ldots, v_t$ such that the average Minkowski sum of $v_i K^\circ$ is 4-Euclidean. Once again, the presence of M in the definition corresponds to the correct normalization.

It is proved in [38] that $t(X) \leq c(b/M)^2$ (we postpone a proof of this fact until Section 4.5). It was recently observed in [151] that a reverse inequality is true in full generality:

Theorem 4.3.3. $t(X) \geq \frac{1}{4}(b/M)^2$.

For the proof of this assertion we shall make use of the following lemma:

Lemma. *Let $x_1, \ldots, x_t \in S^{n-1}$. There exists $y \in S^{n-1}$ such that $\sum_{i=1}^{t} |\langle y, x_i\rangle| \geq \sqrt{t}$.*

Proof: We consider all vectors of the form $z(\varepsilon) = \sum_{i=1}^{t} \varepsilon_i x_i$, where $\varepsilon_i = \pm 1$. If $z = z(\overline{\varepsilon})$ has maximal length among them, the parallelogram law shows that $|z| \geq \sqrt{t}$. Also,

$$\sum_{i=1}^{t} |\langle z, x_i\rangle| \geq \sum_{i=1}^{t} \langle z, \overline{\varepsilon_i} x_i\rangle = |z|^2 \geq |z|\sqrt{t}. \tag{4}$$

Choosing $y = z/|z|$ we conclude the proof. □

Proof of Theorem 4.3.3: Assume that we can find t orthogonal transformations $u_1, \ldots, u_t$ such that $\frac{1}{t}\sum_{i=1}^{t} \|u_i x\| \leq 2M|x|$ for every $x \in \mathbb{R}^n$. We find $x_0 \in S^{n-1}$ with $\|x_0\| = b$ (minimal distance from the origin). It is clear that $1 = \|x_0\|_* \|x_0\| = b\|x_0\|_*$. We set $x_i = u_i^{-1} x_0$ and use the Lemma to find $y \in S^{n-1}$ such that $\sum_{i=1}^{t} |\langle y, x_i\rangle| \geq \sqrt{t}$. Then, we have

$$\sqrt{t} \leq \sum_{i=1}^{t} |\langle y, u_i^{-1} x_0\rangle| = \sum_{i=1}^{t} |\langle u_i y, x_0\rangle| \leq \|x_0\|_* \sum_{i=1}^{t} \|u_i y\| \leq \frac{2Mt}{b}. \tag{5}$$

This shows that $4t \geq (b/M)^2$. □

Combining Theorem 4.3.3 with the upper bound for $t(X)$ we obtain a second asymptotic formula:

Theorem 4.3.4. *For every finite dimensional normed space X we have*

$$t(X) \simeq (b/M)^2. \quad \square$$

Theorems 4.3.2 and 4.3.4 give a very precise asymptotic relation between a local and a global parameter of X [151]:

Fact. *There exists an absolute constant $c > 0$ such that*

$$\frac{1}{c}n \leq k(X)t(X) \leq cn$$

for every n-dimensional normed space X. □

4.4 Applications of the concentration of measure on the sphere

We used the concentration of measure on S^{n-1} for the proof of Dvoretzky's theorem. The same principle applies in very different situations. We shall demonstrate this by two more examples.

(a) **Banach-Mazur distance.** Recall that by John's theorem $d(X, \ell_2^n) \leq \sqrt{n}$ for every n-dimensional space X. Then, the multiplicative triangle inequality for d shows that $d(X,Y) \leq n$ for every pair of spaces X and Y. On the other hand, E.D. Gluskin [64] has proved that the diameter of the Banach-Mazur compactum is roughly equal to n:

There exists an absolute constant $c > 0$ such that for every n we can find two n-dimensional spaces X_n, Y_n with $d(X_n, Y_n) \geq cn$.

The spaces X_n, Y_n in Gluskin's example are random and of the same nature: random symmetric polytopes with αn vertices. We shall show that spaces whose unit balls are geometrically quite different objects have "small" distance [55]:

Theorem 4.4.1. *Let X and Y be two n-dimensional normed spaces such that $\#\mathrm{Extr}(K_X) \leq n^\alpha$ and $\#\mathrm{Extr}(K_{Y^*}) \leq n^\beta$ for some $\alpha, \beta > 0$, where $\#\mathrm{Extr}(\cdot)$ denotes the number of extreme points. Then,*

$$d(X,Y) \leq c\sqrt{\alpha+\beta}\sqrt{n\log n}.$$

[In other words, if a body has few extreme points and a second body has few facets, then their distance is not more than $\sqrt{n \log n}$.]

Proof: We may assume that $\frac{1}{\sqrt{n}}D_n \subseteq K_X \subseteq D_n \subseteq K_Y \subseteq \sqrt{n}D_n$. Then, $K_{Y^*} \subseteq D_n$. If $u \in O(n)$, it is clear that $\|u^{-1} : Y \to X\| \leq n$. We are going to show that $\|u : X \to Y\|$ is small for a random u.

We estimate the norm of u as follows:

$$\|u : X \to Y\| = \sup_{x \in K_X} \|ux\|_Y = \max_{x \in \mathrm{Extr}(K_X)} \max_{y^* \in \mathrm{Extr}(K_{Y^*})} |\langle ux, y^* \rangle|.$$

Observe that if $x \in \mathrm{Extr}(K_X)$ and $y^* \in \mathrm{Extr}(K_{Y^*})$, then $ux, y^* \in D_n$. It follows that

$$\mu(u \in O(n) : |\langle ux, y^* \rangle| \geq \varepsilon) \leq c\exp(-\varepsilon^2 n/2).$$

Therefore, if $cn^{\alpha+\beta}\exp(-\varepsilon^2 n/2) < 1$, we can find $u \in O(n)$ such that $\|u : X \to Y\| \leq \varepsilon$. Solving for ε we see that we can choose

$$\varepsilon \simeq \sqrt{\alpha+\beta}\sqrt{\log n/n}.$$

Hence, there exists $u \in O(n)$ for which

$$d(X,Y) \leq \|u : X \to Y\| \, \|u^{-1} : Y \to X\| \leq c\sqrt{\alpha+\beta}\sqrt{n \log n}. \quad \square$$

(b) **Random projections.** Let $1 \leq k \leq n$, and $E \in G_{n,k}$. A simple computation shows that

$$\int_{S^{n-1}} |P_E(x)|^2 \sigma(dx) = \frac{k}{n},$$

and since P_E is a 1-Lipschitz function, concentration of measure on the sphere shows that

$$\sigma\left(x \in S^{n-1} : \big| \, |P_E(x)| - \sqrt{k/n} \, \big| > \varepsilon\right) \leq c_1 \exp(-c_2\varepsilon^2 n)$$

for every $\varepsilon > 0$. Double integration and the choice $\varepsilon = \delta\sqrt{k/n}$ show that for any fixed subset $\{y_1, \ldots, y_N\}$ of S^{n-1} and any $\delta \in (0,1)$ we have

$$\nu_{n,k}\left(E \in G_{n,k} : (1-\delta)\sqrt{k/n} < |P_E(y_j)| < (1+\delta)\sqrt{k/n} \; , \; j \leq N\right)$$
$$> 1 - c_1 N \exp(-c_2\delta^2 k).$$

If $N \leq c_1^{-1}\exp(c_2\delta^2 k)$, then we can find a k-dimensional subspace E such that $|P_E(y_j)| \simeq \sqrt{\frac{k}{n}}$ for every $j \leq N$. It can be also arranged so that the distances of the y_j's will shrink in a uniform way under P_E (this application comes from [97]).

4.5 The concentration phenomenon: Lévy families

The concentration of measure on the sphere is just an example of the concentration phenomenon of invariant measures on high-dimensional structures. Assume that (X, d, μ) is a compact metric space with metric d and diameter $\mathrm{diam}(X) \geq 1$, which is also equipped with a Borel probability measure μ. We then define the *concentration function* (or "isoperimetric constant") of X by

$$\alpha(X;\varepsilon) = 1 - \inf\{\mu(A_\varepsilon) : A \text{ Borel subset of } X, \mu(A) \geq \frac{1}{2}\},$$

where $A_\varepsilon = \{x \in X : d(x,A) \leq \varepsilon\}$ is the ε-extension of A. As a consequence of the isoperimetric inequality on S^{n+1} we saw that

$$\alpha(S^{n+1};\varepsilon) \leq \sqrt{\pi/8}\exp(-\varepsilon^2 n/2),$$

an estimate which was crucial for the proof of Dvoretzky's theorem and the applications in Section 4.4.

P. Lévy [108] was the first to observe the role of the dimension in this particular example. For this reason, a family (X_n, d_n, μ_n) of metric probability spaces is called a *normal Lévy family with constants* (c_1, c_2) (see [84] and [9]) if

$$\alpha(X_n, \varepsilon) \leq c_1 \exp(-c_2\varepsilon^2 n),$$

or, more generally, a *Lévy family* if for every $\varepsilon > 0$

$$\alpha(X_n;\varepsilon) \to 0$$

as $n \to \infty$. There are many examples of Lévy families which have been discovered and used for local theory purposes. In most cases, new and very interesting techniques were invented in order to estimate the concentration function $\alpha(X;\varepsilon)$. We list some of them (and refer the reader to [175] in this volume for more information; see also [73], [74] for a development in a different direction):

(1) The family of the orthogonal groups $(SO(n), \rho_n, \mu_n)$ equipped with the Hilbert-Schmidt metric and the Haar probability measure is a Lévy family with constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/8$.

(2) The family $X_n = \prod_{i=1}^{m_n} S^n$ with the natural Riemannian metric and the product probability measure is a Lévy family with constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/2$.

(3) All homogeneous spaces of $SO(n)$ inherit the property of forming Lévy families. In particular, any family of Stiefel manifolds W_{n,k_n} or any family of Grassman manifolds G_{n,k_n} is a Lévy family with the same constants as in (1).

[All these examples of normal Lévy families come from [84].]

(4) The space $F^n = \{-1,1\}^n$ with the normalized Hamming distance $d(\eta,\eta') = \#\{i \leq n : \eta_i \neq \eta_i'\}/n$ and the normalized counting measure is a Lévy family with constants $c_1 = 1/2$ and $c_2 = 2$. This follows from an isoperimetric inequality of Harper [90], and was first put in such form and used in [8].

(5) The group Π_n of permutations of $\{1,\dots,n\}$ with the normalized Hamming distance $d(\sigma,\tau) = \#\{i \leq n : \sigma(i) \neq \tau(i)\}/n$ and the normalized counting measure satisfies $\alpha(\Pi_n;\varepsilon) \leq 2\exp(-\varepsilon^2 n/64)$. This was proved by Maurey [119] with a martingale method, which was further developed in [172].

• We shall give two more examples of situations where Lévy families are used. In particular, we shall complete the proof of the global form of Dvoretzky's theorem using the concentration phenomenon for products of spheres.

(*a*) **A topological application.** Let $1 \leq k \leq n$ and $V_k = \{(\xi,x) : \xi \in G_{n,k}, x \in S(\xi)\}$ be the canonical sphere bundle over $G_{n,k}$. Assume that $f : S^{n-1} \to \mathbb{R}$ is a Lipschitz function with the following property:

For every $\xi \in G_{n,k}$ we can find $x \in S(\xi)$ such that $f(x) = 0$.

