Asymptotic Convex Geometry: a short overview

A. A. Giannopoulos and V. D. Milman*

1 Introduction

In this article we outline a rapidly developing theory of high dimensional normed spaces and convex bodies. The classical Convex Geometry, sometimes called Brunn-Minkowski theory, studies the geometry of convex bodies and related geometric inequalities in Euclidean space of a fixed dimension (because of this, it is an *isometric* theory). The classical Functional Analysis is standardly understood as the theory of infinite dimensional spaces. However, it is a relatively recent discovery that there is a theory "in between", which is concerned with the geometric and linear properties of finite dimensional normed spaces or convex bodies, the emphasis now being on the asymptotic behaviour of various quantitative parameters as the dimension grows to infinity. We call it Convex Geometric Analysis, but also Asymptotic Convex Geometry (actually, more names are associated to it: history has not yet selected the right one). In the framework of this theory, very unexpected phenomena, hidden structures and forms of behaviour were discovered, new intuition was built and many new tools were developed. It is now clear that the theory provides the right questions to reveal the underlying "order" and structures which accompany high dimensional spaces.

The quantitative study of high dimensional normed spaces used many of the tools of convex geometry. However, these tools were now used under a different point of view. The isometric questions which were typical in classical convexity were replaced by isomorphic ones, which were most natural for functional analysis but alien to convexity. Isoperimetric type problems provide a bold example of this transformation. Isomorphic versions of such problems, which make sense only from the asymptotic point of view, led to the discovery of the concentration of measure phenomenon, which plays a crucial role in the proof of Dvoretzky type theorems. Later, the method spread and influenced the development of other "asymptotic"

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theories in Probability, Asymptotic Combinatorics and Complexity, where much more general high parametric systems arise.

After this major step on the conceptual level, many unsolved problems of classical convexity were put in asymptotic form and were studied systematically. In this way, the two theories started to interact with many deep consequences in both analysis and geometry. Typical examples are the reverse Brunn-Minkowski inequality and the reverse Santaló inequality, which provides an affirmative answer - at least in the asymptotic sense - to a classical conjecture of Mahler.

The article is organized as follows: Section 2 gives a brief synopsis of the major results of asymptotic convex geometry (the concept of concentration, Dvoretzky type theorems, Pisier's inequality on the Rademacher projection, Milman's low M^*-estimate and the quotient of subspace theorem, entropy estimates) and of some more recent important directions (global theory, asymptotic formulas and phase transition behaviour, "coordinate theory").

Section 3 contains background material from classical convexity: the Brunn-Minkowski inequality and its functional forms, the Alexandrov-Fenchel inequality and related geometric inequalities about mixed volumes of convex bodies, volume preserving transformations (Knöthe and Brenier map).

Section 4 describes classical positions of convex bodies such as John's position, the minimal surface area and the minimal mean width position. They are all characterized as isotropic ones, an observation which relates them to the Brascamp-Lieb inequality and its reverse form. Some sharp geometric inequalities are applications of this point of view, a fact which was first observed and successfully exploited by Ball. We also give a short account on the challenging slicing problem.

Section 5 gives some classical and recent examples of the interaction between the asymptotic convex geometry point of view and classical convexity. The reverse Santaló inequality and the reverse Brunn-Minkowski inequality are proved with the method of isomorphic symmetrization. This discussion introduces M-ellipsoids and their basic properties. Recent results of Klartag and Milman on the number of Minkowski or Steiner symmetrizations that are needed in order to bring an arbitrary convex body close to a ball give another example of use of the asymptotic theory in questions with classical convexity flavor.

Section 6, which is closer to the spirit of geometric functional analysis, is devoted to the geometry of the Banach-Mazur compactum and some questions on the local structure of high-dimensional normed spaces. Random spaces, which were first introduced by Gluskin, play an important role in this discussion.