One can easily check that V_k is a homogeneous space of $SO(n)$ whose concentration function satisfies

$$\alpha(V_k;\varepsilon) \leq \sqrt{\pi/8}\exp(-\varepsilon^2 n/8).$$

A standard argument shows that given $\delta > 0$, if $k \leq c\delta^2 n/\log(3/\delta)$ then we can find a subspace $\xi \in G_{n,k}$ and a δ-net $\mathcal{N}$ of $S(\xi)$ such that $f(x) = 0$ for every $x \in \mathcal{N}$. Assuming that the Lipschitz constant of f is not large, we get [84]:

There exists $\xi \in G_{n,k}$ such that $|f(x)| \leq c\delta$ for every $x \in S(\xi)$.

(b) **Global form of Dvoretzky's Theorem.** Recall that $t(X)$ is the least positive integer for which there exist $u_1, \ldots, u_t \in O(n)$ such that $\frac{1}{2}M|x| \leq \frac{1}{t}\sum_{i=1}^{t}\|u_i x\| \leq 2M|x|$ for every $x \in \mathbb{R}^n$.

We saw that $4t(X) \geq (b/M)^2$. We shall now prove the reverse inequality (which is stated in Theorem 4.3.4) following [118]:

Consider the space $\tilde{S}^t = \{\overline{x} = (x_1, \ldots, x_t) : x_i \in S^{n-1}\}$. Define $f(\overline{x}) = \frac{1}{t}\sum_{i=1}^{t}\|x_i\|$. Then, for every $\overline{x}, \overline{y} \in \tilde{S}^t$ we have:

$$|f(\overline{x}) - f(\overline{y})| \leq \frac{1}{t}\sum_{i=1}^{t}\|x_i - y_i\| \leq \left(\frac{1}{t}\sum_{i=1}^{t}\|x_i - y_i\|^2\right)^{1/2} \leq \frac{b}{\sqrt{t}}\rho(\overline{x}, \overline{y}).$$

The concentration estimate for products of spheres gives

$$\mathrm{Prob}\left(\Big|\ \frac{1}{t}\sum_{i=1}^{t}\|x_i\| - L_f\ \Big| > \delta L_f\right) \leq \exp(-c\delta^2 t L_f^2 n/b^2)$$

for every $\delta \in (0,1)$. Equivalently, if $x \in S^{n-1}$ then

$$(1-\delta)L_f \leq \frac{1}{t}\sum_{i=1}^{t}\|u_i x\| \leq (1+\delta)L_f$$

for all $(u_i)_{i\leq t}$ in a subset of $[O(n)]^t$ of measure greater than $1 - \exp(-c\delta^2 t L_f^2 n/b^2)$. If $\mathcal{N}$ is a δ-net for S^{n-1}, we can find $u_1, \ldots, u_t \in O(n)$ such that $\frac{1}{t}\sum\|u_i x\| \simeq L_f$ for all $x \in \mathcal{N}$, provided that $n/\log(3/\delta) \leq c\delta^2 t L_f^2 n/b^2$. We choose $\delta > 0$ small enough so that successive approximation will give $\frac{1}{t}\sum\|u_i x\| \simeq L_f$ for all $x \in S^{n-1}$, and we verify that the condition will be satisfied for some $t \leq c'(b/L_f)^2$. Since $M \simeq L_f$ up to a multiplicative constant, the proof is complete. $\square$

4.6 Dvoretzky's theorem and duality

4.6.1. Recall that if $X = (\mathbb{R}^n, \|\cdot\|)$ is a normed space, then the dual norm is defined by $\|x\|_* = \sup\{|\langle x, y\rangle| : \|y\| \leq 1\}$. It is clear that $\frac{1}{b}|x| \leq \|x\|_* \leq a|x|$, hence if we define $k^* = k(X^*)$ and $M^* = M(X^*)$ then Theorem 4.3.2 shows that

$$k^* \simeq n(M^*/a)^2.$$

On the other hand, it is a trivial consequence of the Cauchy-Schwarz inequality that

$$(1) \qquad MM^* \geq \left(\int_{S^{n-1}}\|x\|_*^{\frac{1}{2}}\|x\|^{\frac{1}{2}}\sigma(dx)\right)^2 \geq \left(\int_{S^{n-1}}|\langle x, x\rangle|^{\frac{1}{2}}\sigma(dx)\right)^2 = 1.$$

Multiplying the estimates for k and k^* we obtain

$$(2) \qquad kk^* \geq cn^2\frac{(MM^*)^2}{(ab)^2} \geq cn^2/(ab)^2.$$

Since we can always assume that $ab \leq \sqrt{n}$, we have proved:

Theorem 4.6.1. [61] *Let X be an n-dimensional normed space. Then,*

$$k(X)k(X^*) \geq cn. \quad \Box$$

This already shows that for every pair (X, X^*) at least one of the quantities k, k^* is greater than $c\sqrt{n}$. Recall that for $X = \ell_\infty^n$ we have $k(\ell_\infty^n) \simeq \log n$, therefore $k(\ell_1^n) \geq cn/\log n$ – almost proportional to n. In fact, a direct computation shows that $M(\ell_1^n) \simeq b(\ell_1^n) \simeq \sqrt{n}$, therefore $k(\ell_1^n) \simeq n$. Although $d(X, \ell_1^n)$ is the maximal possible, ℓ_1^n has Euclidean sections of dimension proportional to n.

4.6.2. Let $\overline{k} = \min\{k, k^*\}$. Since Dvoretzky's theorem holds for random subspaces of the appropriate dimension, we can find a subspace $E \in G_{n,\overline{k}}$ on which we have

$$\frac{1}{2}M|x| \leq \|x\| \leq 2M|x| \quad , \quad \frac{1}{2}M^*|x| \leq \|x\|_* \leq 2M^*|x| \tag{3}$$

simultaneously. This implies that $\|P_E : X \to E\| \leq 4MM^*$. We see this as follows: let $x \in \mathbb{R}^n$. Then,

$$|P_E(x)|^2 = \langle P_E(x), x\rangle \leq \|P_E(x)\|_*\|x\| \leq 2M^*|P_E(x)|\,\|x\|, \tag{4}$$

since $P_E(x) \in E$. For the same reason,

$$\|P_E(x)\| \leq 2M|P_E(x)| \leq 4MM^*\|x\|. \tag{5}$$

But then,

$$kk^* \simeq n^2\frac{(MM^*)^2}{(ab)^2} \geq cn^2\frac{\|P_E\|^2}{(ab)^2}, \tag{6}$$

which is a strengthening of Theorem 4.6.1 [61]. In the example of $X = \ell_\infty^n$ we know that $\overline{k} \simeq \log n$, therefore our estimate shows that for a random subspace $E(\log n)$ of dimension roughly equal to $\log n$ we must have

$$k(\ell_1^n)\log n \geq cn\|P_{E(\log n)}\|^2.$$

On the other hand, the norm of a random projection of ℓ_∞^n of rank $\log n$ is known to exceed $\sqrt{\log n}$, so we get the correct estimate $k(\ell_1^n) \geq cn$.

4.6.3. Another example where the preceding computation gives precise information on several parameters of X is the case $X = \ell_p^n, 1 < p < 2$. Let q be the conjugate exponent of p. We need the following result [43] (see also [149, pp. 22]):

Theorem 4.6.2. $k(\ell_q^n) \leq c(q)n^{2/q}$. $\Box$

It is a simple consequence of Hölder's inequality that $(ab)^2 \leq n^{1-\frac{2}{q}}$ for $X = \ell_p^n$. Our computation in 4.6.2 and Theorem 4.6.2 show that if $k = \min\{k(\ell_p^n), k(\ell_q^n)\}$, then

$$c(q)n^{2/q}k(\ell_p^n) \geq n^{1+\frac{2}{q}}\|P_{E(k)}\|^2. \tag{7}$$

Since $k(\ell_p^n) \leq n$ (!), we immediately get:

Theorem 4.6.3. *Let $1 < p < 2$ and q be its conjugate exponent. Then,*

$$k(\ell_p^n) \simeq n \quad , \quad k(\ell_q^n) \simeq \sqrt{q}n^{2/q} \quad , \quad d(\ell_p^n, \ell_2^n) = d(\ell_q^n, \ell_2^n) \simeq n^{\frac{1}{2}-\frac{1}{q}}. \quad \square$$

4.6.4. A combinatorial application. We saw that the $\log n$ estimate in Dvoretzky's theorem is optimal by studying the example of ℓ_∞^n. The argument we used for the cube shows something more general: Let P be a symmetric polytope, and denote its number of facets by $f(P)$ and its number of vertices by $v(P)$. Then, $k \leq \log f(P)$ and since $v(P) = f(P^\circ)$ we also get $k^* \leq \log v(P)$. We have seen that $kk^* \geq cn$, and this proves the following fact [61]:

Theorem 4.6.4. *Let P be a symmetric polytope in $\mathbb{R}^n$. Then,*

$$\log f(P) \log v(P) \geq cn. \quad \square$$

4.7 Isomorphic versions of Dvoretzky's Theorem

4.7.1. Bounded volume ratio. Let K be a body in $\mathbb{R}^n$. The *volume ratio* of K is the quantity

$$vr(K) = \inf\Big\{ \Big(\frac{|K|}{|E|} \Big)^{1/n} : E \subseteq K \Big\},$$

where the inf is over all ellipsoids contained in K. It is easily checked that $vr(K)$ is an affine invariant.

We shall show that if a body K has small volume ratio, then the space X_K has subspaces F of dimension proportional to n which are "well-isomorphic" to $\ell_2^{\dim F}$:

Theorem 4.7.2. *Let K be a body in $\mathbb{R}^n$ with $vr(K) = A$. Then, for every $k \leq n$ there exists a k-dimensional subspace F of X_K such that*

$$d(F, \ell_2^k) \leq (cA)^{\frac{n}{n-k}}.$$

Proof: We may assume that D_n is the maximal volume ellipsoid of K. Then, $\|x\| \leq |x|$ for every $x \in \mathbb{R}^n$. Given $k \leq n$, double integration shows that there exists $F \in G_{n,k}$ satisfying

$$(1) \qquad \int_{S^{n-1}\cap F} \|x\|^{-n} \sigma_k(dx) \leq vr(K)^n = A^n.$$

Then, Markov's inequality shows that for any $r > 0$, $\sigma_k\{x \in S^{n-1} \cap F : \|x\| < r\} \leq (rA)^n$. If we consider just one point x in $S^{n-1} \cap F$, then the $r/2$ neighbourhood of x with respect to $|\cdot|$ has σ_k measure greater than $(cr)^k$, for some absolute constant $c > 0$. This means that if $(rA)^n < (cr)^k$ then the set $\{x \in S^{n-1} \cap F : \|x\| \geq r\}$ is an $r/2$ net for $S^{n-1} \cap F$: if $y \in S^{n-1} \cap F$, we can find x with $|x - y| \leq r/2$ and $\|x\| \geq r$, and the triangle inequality shows that

$$(2) \qquad \|y\| \geq \|x\| - \|x - y\| \geq r - |x - y| \geq r/2.$$

This shows that $d(F,\ell_2^k)\le\frac{2}{r}$. Analyzing the necessary condition on r we obtain

(3) $$d(F,\ell_2^k)\le(cA)^{\frac{n}{n-k}}.\quad\square$$

Theorem 4.7.2 has its origin in the work of Kashin [100], who proved that there exist $c(\alpha)$-Euclidean subspaces of ℓ_1^n of dimension $[\alpha n]$, for every $\alpha\in(0,1)$. Szarek [180] realized the fact that bounded volume ratio is responsible for this property of ℓ_1^n, while the notion of volume ratio was formally introduced somewhat later in [187].