A number of surveys on different aspects of the theory were recently published (see [14], [86], [87], [109] and [110]). In particular, [54] gives a more geometrically directed point of view on the theory. However, this article was written before 1999 and a new stream of results is now available. We cannot avoid repeating the very basic and already classical line of development we described, but we refer to [54] for many proofs which are outlined there in a very condense form. General references on the Brunn-Minkowski theory and geometric inequalities are the books of Schneider [134] and Burago-Zalgaller [34]. The reader may consult the books of

Milman-Schechtman [113], Pisier [121] and Tomczak-Jaegermann [147] for various aspects of the asymptotic theory of finite dimensional normed spaces.

2 Asymptotic Convex Geometry

We study finite-dimensional real normed spaces $X = (\mathbb{R}^n, \|\cdot\|)$. The unit ball K_X of such a space is a symmetric (with respect to the origin) convex body in $\mathbb{R}^n$. Conversely, if K is a symmetric convex body, then $\|x\|_K = \min\{\lambda \geq 0 : x \in \lambda K\}$ is a norm defining a space X_K with K as its unit ball. If K_1 and K_2 are symmetric convex bodies in $\mathbb{R}^n$, their geometric distance $d(K_1, K_2)$ is defined by

$$d(K_1, K_2) = \inf\{ab : a, b > 0, K_1 \subseteq bK_2, K_2 \subseteq aK_1\}.$$

The natural distance between the n-dimensional spaces X_{K_1} and X_{K_2} is the *Banach-Mazur distance*

$$d(X_{K_1}, X_{K_2}) = \inf\{d(K_1, T(K_2)) : T \in GL(n)\}.$$

Note that $d(X_{K_1}, X_{K_2})$ is the smallest positive number d for which we can find $T \in GL(n)$ such that $K_1 \subseteq T(K_2) \subseteq dK_1$. In the language of geometric functional analysis, if X and Y are two n-dimensional normed spaces, then

$$d(X, Y) = \min\{\|T\|\,\|T^{-1}\| \;:\; T : X \to Y \text{ is an isomorphism}\}.$$

We assume that $\mathbb{R}^n$ is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$ and denote the corresponding Euclidean norm by $|\cdot|$. B_2^n is the Euclidean unit ball and S^{n-1} is the unit sphere. The rotationally invariant probability measure on S^{n-1} will be denoted by σ. The unit ball of ℓ_p^n is denoted by B_p^n. By a classical theorem of John [73] one has $d(X, \ell_2^n) \leq \sqrt{n}$ for every n-dimensional normed space X (see also §4.1).

If K is a symmetric convex body in $\mathbb{R}^n$, its polar body is defined by $\|y\|_{K^\circ} = \max_{x\in K} |\langle x, y\rangle|$. Note that $X_{K^\circ} = X_K^*$: K° is the unit ball of the dual space of X.

Let K be a convex body in $\mathbb{R}^n$ with $0 \in \text{int}(K)$. The radial function $\rho_K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^+$ of K is defined by $\rho_K(x) = \max\{\lambda > 0 : \lambda x \in K\}$. The support function $h_K : \mathbb{R}^n \to \mathbb{R}$ of K is defined by $h_K(x) = \max\{\langle x, y\rangle : y \in K\}$. The width of K in the direction of $\theta \in S^{n-1}$ is the quantity $w(K, \theta) = h_K(\theta) + h_K(-\theta)$, and the mean width of K is defined by

$$w(K) = \frac{1}{2}\int_{S^{n-1}} w(K, \theta)\sigma(d\theta) = \int_{S^{n-1}} h_K(\theta)\sigma(d\theta).$$

Note that if K is symmetric then $\rho_K(x) = 1/\|x\|_K$ and $h_K(x) = \|x\|_{K^\circ}$.

2.1 Isomorphic isoperimetric inequalities and concentration of measure

Concentration of measure was understood and developed as a method for the goals of geometric functional analysis, but it was soon realized that it was very well adapted to the needs of probability theory, asymptotic combinatorics and complexity. General references on concentration, from various viewpoints, are the following surveys and books: [14], [68], [69], [80], [81], [104], [110], [132].

The general framework is a probability space $(X, \mathcal{A}, d, \mu)$, where $\mathcal{A}$ is the Borel σ-algebra with respect to a given metric d on X. For every $A \in \mathcal{A}$, we consider the t-extension $A_t = \{x \in X : d(x, A) \leq t\}$ of A. One can then formulate the abstract isoperimetric problem for metric probability spaces as follows: Given $0 < \alpha < 1$ and $t > 0$, find $\inf\{\mu(A_t) : A \in \mathcal{A}, \mu(A) \geq \alpha\}$ and describe the sets A on which this infimum is possibly attained. The complete answer to the isoperimetric problem is available for a few but very important geometric examples.

Spherical isoperimetric inequality: Consider the sphere S^{n-1} as a metric probability space, with the geodesic distance ρ and the $O(n)$-invariant probability measure σ. The spherical isoperimetric inequality states that spherical caps of the form $B(x, r)$ are the extremal sets: if A is a Borel subset of S^{n-1} and $\sigma(A) = \sigma(B(x_0, r))$ for some $x_0 \in S^{n-1}$ and $r > 0$, then

$$\sigma(A_t) \geq \sigma(B(x_0, r + t)) \tag{2.1.1}$$

for every $t > 0$. This is proved by spherical symmetrization (see e.g. [44]). Since spherical caps are easy to work with, one can use (2.1.1) to obtain a good lower bound for the measure of the t-extension of an arbitrary subset of the sphere in terms of its measure. The most important case is when $\sigma(A) = 1/2$ (see [113]).

Theorem 2.1 *If A is a Borel subset of S^{n+1} and $\sigma(A) = 1/2$, then*

$$\sigma(A_t) \geq 1 - \sqrt{\pi/8}\exp(-t^2 n/2) \tag{2.1.2}$$

for every $t > 0$. □

Isoperimetric inequality in Gauss space: Consider $\mathbb{R}^n$ as a metric probability space, with the Euclidean distance $|\cdot|$ and the standard Gaussian probability measure γ_n. The isoperimetric inequality in Gauss space (proved by Borell and Sudakov-Tsirelson, see [80] or [81] for references) states that halfspaces are the extremal sets: if $\alpha \in (0, 1)$, $\theta \in S^{n-1}$ and $H = \{x \in \mathbb{R}^n : \langle x, \theta\rangle \leq s\}$ is a halfspace in $\mathbb{R}^n$ with $\gamma_n(H) = \alpha$, then, for every $t > 0$ and every Borel subset A of $\mathbb{R}^n$ with $\gamma_n(A) = \alpha$, one has

$$\gamma_n(A_t) \geq \gamma_n(H_t). \tag{2.1.3}$$

A direct computation shows the following.

Theorem 2.2 *If* $\gamma_n(A) \geq 1/2$ *then for every* $t > 0$

$$\gamma_n(A_t) \geq 1 - \frac{1}{2}\exp(-t^2/2). \qquad \square \tag{2.1.4}$$

These examples lead to the definition of the concentration function of a metric probability space. For every $t \geq 0$ we set

$$\alpha(X,t) := 1 - \inf\{\mu(A_t) : \mu(A) \geq 1/2\}. \tag{2.1.5}$$

P. Lévy [82] realized the role of the dimension in the spherical isoperimetrical inequality (2.1.2): if we fix $\alpha = 1/2$ and $t > 0$, as the dimension n increases to infinity the measure of the complement of A_t decreases exponentially to zero for every subset A of S^{n-1} with $\sigma(A) = 1/2$. Following this basic example, we say that a sequence $(X_n, \mathcal{A}_n, d_n, \mu_n)$ of metric probability spaces is a normal Lévy family with constants (c_1, c_2) if

$$\alpha(X_n, t) \leq c_1 \exp(-c_2 t^2 n). \tag{2.1.6}$$

There are many examples of normal Lévy families which have found applications in the asymptotic theory of finite dimensional normed spaces. For some important metric probability spaces X, the exact solution to the isoperimetric problem was (and still is) unknown: new and very interesting techniques were invented in order to estimate the concentration function $\alpha(X,t)$. Natural families of obvious geometric importance are the following.