4.7.3. A natural question related to Dvoretzky's theorem is to give an estimate for

$$\max_{\dim X=n}\min\{d(F,\ell_2^k):F\subset X,\dim F=k\}.$$

for each $1\le k\le n$. Such an "isomorphic" version was proved by Milman and Schechtman [150] who showed the following:

Theorem 4.7.4. *There exists an absolute constant $C>0$ such that, for every n and every $k\ge C\log n$, every n-dimensional normed space X contains a k-dimensional subspace F for which*

$$d(F,\ell_2^k)\le C\sqrt{k/\log(n/k)}.\quad\square$$

For an extension to the non-symmetric case, see [75], [86].

5 The Low M^*-estimate and the Quotient of Subspace Theorem

5.1 The Low M^*-estimate

Dvoretzky's theorem gives very strong information about the Euclidean structure of k-dimensional subspaces of an arbitrary n-dimensional space when their dimension k is up to the order of $\log n$. In some cases one can find Euclidean subspaces of dimension even proportional to n, but no "proportional theory" can be expected in such a strong sense. However, surprisingly enough, there is non trivial Euclidean structure in subspaces of dimension λn, $\lambda\in(0,1)$, even for λ very close to 1. The first step in this direction is the Low M^*-estimate:

Theorem 5.1.1. *There exists a function $f:(0,1)\to\mathbb{R}^+$ such that for every $\lambda\in(0,1)$ and every n-dimensional normed space X, a random subspace $E\in G_{n,[\lambda n]}$ satisfies*

(1) $$\frac{f(\lambda)}{M^*}|x|\le\|x\|\quad,\quad x\in E,$$

where $c>0$ is an absolute constant.

Theorem 5.1.1 was originally proved in [132] and a second proof using the isoperimetric inequality on S^{n-1} was given in [133], where (1) was shown to hold with $f(\lambda) \geq c(1-\lambda)$ for some absolute constant $c > 0$ (and with an estimate $f(\lambda) \geq 1-\lambda+o(\lambda)$ as $\lambda \to 0^+$). This was later improved to $f(\lambda) \geq c\sqrt{1-\lambda}$ in [164] (see also [139] for a different proof with this best possible $\sqrt{1-\lambda}$ dependence). Finally, it was proved in [69] that one can have

$$f(\lambda) \geq \sqrt{1-\lambda}\left(1+O(\frac{1}{(1-\lambda)n})\right). \tag{2}$$

Geometrically speaking, Theorem 5.1.1 says that for a random λn-dimensional section of K_X we have

$$K_X \cap E \subseteq \frac{M^*}{f(\lambda)} D_n \cap E, \tag{3}$$

that is, the diameter of a random section of a body of dimension proportional to n is controlled by the mean width M^* of the body (a random section does not feel the diameter a of K_X but the radius M^* which is roughly the level r at which half of the supporting hyperplanes of rD_n cut the body K_X).

The dual formulation of the theorem has an interesting geometric interpretation. A random λn-dimensional projection of K_X contains a ball of radius of the order of $1/M$. More precisely, for a random $E \in G_{n,\lambda n}$ we have

$$P_E(K_X) \supseteq \frac{f(\lambda)}{M} D_n \cap E. \tag{4}$$

We shall present the proof from [133] which gives linear dependence in λ and is based on the isoperimetric inequality for S^{n-1}:

Proof of the Low M^*-estimate: Consider the set $A = \{y \in S^{n-1} : \|y\|_* \leq 2M^*\}$. We obviously have $\sigma(A) \geq \frac{1}{2}$.

CLAIM:*For every $\lambda \in (0,1)$ there exists a subspace E of dimension $k = [\lambda n]$ such that*

$$E \cap S^{n-1} \subseteq A_{(\frac{\pi}{2}-\delta)}, \tag{5}$$

where $\delta \geq c(1-\lambda)$.

Proof of the claim: We have $\sigma(A_{\pi/4}) \geq 1 - c\sqrt{n}\int_0^{\pi/4} \sin^{n-2} t dt$, and double integration through $G_{n,k}$ shows that a random $E \in G_{n,k}$ satisfies

$$\sigma_k(A_{\pi/4} \cap E) \geq 1 - c\sqrt{n}\int_0^{\pi/4} \sin^{n-2} t dt. \tag{6}$$

On the other hand, for every $x \in S^{n-1} \cap E$ we have

$$\sigma_k(B(x, \frac{\pi}{4} - \delta)) \simeq \sqrt{k}\int_0^{\frac{\pi}{4}-\delta} \sin^{k-2} t dt. \tag{7}$$

This means that if

$$\sqrt{\lambda}\int_0^{\frac{\pi}{4}-\delta}\sin^{k-2} t dt \simeq \int_0^{\frac{\pi}{4}}\sin^{n-2} t dt, \tag{8}$$

then $A_{\pi/4}\cap B(x,\frac{\pi}{4}-\delta)\neq\emptyset$, and hence $x\in A_{\frac{\pi}{2}-\delta}$. Analyzing the sufficient condition (8) we see that we can choose $\delta\geq c(1-\lambda)$. $\square$

We complete the proof of Theorem 5.1.1 as follows: Let $x\in S^{n-1}\cap E$. There exists $y\in A$ such that

$$\sin\delta\leq|\langle x,y\rangle|\leq\|y\|_*\|x\|\leq 2M^*\|x\|, \tag{9}$$

and since $\sin\delta\geq\frac{2}{\pi}\delta\geq c'(1-\lambda)$, the theorem follows. $\square$

5.2 The ℓ-position

Let X be an n-dimensional normed space. Figiel and Tomczak-Jaegermann [60] defined the ℓ-norm of $T\in L(\ell_2^n,X)$ by

$$\ell(T)=\sqrt{n}\left(\int_{S^{n-1}}\|Ty\|^2\sigma(dy)\right)^{1/2}. \tag{1}$$

Alternatively, if $\{e_j\}$ is any orthonormal basis in $\mathbb{R}^n$, and if $g_1,\dots,g_n$ are independent standard Gaussian random variables on some probability space Ω, we have

$$\ell(T)=\left(\mathbb{E}\ \|\sum_{i=1}^n g_iT(e_i)\ \|\right)^{1/2}, \tag{2}$$

where $\mathbb{E}$ denotes expectation.

Let now Rad_nX be the subspace of $L_2(\Omega,X)$ consisting of functions of the form $\sum_{i=1}^n g_i(\omega)x_i$ where $x_i\in X$ (the notation here is perhaps not canonical, but convenient). The natural projection from $L_2(\Omega,X)$ onto Rad_nX is defined by

$$\mathrm{Rad}_nf=\sum_{i=1}^n\left(\int_\Omega g_if\right)g_i. \tag{3}$$

We write $\|\mathrm{Rad}_n\|_X$ for the norm of this projection as an operator in $L_2(\Omega,X)$.

The dual norm ℓ^* is defined on $L(X,\ell_2^n)$ by

$$\ell^*(S)=\sup\{\ \mathrm{tr}(ST):T\in L(\ell_2^n,X),\ell(T)\leq 1\}. \tag{4}$$

From a general result of Lewis [109] it follows that for some $T\in L(\ell_2^n,X)$ one has $\ell(T)\ell^*(T^{-1})=n$. Using this fact, Figiel and Tomczak-Jaegermann proved that for every n-dimensional space X there exists $T:\ell_2^n\to X$ such that

$$\ell(T)\ell((T^{-1})^*)\leq n\|\mathrm{Rad}_n\|_X. \tag{5}$$

The norm of the projection Rad_n was estimated by Pisier [159]: For every n-dimensional space X,

$$\|\mathrm{Rad}_n\|_X \leq c\log[d(X,\ell_2^n)+1]. \tag{6}$$

This implies that for every $X=(\mathbb{R}^n,\|\cdot\|)$ we can define a Euclidean structure $\langle\cdot,\cdot\rangle$ (called the ℓ-structure) on $\mathbb{R}^n$, for which

$$M(X)M^*(X)\leq c\log[d(X,\ell_2^n)+1]. \tag{7}$$

Equivalently, we can state the following theorem:

Theorem 5.2.1. *Let K be a symmetric convex body in $\mathbb{R}^n$. There exists a position $\tilde{K}$ of K for which*

$$M(\tilde{K})M^*(\tilde{K})\leq c\log[d(X_K,\ell_2^n)+1], \tag{8}$$

where $c>0$ is an absolute constant. □

Pisier's argument uses symmetry in an essential way, therefore one cannot transfer directly this line of thinking to the non-symmetric case. For recent progress on the non-symmetric MM^*-estimate, see Appendix 7.2.

5.3 The quotient of subspace theorem

The quotient of subspace theorem [134] states that by performing two operations on an n-dimensional space, taking first a subspace and then a quotient of it, we can always arrive at a new space of dimension proportional to n which is (independently of n) close to Euclidean:

Theorem 5.3.1.(Milman) *Let X be an n-dimensional normed space and $\alpha\in[\frac{1}{2},1)$. Then, there exist subspaces $E\supset F$ of X for which*

$$k=\dim(E/F)\geq\alpha n \quad , \quad d(E/F,\ell_2^k)\leq c(1-\alpha)^{-1}|\log(1-\alpha)|. \tag{1}$$

Geometrically, this means that for every body K in $\mathbb{R}^n$ and any $\alpha\in[\frac{1}{2},1)$, we can find subspaces $G\subset E$ with $\dim G\geq\alpha n$ and an ellipsoid $\mathcal{E}$ such that

$$\mathcal{E}\subset P_G(K\cap E)\subset c(1-\alpha)^{-1}|\log(1-\alpha)|\mathcal{E}. \tag{2}$$

The proof of the theorem is based on the Low M^*-estimate and an iteration procedure which makes essential use of the ℓ-position.

Proof: We may assume that K_X is in ℓ-position: then, by Theorem 5.2.1 we have $M(X)M^*(X)\leq c\log[d(X,\ell_2^n)+1]$.

Step 1: Let $\lambda\in(0,1)$. We shall show that there exist a subspace E of X, $\dim E\geq\lambda n$, and a subspace F of E^*, $\dim F=k\geq\lambda^2 n$, such that $d(F,\ell_2^k)\leq c(1-\lambda)^{-1}\log[d(X,\ell_2^n)+1]$.

The proof of this fact is a double application of the Low M^*-estimate. By Theorem 5.1.1, a random λn-dimensional subspace E of X satisfies

$$\frac{c_1\sqrt{1-\lambda}}{M^*(X)}|x| \leq \|x\| \leq b|x| \quad , \quad x \in E. \tag{3}$$

Moreover, since (3) holds for a random $E \in G_{n,\lambda n}$, we may also assume that $M(E) \leq c_2 M(X)$. Therefore, repeating the same argument for E^*, we may find a subspace F of E^* with $\dim F = k \geq \lambda^2 n$ and

$$\frac{c_3\sqrt{1-\lambda}}{M(X)}|x| \leq \frac{c_1\sqrt{1-\lambda}}{M^*(E^*)}|x| \leq \|x\|_{E^*} \leq \frac{M^*(X)}{c_1\sqrt{1-\lambda}}|x| \tag{4}$$

for every $x \in F$. Since K_X is in ℓ-position, we obtain

$$d(F, \ell_2^k) \leq c_4(1-\lambda)^{-1} M(X)M^*(X) \leq c(1-\lambda)^{-1}\log[d(X,\ell_2^n)+1]. \tag{5}$$

STEP 2: Denote by $QS(X)$ the class of all quotient spaces of a subspace of X, and define a function $f : (0,1) \to \mathbb{R}^+$ by

$$f(\alpha) = \inf\{d(F,\ell_2^k) : F \in QS(X), \dim F \geq \alpha n\}. \tag{6}$$

Then, what we have really proved in Step 1 is the estimate

$$f(\lambda^2\alpha) \leq c(1-\lambda)^{-1}\log f(\alpha). \tag{7}$$

An iteration lemma (see [134] or [162, pp. 130]) allows us to conclude that

$$f(\alpha) \leq c(1-\alpha)^{-1}|\log(1-\alpha)|. \quad \square$$

5.4 Variants and applications of the Low M^*-estimate

1. An almost direct consequence of the Low M^*-estimate is the existence of a function $f : (0,1) \to \mathbb{R}^+$ with the following property [141]:

If K is a body in $\mathbb{R}^n$ and if $\lambda \in (0,1)$, then a random λn-dimensional section $K \cap F$ of K satisfies $\mathrm{diam}(K \cap F) \leq 2r$, *where r is the solution of the equation*

$$M^*(K \cap rD_n) = f(\lambda)r. \tag{1}$$

One can choose $f(\lambda) = (1-\varepsilon)\sqrt{1-\lambda}$ for any $\varepsilon \in (0,1)$, and then (1) is satisfied for all F in a subset of $G_{n,[\lambda n]}$ of measure greater than $1 - c_1\exp(-c_2\varepsilon^2(1-\lambda)n)$.

2. Let $t(r) = t(X_K; r)$ be the greatest integer k for which a random subspace $F \in G_{n,k}$ satisfies $\mathrm{diam}(K \cap F) \leq 2r$. The following linear duality relation was proved in [140]:

If $t^*(r) = t(X^*; r)$, then for any $\zeta > 0$ and any $r > 0$ we have

$$t(r) + t^*\left(\frac{1}{\zeta r}\right) \geq (1-\zeta)n - C, \tag{2}$$

where $C > 0$ is an absolute constant.

This surprisingly precise connection between the structure of proportional sections of a body and its polar is also expressed as follows [81]:

Let $\zeta > 0$ and k, l be integers with $k + l \leq (1-\zeta)n$. Then, for every body K in $\mathbb{R}^n$ we have

$$\int_{G_{n,k}} M^*(K \cap F)d\nu_{n,k}(F) \int_{G_{n,l}} M^*(K^\circ \cap F')d\nu_{n,l}(F') \leq \frac{C}{\zeta}, \tag{3}$$

where $C > 0$ is an absolute constant.

3. An estimate dual to (1) was established in [79]. There exists a second function $g : (0,1) \to \mathbb{R}$ such that: for every body K in $\mathbb{R}^n$ and every $\lambda \in (\frac{1}{2}, 1)$, a random λn-dimensional section $K \cap F$ of K satisfies $\mathrm{diam}(K \cap F) \geq 2r$, where r is the solution of the equation

$$M^*(K \cap rD_n) = g(\lambda)r. \tag{4}$$

This double sided estimate provided by (1) and (4) may be viewed as an (incomplete) asymptotic formula for the diameter of random proportional sections of K, which is of interest from the computational geometry point of view since the function $r \to M^*(K \cap rD_n)$ is easily computable.

4. The diameter of proportional dimensional sections of K is connected with the following global parameter of K: For every integer $t \geq 2$ we define $r_t(K)$ to be the smallest $r > 0$ for which there exist rotations $u_1, \ldots, u_t$ such that $u_1(K) \cap \ldots \cap u_t(K) \subseteq rD_n$.

If $R_t(K)$ is the smallest $R > 0$ for which most of the $[n/t]$-dimensional sections of K satisfy $\mathrm{diam}(K \cap F) \leq 2R$, then it is proved in [141] that $r_{2t}(K) \leq \sqrt{t}R_t(K)$. The fact that a reverse comparison of these two parameters is possible was established in [80]: There exists an absolute constant $C > 1$ such that

$$R_{C^t}(K) \leq C^t r_t(K) \tag{5}$$

for every $t \geq 2$.

5. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. Then, for every non empty $\sigma \subseteq \{1, \ldots, n\}$ we define the coordinate subspace $\mathbb{R}^\sigma = \mathrm{span}\{e_j : j \in \sigma\}$.

We are often interested in analogues of the Low M^*-estimate with the additional restriction that the subspace E should be a coordinate subspace of a given proportional dimension (see [63] for applications to Dvoretzky-Rogers factorization questions). Such estimates are sometimes possible [78]:

If K is an ellipsoid in $\mathbb{R}^n$, then for every $\lambda \in (0,1)$ we can find $\sigma \subseteq \{1, \ldots, n\}$ of cardinality $|\sigma| \geq (1-\lambda)n$ such that

$$P_{\mathbb{R}^\sigma}(K) \supseteq \frac{[\lambda/\log(1/\lambda)]^{1/2}}{M_K} D_n \cap \mathbb{R}^\sigma. \tag{6}$$

Analogues of this hold true if the volume ratio of K or the cotype-2 constant of X_K is small.

Finally, let us mention that Bourgain's solution of the $\Lambda(p)$ problem [23] (see also [189] and [25]) is closely related to the following "coordinate" result:

Let $(\phi_i)_{i\leq n}$ be a sequence of functions on $[0,1]$ which is orthogonal in L_2. If $\|\phi_i\|_\infty \leq 1$ and $\|\phi_i\|_2 \geq c > 0$ for every $i \leq n$, then for every $p > 2$ most of the subsets $\sigma \subseteq \{1,\dots,n\}$ of cardinality $[n^{2/p}]$ satisfy

$$c\left(\sum_{i\in\sigma} t_i^2\right)^{1/2} \leq \Big\| \sum_{i\in\sigma} t_i\phi_i \Big\|_p \leq K(p)\left(\sum_{i\in\sigma} t_i^2\right)^{1/2} \tag{7}$$

for every choice of reals $(t_i)_{i\in\sigma}$. We refer the reader to the article [99] in this collection for the results of Bourgain-Tzafriri on restricted invertibility, which are closely related to the above.

6 Isomorphic symmetrization and applications to classical convexity

6.1 Estimates on covering numbers

Let K_1 and K_2 be convex bodies in $\mathbb{R}^n$. The *covering number* $N(K_1,K_2)$ of K_1 by K_2 is the least positive integer N for which there exist $x_1,\dots,x_N \in \mathbb{R}^n$ such that

$$K_1 \subseteq \bigcup_{i=1}^{N}(x_i + K_2). \tag{1}$$

We shall formulate and sketch the proofs of a few important results on covering numbers which we need in the next section.

The well known Sudakov inequality [179] estimates $N(K,tD_n)$:

Theorem 6.1.1. *Let K be a body in $\mathbb{R}^n$. Then,*

$$N(K,tD_n) \leq \exp(cn(M^*/t)^2) \tag{2}$$

for every $t>0$, where $c>0$ is an absolute constant.

The dual Sudakov inequality, proved by Pajor and Tomczak-Jaegermann [163], gives an upper bound for $N(D_n,tK)$:

Theorem 6.1.2. *Let K be a symmetric convex body in $\mathbb{R}^n$. Then,*

$$N(D_n,tK) \leq \exp(cn(M/t)^2) \tag{3}$$

for every $t>0$, where $c>0$ is an absolute constant.

We shall give a simple proof of Theorem 6.1.2 which is due to Talagrand (see [115, pp. 82]).

Proof of Theorem 6.1.2: We consider the standard Gaussian probability measure γ_n on $\mathbb{R}^n$, with density

$$d\gamma_n = (2\pi)^{-n/2}\exp(-|x|^2/2)dx.$$

A direct computation shows that $\int \|x\| d\gamma_n(x) = \alpha_n M$, where $\alpha_n/\sqrt{n} \to 1$ as $n \to \infty$. Markov's inequality shows that

$$\gamma_n(x : \|x\| \leq 2M\alpha_n) \geq \frac{1}{2}. \tag{4}$$

Let $\{x_1, \dots, x_N\}$ be a subset of D_n which is maximal under the requirement that $\|x_i - x_j\| \geq t$, $i \neq j$. Then, the sets $x_i + \frac{t}{2}K$ have disjoint interiors. The same holds true for the sets $y_i + 2M\alpha_n K$, $y_i = (4M\alpha_n/t)x_i$. Therefore,

$$\sum_{i=1}^{N} \gamma_n(y_i + 2M\alpha_n K) \leq 1. \tag{5}$$

Using the convexity of e^{-s}, the symmetry of K and (4), we can then estimate $\gamma_n(y_i + 2M\alpha_n K)$ from below as follows:

$$\gamma_n(y_i + 2M\alpha_n K) \geq \frac{1}{2}\exp(-(4M\alpha_n/t)^2). \tag{6}$$

Now, (5) shows that

$$N \leq 2\exp((4M\alpha_n/t)^2), \tag{7}$$

and since $\alpha_n \simeq \sqrt{n}$ we conclude the proof. □

Sudakov's inequality (Theorem 6.1.1) can be deduced from Theorem 6.1.2 with a duality argument of Tomczak-Jaegermann [194]: Let

$$A = \sup_{t>0} t(\log N(D_n, tK^\circ))^{1/2}. \tag{8}$$

We check that $2K \cap (\frac{t^2}{2}K^\circ) \subseteq tD_n$ for every $t > 0$, and this implies that

$$\begin{aligned} N(K, tD_n) &\leq N(K, 2K \cap (\frac{t^2}{2}K^\circ)) = N(K, \frac{t^2}{4}K^\circ) \\ &\leq N(K, 2tD_n)N(D_n, \frac{t}{8}K^\circ). \end{aligned} \tag{9}$$

This shows that

$$t(\log N(K, tD_n))^{1/2} \leq t(\log N(K, 2tD_n))^{1/2} + 8A, \tag{10}$$

from which we easily get

$$\sup_{t>0} t(\log N(K, tD_n))^{1/2} \leq 16A. \tag{11}$$

This is equivalent to the assertion of Theorem 6.1.1 (just observe that $M^*(K) = M(K^\circ)$). $\square$

A weaker version of Sudakov's inequality can be proved if we use Urysohn's inequality: For every body K and any $t > 0$, we have

$$N(K, tD_n) \leq \exp(2nM^*/t). \tag{12}$$

Proof: Consider a set $\{x_1, \ldots, x_N\} \subset K$ which is maximal under the requirement $\mathrm{int}(x_i + \frac{t}{2}D_n) \cap \mathrm{int}(x_j + \frac{t}{2}D_n) = \emptyset$. Then,

$$N(K, tD_n) \leq N \leq \frac{|K + \frac{t}{2}D_n|}{|\frac{t}{2}D_n|} = \left(\frac{2}{t}\right)^n \frac{|K + \frac{t}{2}D_n|}{|D_n|}, \tag{13}$$

and Urysohn's inequality shows that

$$N(K, tD_n) \leq \left(\frac{2}{t}\right)^n (M^*(K + (t/2)D_n))^n \tag{14}$$

$$= \left(\frac{2}{t}\right)^n \left(M^* + \frac{t}{2}\right)^n = \left(1 + \frac{2M^*}{t}\right)^n \leq \exp(2nM^*/t). \quad \square$$

Using the covering numbers one can compare volumes of convex bodies in various situations. A main ingredient of the proof of the lemmas below (which may be found in [138]) is the Brunn-Minkowski inequality:

Lemma 1. *Let K, T, and P be symmetric convex bodies in $\mathbb{R}^n$. Then,*

$$|K \cap (T + x) + P| \leq |K \cap T + P| \tag{15}$$

for every $x \in \mathbb{R}^n$.