1. The family of the orthogonal groups $(SO(n), \rho_n, \mu_n)$ equipped with the Hilbert-Schmidt metric and the Haar probability measure is a Lévy family with constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/8$.

2. The family $X_n = \prod_{i=1}^{m_n} S^n$ with the natural Riemannian metric and the product probability measure is a Lévy family with constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/2$.

3. All homogeneous spaces of $SO(n)$ inherit the property of forming Lévy families. In particular, any family of Stiefel manifolds W_{n,k_n} or any family of Grassman manifolds G_{n,k_n} is a Lévy family with the same constants as in 1. These first three examples appear in [70].

4. The spaces $E_2^n = \{-1,1\}^n$ with the normalized Hamming distance $d(\eta,\eta') = \#\{i \leq n : \eta_i \neq \eta_i'\}/n$ and the normalized counting measure form a Lévy family with constants $c_1 = 1/2$ and $c_2 = 2$. This follows from an isoperimetric inequality of Harper [72] and it was first stated in this form and used in [5].

5. The group Π_n of permutations of $\{1, \ldots, n\}$ equipped with the normalized Hamming distance $d(\sigma,\tau) = \#\{i \leq n : \sigma(i) \neq \tau(i)\}/n$ and the normalized counting measure satisfies $\alpha(\Pi_n, t) \leq 2\exp(-t^2 n/64)$. This was proved by Maurey [94] with a martingale method, which was further developed by Schechtman [131].

An equivalent way to express concentration is by means of Lipschitz functions (see [80] or [113]).

Theorem 2.3 *Let $(X, \mathcal{A}, d, \mu)$ be a metric probability space. If $f : X \to \mathbb{R}$ is a Lipschitz function with constant 1, then*

$$\mu\left(\{x \in X : |f(x) - M_f| > t\}\right) \leq 2\alpha(X, t) \tag{2.1.7}$$

where M_f is the Lévy median of f. □

Therefore, if the concentration function of X is small, Lipschitz functions are almost constant on almost all space. This observation has very important applications to the study of the normal Lévy families above.

Many problems which arise in the asymptotic geometric analysis require the proof of the existence of some geometric structure with prescribed behaviour. The basic idea of the probabilistic method is to show that a random element of a suitable metric probability space has the required properties. The method (which was first used in combinatorial geometry and graph theory) works because the desirable structure is quite often the typical one. The concentration phenomenon provides a powerful tool for the probabilistic method, since it enables us to identify the typical structure in many situations. The first appearance of this idea in Analysis was in the proof of Dvoretzky's theorem in [97], which we discuss in the next subsection.

2.2 Dvoretzky type theorems

Dvoretzky's theorem [40], [41] states that every high-dimensional normed space has a subspace of "large dimension" which is well isomorphic to the Euclidean space. We use the terminology "Dvoretzky type theorems" for a wide family of results which exhibit large nice substructures inside normed spaces of sufficiently high dimension. The concrete estimates regarding the different parameters which enter in this type of results have become a crucial and important topic in the theory. There are many theorems which provide such estimates and even asymptotic formulas depending on different parameters.

The starting point for Dvoretzky's original theorem is a lemma of Dvoretzky and Rogers [42], which shows that for every symmetric convex body K whose maximal volume ellipsoid is B_2^n (see §4.1), there exist $k \simeq \sqrt{n}$ and a k-dimensional subspace E_k of $\mathbb{R}^n$ such that $B_2^n \cap E_k \subseteq K \cap E_k \subseteq 2Q_k$, where Q_k is the unit cube in E_k with respect to a suitable coordinate system. Grothendieck asked whether it is possible to replace Q_k by $B_2^n \cap E_k$ in this statement, so that k will be still increasing to infinity with n. Dvoretzky's theorem provides an affirmative answer to this question. The best known version can be stated in the language of geometric functional analysis as follows.