Proof: Let $T_x = K \cap (T + x) + P$. We easily check that $T_x + T_{-x} \subseteq 2T_0$, and then apply the Brunn-Minkowski inequality. $\square$

Lemma 2. *Let K and P be symmetric convex bodies in $\mathbb{R}^n$. If $t > 0$, then*

$$|K + P| \leq N(K, tD_n)|(K \cap tD_n) + P|. \tag{16}$$

Proof: If $K \subseteq \bigcup_{i \leq N} K \cap (x_i + tD_n)$, then $K + P \subseteq \bigcup_{i \leq N}[(x_i + tD_n) \cap K + P]$. We compare volumes using the information from Lemma 1. $\square$

Lemma 3. *Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Assume that $L \subseteq bK$ for some $b \geq 1$. Then,*

$$N\left(\mathrm{co}(K \cup L), (1 + \frac{1}{n})K\right) \leq 2bnN(L, K). \quad \square \tag{17}$$

Using Lemma 3 with $L = \frac{1}{t}D_n$ and combining with Lemma 2, we have:

Lemma 4. *Let K and P be symmetric convex bodies in $\mathbb{R}^n$. Assume that $D_n \subseteq tbK$ for some $t > 0$. Then,*

$$|\mathrm{co}(K \cup (1/t)D_n) + P) \leq 2ebnN(D_n, tK)|K + P|. \quad \square \tag{18}$$

6.2 Isomorphic symmetrization and applications to classical convexity

The functional analytic approach and the methods of local theory lead to new isomorphic geometric inequalities. In this way, the ideas we described in previous sections find applications to classical convexity theory in $\mathbb{R}^n$. We shall describe two results in this direction:

6.2.1. The inverse Blaschke-Santaló inequality[32] *There exists an absolute constant $c > 0$ such that*

$$0 < c \leq \left(\frac{|K||K^\circ|}{|D_n||D_n|} \right)^{\frac{1}{n}} \leq 1 \tag{1}$$

for every body in $\mathbb{R}^n$.

The inequality on the right is the Blaschke-Santaló inequality: the volume product $s(K) = |K||K^\circ|$ is maximized (among symmetric convex bodies) exactly when K is an ellipsoid. A well-known conjecture of Mahler states that $s(K) \geq 4^n/n!$ for every K. This has been verified for some classes of bodies, e.g. zonoids and 1-unconditional bodies (see [165], [128], [171], [87]). The left handside inequality comes from [32] and answers the question of Mahler in the asymptotic sense: For every body K, the affine invariant $s(K)^{1/n}$ is of the order of $1/n$.

6.2.2. The inverse Brunn-Minkowski inequality[135] *There exists an absolute constant $C > 0$ with the following property: For every body K in $\mathbb{R}^n$ there exists an ellipsoid M_K such that $|K| = |M_K|$ and for every body T in $\mathbb{R}^n$*

$$\frac{1}{C}\big|M_K + T\big|^{1/n} \leq \big|K + T\big|^{1/n} \leq C\big|M_K + T\big|^{1/n}. \tag{2}$$

This implies that for every body K in $\mathbb{R}^n$ there exists a position $\tilde{K} = u_K(K)$ of volume $|\tilde{K}| = |K|$ such that the following reverse Brunn-Minkowski inequality holds true:

"If K_1 and K_2 are bodies in $\mathbb{R}^n$, then

$$|t_1\tilde{K_1} + t_2\tilde{K_2}|^{1/n} \leq C\left(t_1|\tilde{K_1}|^{1/n} + t_2|\tilde{K_2}|^{1/n}\right), \tag{3}$$

for all $t_1, t_2 > 0$, where $C > 0$ is an absolute constant".

The ellipsoid M_K in 6.2.2 is called an *M-ellipsoid* for K. Analogously, the body $\tilde{K} = u_K(K)$ is called an *M-position* of K (and then, one may take $M_{\tilde{K}} = \rho D_n$). The symmetry of K is not really needed in 6.2.1 and 6.2.2 (see e.g. [147]).

Both results were originally proved by a dimension descending procedure which was based on the quotient of subspace theorem. We shall present a second approach, which appeared in [138] and introduced an "isomorphic symmetrization" technique. This is a symmetrization scheme which is in many ways different from the classical symmetrizations. In each step, none of the natural parameters of the body is being preserved, but the ones which are of interest remain under control. After a finite number of steps, the body has come close to an ellipsoid and this is sufficient for our purposes, but there is no natural notion of *convergence* to an ellipsoid.

6.2.3. Remarks. Applying (2) for $T = M_K$ we get

$$|K + M_K|^{1/n} \leq C|K|^{1/n}. \tag{4}$$

This is equivalent to Theorem 6.2.2 and to each one of the following statements:

(i) *There exists a constant $C > 0$ such that for every body K we can find an ellipsoid M_K with $|M_K| = |K|$ and*

$$N(K, M_K) \leq \exp(Cn).$$

(ii) *There exists a constant $C > 0$ such that for every body K we can find an ellipsoid M_K with $|M_K| = |K|$ and*

$$N(M_K, K) \leq \exp(Cn).$$

We can also pass to polars and show that for every body T in $\mathbb{R}^n$,

$$\frac{1}{C}|M_K^\circ + T|^{1/n} \leq |K^\circ + T|^{1/n} \leq C|M_K^\circ + T|^{1/n}.$$

Since the M-position is isomorphically defined, one may ask for stronger regularity on the covering numbers estimates (i) and (ii): Pisier proved (see [162, Chapter 7]) that, for every $\alpha > 1/2$ and every body K there exists an affine image $\tilde{K}$ of K which satisfies $|\tilde{K}| = |D_n|$ and

$$\max\{N(K, tD_n), N(D_n, tK), N(K^\circ, tD_n), N(D_n, tK^\circ)\} \leq \exp(c(\alpha)nt^{-1/\alpha})$$

for every $t \geq 1$, where $c(\alpha)$ is a constant depending only on α, with $c(\alpha) = O((\alpha - \frac{1}{2})^{-1/2})$ as $\alpha \to \frac{1}{2}$. We then say that K is in *M-position of order α* (*α-regular* in the terminology of [162]).

Proof of the Theorems: Since $s(K)$ is an affine invariant, we may assume that K is in a position such that $M(K)M^*(K) \leq c\log[d(X_K, \ell_2^n)+1]$. We may also normalize so that $M(K) = 1$. We define

$$\lambda_1 = M^*(K)a_1 \quad , \quad \lambda_1' = M(K)a_1, \tag{5}$$

for some $a_1 > 1$, and consider the new body

$$K_1 = \mathrm{co}[(K \cap \lambda_1 D_n) \cup \frac{1}{\lambda_1'} D_n]. \tag{6}$$

Using Sudakov's inequality and Lemma 2 with $P = \{0\}$, we see that

$$|K_1| \geq |K \cap \lambda_1 D_n| \geq |K|/N(K, \lambda_1 D_n) \geq |K|\exp(-cn/a_1^2), \tag{7}$$

while using the dual Sudakov inequality and Lemma 3 we get

$$(8)\qquad |K_1| \leq |\mathrm{co}(K \cup \frac{1}{\lambda_1'} D_n)| \leq 2e \frac{b}{\lambda_1'} n N(D_n, \lambda_1' K)|K| \leq \exp(cn/a_1^2).$$

The same computation can be applied to K_1°, and this shows that

$$(9)\qquad \exp(-cn/a_1^2) \leq \frac{s(K_1)}{s(K)} \leq \exp(cn/a_1^2).$$

We continue in the same way. We now know that $d(X_{K_1}, \ell_2^n) \leq M(K)M^*(K)a_1^2$ and, since $s(K_1)$ is an affine invariant, we may assume that $M(K_1)M^*(K_1) \leq c\log[d(X_{K_1}, \ell_2^n) + 1]$ and $M(K_1) = 1$. We then define

$$(10)\qquad \lambda_2 = M^*(K_1)a_2 \quad , \quad \lambda_2' = M(K_1)a_2,$$

and consider the body $K_2 = \mathrm{co}[(K_1 \cap \lambda_2 D_n) \cup \frac{1}{\lambda_2'} D_n]$. Estimating volumes, we see that

$$(11)\qquad \exp(-cn/a_2^2) \leq \frac{s(K_2)}{s(K_1)} \leq \exp(cn/a_2^2).$$

We iterate this scheme, choosing $a_1 = \log n$, $a_2 = \log\log n, \ldots, a_t = \log^{(t)} n$ – the t-iterated logarithm of n, and stop the procedure at the first t for which $a_t < 2$. It is easy to check that $d(X_{K_t}, \ell_2^n) \leq C$, therefore

$$(12)\qquad \frac{1}{C} \leq s(K_t)^{1/n} \leq C.$$

On the other hand, combining our volume estimates we see that

$$(13)\qquad c_1 \leq \exp(-c(\frac{1}{a_1^2} + \ldots + \frac{1}{a_t^2})) \leq \frac{s(K_t)^{1/n}}{s(K)^{1/n}} \leq \exp(c(\frac{1}{a_1^2} + \ldots + \frac{1}{a_t^2})),$$

which proves Theorem 6.1.1 since the series $\frac{1}{a_1^2} + \ldots + \frac{1}{a_t^2} + \ldots$ remains bounded by an absolute constant. □

The proof of Theorem 6.2.2 follows the same pattern. In each step, we verify that for every convex body T

$$(14)\qquad \exp(-cn/a_s^2) \leq \frac{|K_s + T|}{|K_{s-1} + T|} \leq \exp(cn/a_s^2),$$

and the same holds true for K_s°. At the t-th step, we arrive at a body K_t which is C-isomorphic to an ellipsoid M, and (14) shows that $|K_t|^{1/n} \simeq |K|^{1/n}$ up to an absolute constant. If we define $M_K = \rho M$ where $\rho > 0$ is such that $|M_K| = |K|$, then $\rho \simeq 1$ and using (14) we conclude the proof. □

Note. The existence of the M-ellipsoid M_K of K in the non-symmetric case was established in [147]. The key lemma is the observation that if 0 is the centroid of the convex body K, then $|K \cap (-K)| \geq 2^{-n}|K|$.

We close this section with a few geometric consequences of the M-position:

1. Every body K has a position $\tilde{K}$ with the following property: there exist $u, v \in SO(n)$ such that if we set $P = \tilde{K} + u(\tilde{K})$ and $Q = P^\circ + v(P^\circ)$, then Q is equivalent to a Euclidean ball up to an absolute constant. Actually, this statement is satisfied for a random pair $(u, v) \in SO(n) \times SO(n)$. This double operation may be called *isomorphic Euclidean regularization.*

Compare with the following examples: If K is the unit cube, then P is already equivalent to a ball for most $u \in SO(n)$ (this follows from [100], see 4.7.1). If K is the unit ball of ℓ_1^n, the second operation is certainly needed.

A closely related result from [141] is the following isomorphic inequality connecting K with K°:

Let $\rho_t(K) = \max\{\rho > 0 : \rho D_n \subset \frac{1}{t}\sum_{i=1}^t u_i(K) \ , \ u_i \in O(n)\}$. Then, there exists an absolute constant $c > 0$ such that

$$\rho_2(K)\rho_3(K^\circ) \geq c$$

for every body K in $\mathbb{R}^n$. Observe that Kashin's result is a consequence of this fact: if K is the cube, then $\rho_3(K^\circ) \leq c/\sqrt{n}$. Therefore, $K + u(K) \supset c\sqrt{n}D_n$ for some $u \in O(n)$. It is not clear if *two* rotations of K° suffice for a similar statement.

2. One may use the M-position in order to obtain a random version of the quotient of subspace theorem: If K is in M-position, then using Remark 6.2.3(i) we see that every λn-dimensional projection $P_E(K)$ of K has finite volume ratio (which depends on λ). We can therefore apply Theorem 4.7.2 to conclude that a random $\lambda^2 n$-dimensional section $P_F(K) \cap E$ of $P_F(K)$ has distance depending only on λ from the corresponding Euclidean ball.

7 Appendix

7.1 The hyperplane conjecture

In 2.3 we saw that every body in $\mathbb{R}^n$ has an isotropic position K with $|K| = 1$, which satisfies

$$(1) \qquad \int_K \langle x, \theta\rangle^2 dx = L_K^2$$

for every $\theta \in S^{n-1}$. This position is uniquely determined up to orthogonal transformations, and the affine invariant L_K is called the isotropic constant of K. It is an open problem whether there exists an absolute constant $C > 0$ such that $L_K \leq C$ for every body K.

Let K be a body in $\mathbb{R}^n$. Using Theorem 2.3.6, one can easily check that

$$(2) \qquad nL_K^2 \leq \frac{|\mathrm{det} u|}{|uK|^{1+\frac{2}{n}}} \int_K |ux|^2 dx$$

for every invertible linear transformation u. For the same reason,

$$nL_{K^\circ}^2 \leq \frac{|\det(u^{-1})^*|}{|(u^{-1})^*(K^\circ)|^{1+\frac{2}{n}}} \int_{K^\circ} |(u^{-1})^*(x)|^2 dx. \tag{3}$$

We may choose $u : X_K \to \ell_2^n$ such that $d(X_K, \ell_2^n) = \|u\|\,\|u^{-1}\|$. Then, (2) and (3) imply that

$$n^2 L_K^2 L_{K^\circ}^2 \leq d^2(X_K, \ell_2^n)\left(|uK|\,|(u^{-1})^*(K^\circ)|\right)^{-2/n}, \tag{4}$$

and an application of the inverse Santaló inequality shows that

$$L_K L_{K^\circ} \leq c d(X_K, \ell_2^n). \tag{5}$$

Therefore, duality gives the following first estimates on the isotropic constant:

Theorem 7.1.1. *Let K be a body in $\mathbb{R}^n$. Then, $L_K \leq cd(X_K, \ell_2^n) \leq c\sqrt{n}$. Moreover, either $L_K \leq c\sqrt[4]{n}$ or $L_{K^\circ} \leq c\sqrt[4]{n}$.* □

Bourgain [24] has proved that $L_K \leq c\sqrt[4]{n}\log n$, where $c > 0$ is an absolute constant, for every body K. We shall give a proof of this fact following Dar's presentation in [46]. Recall that for every $\theta \in S^{n-1}$ and $p > 1$ we have

$$\left(\frac{1}{|K|}\int_K |\langle x, \theta\rangle|^p dx\right)^{1/p} \leq cp\,\frac{1}{|K|}\int_K |\langle x, \theta\rangle| dx, \tag{6}$$

where $c > 0$ is an absolute constant. This is a consequence of Borell's lemma (see 2.3). It follows from 2.3 (25) that if K is isotropic, then

$$\int_K \exp(|\langle x, \theta\rangle|/cL_K) dx \leq 2, \tag{7}$$

for every $\theta \in S^{n-1}$, where $c > 0$ is an absolute constant. We shall use this information in the following form:

Lemma 1. *Let K be an isotropic body. If N is a finite subset of S^{n-1}, then*

$$\int_K \max_{\theta \in N} |\langle x, \theta\rangle| dx \leq cL_K \log|N|. \quad \square \tag{8}$$

Starting with an isotropic body K, we see from Theorem 2.3.6 that

$$\begin{aligned} nL_K^2 &\leq \frac{\mathrm{tr}T}{n}\int_K |x|^2 dx = \int_K \langle x, Tx\rangle dx \\ &\leq \int_K \|Tx\|_{K^\circ} dx = \int_K \max_{y \in TK} |\langle x, y\rangle| dx \end{aligned} \tag{9}$$

for every symmetric, positive-definite volume preserving transformation T of $\mathbb{R}^n$. In order to estimate this last integral, we first reduce the problem to a discrete one using the Dudley-Fernique decomposition:

Lemma 2. *Let A be a body in $\mathbb{R}^n$, and R be its diameter. For every r and $j = 1, \ldots, r$, we can find finite subsets N_j of A with $\log |N_j| \leq cn(w(A)2^j/R)^2$ with the following property: every $x \in A$ can be written in the form*

$$x = z_1 + \ldots + z_r + w_r,$$

where $z_j \in Z_j = (N_j - N_{j-1}) \cap (3R/2^j)D_n$ and $w_r \in (R/2^r)D_n$ (we set $N_0 = \{o\}$). $\square$

The proof of this decomposition is simple. The estimate on the cardinality of N_j comes from Sudakov's inequality (Theorem 6.1.1). We now choose T in (9) so that $A = TK$ will have minimal mean width: Theorem 5.2.1 allows us to assume that $w(TK) \leq c\sqrt{n} \log n$.

From Lemma 2, we see that for every $x \in K$,

$$\max_{y \in TK} |\langle y, x\rangle| \leq \sum_{j=1}^{r} \max_{z \in Z_j} |\langle z, x\rangle| + \max_{w \in (R/2^r)D_n} |\langle w, x\rangle| \tag{10}$$

$$\leq \sum_{j=1}^{r} \frac{3R}{2^j} \max_{z \in Z_j} |\langle \tilde{z}, x\rangle| + \frac{R}{2^r}|x|,$$

where $\tilde{z} = z/|z| \in S^{n-1}$. Now, Lemma 1 and the estimate on $|N_j|$ imply that

$$\int_K \max_{z \in Z_j} |\langle \tilde{z}, x\rangle| dx \leq cL_K \log |Z_j| \leq cnL_K \left(\frac{w(TK)2^j}{R}\right)^2 \tag{11}$$

for every $j = 1, \ldots, r$. Going back to (9), we conclude that

$$nL_K^2 \leq cL_K \left(\sum_{j=1}^{r} nw^2(TK)\frac{2^j}{R} + \frac{R}{2^r}\sqrt{n}\right) \tag{12}$$

$$\leq c' L_K \left(nw^2(TK)\frac{2^r}{R} + \frac{R}{2^r}\sqrt{n}\right),$$

and the optimal choice for r gives

$$nL_K^2 \leq c\sqrt[4]{n}w(TK)\sqrt{n}L_K. \tag{13}$$

Since $w(TK) \leq c\sqrt{n} \log n$, the proof is complete:

Theorem 7.1.2. *For every body K in $\mathbb{R}^n$ we have $L_K \leq c\sqrt[4]{n} \log n$.* $\square$

Remark: The same holds true for non-symmetric convex bodies as well (see [155]).

7.2 Geometry of the Banach-Mazur compactum

1. Consider the set $\mathcal{B}_n$ of all equivalence classes of n-dimensional normed spaces $X = (\mathbb{R}^n, \|\cdot\|)$, where X is equivalent to X' if and only if X and X' are isometric. Then, $\mathcal{B}_n$ becomes a compact metric space with the metric $\log d$, where d is the Banach-Mazur distance (the *Banach-Mazur compactum*).

There are many interesting questions about the structure of the Banach-Mazur compactum, and most of them remain open. Below, we describe some fundamental results and problems in this area. The interested reader will find more information in the book [195] and the surveys [67], [183].

2. John's theorem shows that $d(X,Y) \leq n$ for every $X, Y \in \mathcal{B}_n$. Therefore, $\mathrm{diam}(\mathcal{B}_n) \leq n$. The natural question of the exact order of $\mathrm{diam}(\mathcal{B}_n)$ remained open for many years and was finally answered by Gluskin [64]: $\mathrm{diam}(\mathcal{B}_n) \geq cn$.

Gluskin does not describe a pair $X, Y \in \mathcal{B}_n$ with $d(X,Y) \geq cn$ explicitely (in fact, there is no concrete example of spaces with distance of order greater than $\sqrt{n}$). The idea of the proof is probabilistic: a random $T : \ell_1^n \to \ell_1^n$ satisfies $\|T\| \, \|T^{-1}\| \geq cn$, and this suggests that by "spoiling" ℓ_1^n it is possible to obtain X and Y with distance cn. The spaces which were used in [64] have as their unit ball a body of the form $K = \mathrm{co}\{\pm e_i, \pm x_j : 1 \leq j \leq 2n\}$, where $\{e_i\}$ is the standard orthonormal basis of $\mathbb{R}^n$ and the x_j's are chosen uniformly and independently from the unit sphere S^{n-1}. A random pair of such spaces has the desired property.

This method of considering random spaces proved to be very fruitful in problems where one needed to establish "pathological behavior". We mention Szarek's finite dimensional analogue of Enflo's example [56] of a space failing the approximation property: there exist n-dimensional normed spaces whose basis constant is of the order of $\sqrt{n}$ [181]. See also [65], [124] and subsequent work of Szarek and Mankiewicz where random spaces play a central role. The article [152] in this collection covers this topic.