Theorem 2.4 *Let X be an n-dimensional normed space and $\varepsilon > 0$. There exist an integer $k \geq c\varepsilon^2 \log n$ and a k-dimensional subspace E_k of X which satisfies $d(E_k, \ell_2^k) \leq 1 + \varepsilon$.* □

The example of ℓ_∞^n shows that the logarithmic dependence of k on n is best possible for small values of ε. The exact relation between n, ε and k has not been settled. It seems reasonable that ℓ_∞^n represents the worst case. This would mean that, for fixed k and ε, every n-dimensional normed space has a k-dimensional subspace which is $(1+\varepsilon)$-isomorphic to ℓ_2^k, provided that $n \geq c(k)\varepsilon^{-\frac{k-1}{2}}$. The problem is very interesting even for small values of k (actually, it is completely understood only in the case $k=2$) and has connections with other branches of mathematics (algebraic topology, number theory, harmonic analysis, see [103] for a discussion).

The proof of Theorem 2.4 given in [97] (with a slightly worse dependence on ε) uses the concentration of measure on S^{n-1}. We start with an n-dimensional normed space X, and we may assume that B_2^n is the ellipsoid of maximal volume inscribed in the unit ball K of X. Then, the function $r : S^{n-1} \to \mathbb{R}$ defined by $r(x) = \|x\|$ is Lipschitz continuous with constant 1. If L_r is the Lévy median of r, Theorems 2.1 and 2.3 imply that for every $t \in (0,1)$,

$$\sigma\left(x \in S^{n-1} : |r(x) - L_r| \geq tL_r\right) \leq 2c_1 \exp(-c_2 t^2 L_r^2 n). \tag{2.2.1}$$

where $c_1, c_2 > 0$ are absolute constants. Since the function $r(x) = \|x\|$ is almost constant and equal to L_r on a subset of the sphere whose measure is practically equal to 1, one can extract a subsphere on which r is almost constant. This is done by a discretization argument via nets of spheres (see [54] for an outline of the argument).

Theorem 2.5 *Let $X = (\mathbb{R}^n, \|\cdot\|)$ and assume that $\|x\| \leq |x|$ for all $x \in \mathbb{R}^n$. For every $\varepsilon \in (0,1)$ we can find $k \geq c_3 \varepsilon^2 L_r^2 n$ and a k-dimensional subspace F of $\mathbb{R}^n$ such that*

$$(1+\varepsilon)^{-1/2} L_r |x| \leq \|x\| \leq L_r (1+\varepsilon)^{1/2} |x| \tag{2.2.3}$$

for every $x \in F$. □

If $Y = (F, \|\cdot\|)$, it is clear that $d(Y, \ell_2^k) \leq 1+\varepsilon$, and what remains is to give a lower bound for L_r. It is easier to work with the expectation

$$M = M(X) = \int_{S^{n-1}} \|x\|\, \sigma(dx), \tag{2.2.4}$$

of the norm on the sphere, and a simple computation shows that $L_r \simeq M$.

We now make full use of the fact that B_2^n is the ellipsoid of maximal volume inscribed in K. By the Dvoretzky-Rogers lemma, we can find an orthonormal basis $\{v_1, \ldots, v_n\}$ with $\|v_i\| \geq 1/2$ for all $i \leq n/2$. One may check that

$$M = \int_{S^{n-1}} \|\sum_{i=1}^n a_i v_i\| \sigma(da) \geq \frac{1}{2} \int_{S^{n-1}} \max_{1 \leq i \leq n/2} |a_i| \sigma(da) \geq c\sqrt{\log n / n}, \tag{2.2.5}$$

where $c > 0$ is an absolute constant. Going back to Theorem 2.5 we conclude the proof of Theorem 2.4. $\square$

Let $X = (\mathbb{R}^n, \|\cdot\|)$ be an n-dimensional normed space. We denote by b the smallest constant for which $\|x\| \leq b|x|$ holds for every $x \in \mathbb{R}^n$. Let $k(X)$ be the largest positive integer $k \leq n$ for which $E_k \in G_{n,k}$ satisfies

$$(M/2)|x| \leq \|x\| \leq (2M)|x|, \qquad x \in E_k \tag{2.2.6}$$

with probability greater than $1 - e^{-k}$. The proof of Theorem 2.4 shows that there exists $k \geq c_1 n(M/b)^2$ such that a random k-dimensional subspace E_k of X has this property. In other words, $k(X) \geq c_1 n(M/b)^2$. It was observed in [115] that this inequality is in fact an "asymptotic formula": for every n-dimensional normed space X one has $k(X) \leq Cn(M/b)^2$.