3. Another natural question about the geometry of the Banach-Mazur compactum is that of the uniqueness of its center: If $\dim X = n$ and $d(X,Y) \leq c\sqrt{n}$ for every $Y \in \mathcal{B}_n$, is it then true that X is "close" (depending on c) to ℓ_2^n? This question was answered in the negative by Bourgain and Szarek [33]: Let $X_0 = \ell_2^s \oplus \ell_1^{n-s}$, where $s = [n/2]$. Then, $d(X_0, Y) \leq c\sqrt{n}$ for every $Y \in \mathcal{B}_n$ (and, clearly, $d(X_0, \ell_2^n) \geq c'\sqrt{n}$). The proof of the fact that X_0 is an asymptotic center of the compactum is based on the proportional version of the Dvoretzky-Rogers lemma (see 4.1).

4. Fix $X \in \mathcal{B}_n$. Then, one can define the *radius* of $\mathcal{B}_n$ with respect to X by $R(X) = \max\{d(X,Y) : Y \in \mathcal{B}_n\}$. Many problems of obvious geometric interest arise if one wants to give the order of the radius with respect to important concrete centers. For example, the problem of the distance to the cube $R(\ell_\infty^n)$ remains open. It is known that $R(\ell_\infty^n) \leq cn^{5/6}$ (see [33], [186] and [62]). On the other hand, Szarek has proved [182] that $R(\ell_\infty^n) \geq c\sqrt{n}\log n$, therefore ℓ_1^n and ℓ_∞^n are not asymptotic centers of the compactum (these are actually the only concrete examples of spaces for which this property has been established).

5. If we restrict ourselves to subclasses of $\mathcal{B}_n$, then the diameter may be significantly smaller than n: Let $\mathcal{A}_n$ be the family of all 1-*symmetric spaces.* Tomczak-Jaegermann [192] (see also [66]) proved that $d(X,Y) \leq c\sqrt{n}$ whenever $X, Y \in \mathcal{A}_n$. This result is clearly optimal: recall that $d(\ell_1^n, \ell_2^n) = \sqrt{n}$. The analogous problem for the family of 1-*unconditional spaces* remains open. Lindenstrauss and Szankowski [114] have shown that in this case $d(X,Y) \leq cn^{\alpha}$, where α is a constant close to $2/3$. It is conjectured that the right order is close to $\sqrt{n}$.

The diameter of other subclasses of $\mathcal{B}_n$ was estimated with the method of *random orthogonal factorizations.* The idea (which has its origin in work of Tomczak-Jaegermann [190], and was later developped and used by Benyamini and Gordon [27]) is to use the average of $\|T\|_{X\to Y}\|T^{-1}\|_{Y\to X}$ with respect to the probability Haar measure on $SO(n)$ as an upper bound for $d(X,Y)$. Using this method one can prove a general inequality in terms of the type-2 constants of the spaces [27], [55]:

$$d(X,Y) \leq c\sqrt{n}[T_2(X) + T_2(Y^*)]$$

for every $X, Y \in \mathcal{B}_n$. This was further improved by Bourgain and Milman [31] to

$$d(X,Y) \leq c\Big(d(Y,\ell_2^n)T_2(X) + d(X,\ell_2^n)T_2(Y^*)\Big).$$

In [31] it is also shown that $d(X,X^*) \leq c(\log n)^{\gamma} n^{5/6}$ for every $X \in \mathcal{B}_n$. All these results indicate that the distance between spaces whose unit balls are "quite different" should be significantly smaller than $\mathrm{diam}(\mathcal{B}_n)$.

6. The Banach-Mazur distance $d(K,L)$ between two not necessarily symmetric convex bodies K and L is the smallest $d > 0$ for which there exist $z_1, z_2 \in \mathbb{R}^n$ and $T \in GL_n$ such that $K - z_1 \subseteq T(L - z_2) \subseteq d(K - z_1)$.

The question of the maximal distance between non-symmetric bodies is open. John's theorem implies that $d(K,L) \leq n^2$. Better estimates were obtained with the method of random orthogonal factorizations and recent progress on the non-symmetric analogue of the MM^*-estimate (Theorem 5.2.1). In [42] it was proved that every convex body K has an affine image K_1 such that $M(K_1)M^*(K_1) \leq c\sqrt{n}$, a bound which was improved to $cn^{1/3}\log^{\beta} n$, $\beta > 0$ in [170]. Using this fact, Rudelson showed that $d(K,L) \leq cn^{4/3}\log^{\beta} n$ for any $K, L \in \mathcal{K}_n$. See also recent work of Litvak and Tomczak-Jaegermann [116] for related estimates in the non-symmetric case.

7. Milman and Wolfson [153] studied spaces X whose distance from ℓ_2^n is extremal. They showed that if $d(X,\ell_2^n) = \sqrt{n}$, then X has a k-dimensional subspace F with $k \geq c\log n$ which is isometric to ℓ_1^k. The example of $X = \ell_\infty^n$ shows that this estimate is exact.

An isomorphic version of this result is also possible [153]: If $d(X,\ell_2^n) \geq \alpha\sqrt{n}$ for some $\alpha \in (0,1)$, then X has a k-dimensional subspace F (with $k = h(n) \to \infty$ as $n \to \infty$) which satisfies $d(F,\ell_1^k) \leq c(\alpha)$, where $c(\alpha)$ depends only on α. The original estimate for k in [153] was later improved to $k \geq c_1(\alpha)\log n$ through work of Kashin, Bourgain and Tomczak-Jaegermann (see [195, Section 31] for details).

An extension of this fact appears in [158]: Recall that a Banach space X *contains ℓ_1^n's uniformly* if X contains a sequence of subspaces $F_n, n \in \mathbb{N}$ with $d(F_n, \ell_1^n) \leq C$. Then, the following are equivalent:

(i) X does not contain ℓ_1^n's uniformly.

(ii) $\sup\{d(F, \ell_2^n) : F \subset X \ , \ \dim F = n\} = o(\sqrt{n})$.

(iii) There exists a sequence $\alpha_n = o(\sqrt{n})$ with the following property: If F is an n-dimensional subspace of X, there exists a projection $P : X \to F$ with $\|P\| \leq \alpha_n$.

In the non-symmetric case the extremal distance to the ball is n. Palmon [156] showed that $d(K, D_n) = n$ if and only if K is a simplex.

8. Tomczak-Jaegermann [193] defined the *weak distance* $wd(X, Y)$ of two n-dimensional normed spaces X and Y by $wd(X, Y) = \max\{q(X, Y), q(Y, X)\}$, where

$$q(X, Y) = \inf \int_{\Omega} \|S(\omega)\| \, \|T(\omega)\| \, d\omega,$$

and the inf is taken over all measure spaces Ω and all maps $T : \Omega \to L(X, Y)$, $S : \Omega \to L(Y, X)$ such that $\int_{\Omega} S(\omega) \circ T(\omega) d\omega = \mathrm{id}_X$. It is not hard to check that $wd(X, Y) \leq d(X, Y)$ and that with high probability the weak distance between two Gluskin spaces is bounded by $c\sqrt{n}$. In fact, Rudelson [168] has proved that $wd(X, Y) \leq cn^{13/14} \log^{15/7} n$ for all $X, Y \in \mathcal{B}_n$. It is conjectured that the weak distance in $\mathcal{B}_n$ is always bounded by $c\sqrt{n}$.

7.3 Symmetrization and approximation

Symmetrization procedures play an important role in classical convexity. The question of how many successive symmetrizations of a certain type are needed in order to obtain from a given body K a body $\tilde{K}$ which is close to a ball was extensively studied with the methods of local theory. This study led to the surprising fact that only few such operations suffice:

Let $K \in \mathcal{K}_n$ and $u \in S^{n-1}$. Consider the reflection π_u with respect to the hyperplane orthogonal to u. The Minkowski symmetrization of K with respect to u is the convex body $\frac{1}{2}(K + \pi_u K)$. Observe that this operation is linear and preserves mean width. A random Minkowski symmetrization of K is a body $\pi_u K$, where u is chosen randomly on S^{n-1} with respect to the probability measure σ.

In [38] it was proved that for every $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for every $n \geq n_0$ and $K \in \mathcal{K}_n$, if we perform $N = Cn \log n + c(\varepsilon)n$ independent random Minkowski symmetrizations on K we receive a convex body $\tilde{K}$ such that

$$(1 - \varepsilon) w(K) D_n \subset \tilde{K} \subset (1 + \varepsilon) w(K) D_n$$

with probability greater than $1 - \exp(-c_1(\varepsilon)n)$. The method of proof is closely related to the concentration phenomenon for $SO(n)$.

The same question for Steiner symmetrization was studied in [39]. Mani [123] has proved that, starting with a body $K \in \mathcal{K}_n$, if we choose an infinite random sequence of directions $u_j \in S^{n-1}$ and apply successive Steiner symmetrizations σ_{u_j}

of K in these directions, then we almost surely get a sequence of convex bodies converging to a ball. The number of steps needed in order to bring K at a fixed distance from a ball is much smaller [39]: If $K \in \mathcal{K}_n$ with $|K| = |D_n|$, we can find $N \leq c_1 n \log n$ and $u_1, \ldots, u_N \in S^{n-1}$ such that

$$c_2^{-1} D_n \subseteq (\sigma_{u_N} \circ \ldots \circ \sigma_{u_1})(K) \subseteq c_2 D_n, \tag{1}$$

where $c_1, c_2 > 0$ are absolute constants. It is not clear what the bound $f(n, \varepsilon)$ on N would be if we wanted to replace c_2 by $1 - \varepsilon$, $\varepsilon \in (0,1)$. The proof of (1) is based on the previous result about Minkowski symmetrizations.

Results of the same nature concern questions about approximation of convex bodies by Minkowski sums. The global form of Dvoretzky's theorem is an isomorphic statement of this type.

Recall that a *zonotope* is a Minkowski sum of line segments, and a *zonoid* is a body in $\mathbb{R}^n$ which is the Hausdorff limit of a sequence of zonotopes. A body is a zonoid if and only if its polar body is the unit ball of an n-dimensional subspace of $L_1(0,1)$ (for this and other characterizations of zonoids, see [20]).

The unit ball of ℓ_p^n is a zonoid if and only if $2 \leq p \leq \infty$ (see [50]). In particular, the Euclidean unit ball D_n can be approximated arbitrarily well by sums of segments. The question of how many segments are needed in order to come $(1+\varepsilon)$-close to D_n is equivalent to the problem of embedding ℓ_2^n into ℓ_1^N. From the results in [61] it follows that $N \leq c(\varepsilon) n$ segments are enough. In [40] it was shown that the same bound on N allows us to choose the segments having the same length. The linear dependence of N on n is optimal, but the best possible answer if we view N as a function of both n and ε is not known (see [28], [30], [40], [111], [196]).

If we replace the ball D_n by an arbitrary zonoid Z, then the same approximation problem is equivalent to the question of embedding an n-dimensional subspace of $L_1(0,1)$ into ℓ_1^N. Bourgain, Lindenstrauss and Milman [40] proved, by an adaptation of the empirical distribution method of Schechtman [173], that for every $\varepsilon \in (0,1)$ there exist $N \leq c\varepsilon^{-2} n \log n$ and segments $I_1, \ldots, I_N$ such that $(1-\varepsilon)Z \subset \sum I_j \subset (1+\varepsilon)Z$. Moreover, if the norm of Z is strictly convex then N can be chosen to be of the order of n up to a factor which depends on ε and the modulus of convexity of $\|\cdot\|_Z$. Later, Talagrand [188] showed (with a considerably simpler approach) that one can have $N \leq c \|\mathrm{Rad}_n\|_X^2 \varepsilon^{-2} n$.

For more information on this topic, we refer the reader to the surveys [110], [113] and [99].

7.4 Quasi-convex bodies

Many of the results that we presented about symmetric convex bodies can be extended to a much wider class of bodies. We have already discussed extensions of the main facts to the non-symmetric convex case. We now briefly discuss extensions to the class of quasi-convex bodies.

Recall that a star body K is called quasi-convex if $K + K \subset cK$ for some constant $c > 0$. Equivalently, if the gauge f of K satisfies (i) $f(x) > 0$ if $x \neq 0$, (ii)

$f(\lambda x) = |\lambda| f(x)$ for any $x \in \mathbb{R}^n$, and (iii) $f \in C(\alpha)$ i.e. there exists $\alpha \in (0,1]$ such that

$$\alpha f(x) \leq (f * f)(x) := \inf\{f(x_1) + f(x_2)\ ,\ x_1 + x_2 = x\} \quad , \quad x \in \mathbb{R}^n .$$

A body K is called p-convex, $p \in (0,1)$, if for any $x, y \in K$ and $\lambda, \mu > 0$ with $\lambda^p + \mu^p = 1$ we have $\lambda x + \mu y \in K$. Every p-convex body K is quasi-convex, and $K + K \subset 2^{1/p} K$. Conversely, for every quasi-convex body K (with constant C) we can find a q-convex body K_1 such that $K \subset K_1 \subset 2K$, where $2^{1/q} = 2C$ (see [166]).

Most of the basic results we described in the previous sections were extended to this case. Versions of the Dvoretzky-Rogers lemma and Dvoretzky's theorem were proved by Dilworth [49]. For the low M^*-estimate and the quotient of subspace theorem in the quasi-convex setting, see [117] and [77] respectively (see also [143] for an isomorphic Euclidean regularization result and the random version of the QS-theorem). The reverse Brunn-Minkowski inequality is shown in [36]. For results on existence of M-ellipsoids, entropy estimates and asymptotic formulas, see [117], [118] and [147]. In most of the cases, the tools which were available from the convex case were not enough, and new techniques had to be invented: some of them provided interesting alternative proofs of the known "convex results".

7.5 Type and cotype

The notions of type and cotype were introduced by Hoffmann-Jorgensen [92] in connection with limit theorems for independent Banach space valued random variables. Their importance for the study of geometric properties of Banach spaces was realized through the work of Maurey and Pisier (see the article [120] in this collection for a discussion of the development of this theory).

Given an n-dimensional normed space X, and $1 \leq p \leq 2$ ($2 \leq q < \infty$, respectively), the type-p (cotype-q) constant $T_p(X)$ ($C_q(X)$) of X is the smallest $T > 0$ ($C > 0$) such that: for every $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$,

$$\left(\int_0^1 \left\| \sum_{i=1}^m r_i(t) x_i \right\|^2 dt \right)^{1/2} \leq T \left(\sum_{i=1}^m \|x_i\|^p \right)^{1/p} .$$

$$\left(\text{respectively,} \quad \left(\sum_{i=1}^m \|x_i\|^q \right)^{1/q} \leq C \left(\int_0^1 \left\| \sum_{i=1}^m r_i(t) x_i \right\|^2 \right)^{1/2} . \right)$$

Results of Tomczak-Jaegermann ([191] when $p = q = 2$), König ([103] for any p and q not equal to 2, up to constants depending on p, q) and Szarek [184] show that in order to determine the (Gaussian) type-p or cotype-q constants of X up to an absolute constant, it is enough to consider n vectors. In the Rademacher case, the definite answer is not yet known. It is clear that $T_2(\ell_2^n) = C_2(\ell_2^n) = 1$ and, conversely, Kwapien [104] proved that $d(X, \ell_2^n) \leq C_2(X) T_2(X)$.

Let $k_p(X;\varepsilon)$, $1 \leq p \leq \infty$, be the largest integer $k \leq n$ for which ℓ_p^k is $1+\varepsilon$-isomorphic to a subspace of X (in this terminology, $k(X) = k_2(X;4)$). The following results show how type and cotype enter in the study of the linear structure of a space:

(i) In [61] it is shown that $k_2(X) \geq cn/C_2^2(X)$ and $k_2(X) \geq cn^{2/q}/C_q^2(X)$. This gives another proof of the facts $k_2(\ell_p^n) \geq cn, 1 \leq p \leq 2$, and $k_2(\ell_q^n) \simeq n^{2/q}, q \geq 2$.

(ii) In [159] it is proved that $k_p(X;\varepsilon) \geq c(p,\varepsilon)T_p(X)^q$, where $1 < p < 2$ and $\frac{1}{p}+\frac{1}{q} = 1$. This generalizes the estimate $k_p(\ell_1^n;\varepsilon) \geq c(p,\varepsilon)n$, $1 \leq p \leq 2$, of Johnson and Schechtman [98].

(iii) A quantitative version of Krivine's theorem [9] states that, for every $A \geq \varepsilon$,

$$k_p(X;\varepsilon) \geq c(\varepsilon, A)[k_p(X;A)]^{c_1(\varepsilon/A)^p}.$$

Gowers [70], [71] obtained related estimates on the length of $(1+\varepsilon)$-symmetric basic sequences in X.

(iv) In [121] it is shown that if no cotype-q constant of X is bounded by a number independent of n, then X contains $(1+\varepsilon)$-isomorphic copies of ℓ_∞^k for large k. Alon and Milman [7], using combinatorial methods, provided a quantitative form of this fact: $k_2(X;1)k_\infty(X;1) \geq \exp(c\sqrt{\log n})$.

Bourgain and Milman [32] proved that $vr(K_X) \leq f(C_2(X))$. Thus, spaces with bounded cotype-2 constant satisfy all consequences of bounded volume ratio (this had been independently observed, see e.g. [61],[54]). Milman and Pisier [148] introduced the class of spaces with the weak cotype 2 property: X is weak cotype 2 if there exists $\delta > 0$ such that $k_2(E) \geq \delta \mathrm{dim} E$ for every $E \subset X$. One can then prove that $vr(E) \leq C(\delta)$ for every $E \subset X$ [148].

In 6.2 we saw that every n-dimensional normed space X has a subspace E with $\mathrm{dim}E \geq n/2$ such that $vr(K_{E^*}) \leq C$. This suffices for a proof of the quotient of subspace theorem. However, the following question remains open: does every X contain a subspace E with $\mathrm{dim}E \geq n/2$ such that $C_2(E^*) \leq C$? This problem is related to many open questions in the local theory (for a discussion see [136], [144]).

Finally, let us mention the connection between Gaussian and Rademacher averages [122]: Let X be an n-dimensional normed space, and $\{x_j\}$ be a finite sequence in X. Then,

$$\sqrt{\frac{2}{\pi}}\left(\int_0^1 \Big\| \sum_j r_j(t)x_j \Big\|^2 dt\right)^{1/2} \leq \left(\int_\Omega \Big\| \sum_j g_j(\omega)x_j \Big\|^2 d\omega\right)^{1/2}$$

$$\leq c(1+\log n)^{1/2}\left(\int_0^1 \Big\| \sum_j r_j(t)x_j \Big\|^2 dt\right)^{1/2}.$$

If X has bounded cotype-q constant $C_q(X)$ for some $q \geq 2$, then the constant in the right hand side inequality may be replaced by $c\sqrt{q}C_q(X)$.

7.6 Non-linear type theory

Let (T, d) be a metric space, and $F^n = \{-1, 1\}^n$ with the normalized counting measure μ_n. An *n-dimensional cube* in T is a function $f : F^n \to T$. For any such f and $i \in \{1, \ldots, n\}$, we define

$$(\Delta_i f)(\varepsilon) = d(f(\varepsilon_1, \ldots, \varepsilon_i, \ldots, \varepsilon_n), f(\varepsilon_1, \ldots, -\varepsilon_i, \ldots, \varepsilon_n)).$$

A metric space (T, d) has metric type p, $1 \leq p \leq 2$, if there exists a constant $C > 0$ such that, for every $n \in \mathbb{N}$ and every $f : F^n \to T$ we have

$$\left(\int_{F^n} d(f(\varepsilon), f(-\varepsilon))^2 d\mu_n \right)^{1/2} \leq C n^{\frac{1}{p} - \frac{1}{2}} \left(\sum_{j=1}^{n} \int_{F^n} (\Delta_j f(\varepsilon))^2 d\mu_n \right)^{1/2}.$$

Every metric space has type 1, and if $1 \leq p_1 \leq p_2 \leq 2$, metric type p_2 implies metric type p_1.

Let $\phi : (T_1, d_1) \to (T_2, d_2)$ be a map between metric spaces. The Lipschitz norm of ϕ is defined by

$$\|\phi\|_{\mathrm{Lip}} = \sup_{t \neq s} \frac{d_2(\phi(t), \phi(s))}{d_1(t, s)}.$$

Let F_p^n be the space F^n equipped with the metric induced by ℓ_p^n. We say that a metric space (T, d) contains F_p^n's $(1 + \varepsilon)$-uniformly if for every $n \in \mathbb{N}$ there exist a subset $T_n \subset T$ and a bijection $\phi_n : F_p^n \to T_n$ such that $\|\phi_n\|_{\mathrm{Lip}} \|\phi_n^{-1}\|_{\mathrm{Lip}} \leq 1 + \varepsilon$.

Bourgain, Milman and Wolfson [41] (see also [154]) proved the following:

Theorem 7.6.1. *A metric space (T, d) has metric type p for some $p > 1$ if and only if there exists $\varepsilon > 0$ such that T does not contain F_1^n's $(1 + \varepsilon)$-uniformly.*

A natural question which arises is to compare the notions of metric type and type in the case where T is a normed space. An answer to this question was given in [41], see also [161]:

Theorem 7.6.2. *Let X be a Banach space and let $1 < p < 2$.*

(i) *If X has type (respectively, metric type) p, then X has metric type (respectively, type) p_1 for all $1 \leq p_1 < p$.*

(ii) *X contains F_1^n's uniformly if and only if X contains ℓ_1^n's uniformly.*

We refer the interested reader to [41], [161] for the proofs of these facts, and a comparison with another notion of metric type which was earlier proposed by Enflo [57]. In [41] and [37] one can find a generalization of Dvoretzky's theorem for metric spaces: For every $\varepsilon > 0$ there exists a constant $c(\varepsilon) > 0$ with the following property: every metric space T of cardinality N contains a subspace S with cardinality at least $c(\varepsilon) \log N$ such that for some $\tilde{S} \subset \ell_2$ with $|S| = |\tilde{S}|$ we can find a bijection $\phi : S \to \tilde{S}$ with $\|\phi\|_{\mathrm{Lip}} \|\phi^{-1}\|_{\mathrm{Lip}} \leq 1 + \varepsilon$ (this means that S is $(1 + \varepsilon)$-isomorphic to a subset of a Hilbert space).

Let us finally mention an interesting connection between non-linear problems and a more advanced form of type and cotype, the so-called Markov type and cotype which was introduced and studied by K. Ball [17].

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