Theorem 2.6 *Let X be an n-dimensional normed space. Then, $k(X) \simeq n(M/b)^2$.* $\square$

The estimate $k(X) \geq cn(M/b)^2$ allows one to check that in several situations the dimension of "spherical sections" of high-dimensional convex bodies may be much larger than logarithmic in the dimension. For example, one has $k(\ell_p^n) \simeq n$ if $1 < p < 2$ and $k(\ell_q^n) \simeq \sqrt{q}n^{2/q}$ if $q > 2$ (see [44] or [113]).

It is interesting to check the strength of Theorem 2.5 in the particular example of ℓ_1^n. For every $\varepsilon \in (0,1)$ there exists $c(\varepsilon) > 0$ such that ℓ_1^n has a subspace E_k of dimension $k \geq c(\varepsilon)n$ with $d(E_k, \ell_2^k) \leq 1+\varepsilon$. Because of the nature of the argument, we have subspaces of ℓ_1^n of *some* dimension proportional to n which are "almost isometric" to Euclidean, but no information on $d(E_k, \ell_2^k)$ if k exceeds some fixed proportion of n. An isomorphic Dvoretzky type theorem for ℓ_1^n was proved by Kashin [74]: there exist $c(\alpha)$-Euclidean subspaces of ℓ_1^n of dimension $[\alpha n]$, for every $\alpha \in (0,1)$. Szarek realized that this property of ℓ_1^n is a consequence of the fact that its unit ball has bounded "volume ratio". This notion was formally introduced in [143]: The volume ratio of a symmetric convex body K in $\mathbb{R}^n$ is the parameter

$$\mathrm{vr}(K) = \inf\left\{ \left(\frac{|K|}{|E|}\right)^{1/n} : E \subseteq K \right\}, \tag{2.2.7}$$

where the inf is taken over all ellipsoids E contained in K. A simple computation shows that $\mathrm{vr}(B_1^n) \leq C$ for some absolute constant $C > 0$. Then, Kashin's theorem admits the following generalization [136], [143].

Theorem 2.7 *Let K be a symmetric convex body in $\mathbb{R}^n$ with $\mathrm{vr}(K) = A$. For every $k \leq n$ there exists a k-dimensional subspace E_k of X_K such that*

$$d(E_k, \ell_2^k) \leq (cA)^{\frac{n}{n-k}}, \tag{2.2.8}$$

where $c > 0$ is an absolute constant. $\square$

Isomorphic versions of Dvoretzky's theorem for arbitrary n-dimensional normed spaces were studied by Milman and Schechtman [114]. There exists an absolute constant $C > 0$ such that if $\dim X = n$ and $C \log n \leq k < n$, then X has a k-dimensional subspace E_k with $d(E_k, \ell_2^k) \leq C\sqrt{k/\log(n/k)}$.

We close this subsection with a recent result of Rudelson and Vershynin [129], which is different in nature but very close in spirit to the Dvoretzky type theorems we discussed. Let (T, μ, d) be a metric probability space whose concentration function satisfies the "normal Lévy estimate" $\alpha(T, t) \leq c_1 \exp(-c_2 t^2 n)$ for some n and all $t > 0$. In order to avoid degenerate cases we also assume that there exist $\varepsilon, \delta > 0$ such that the ε-neighborhood of any point in T has measure smaller than $1 - \delta$ (T is (ε, δ)-regular). We say that (T, d) is K-Lipschitz embedded into a normed space X if there exists $F : T \to X$ such that $d(x, y) \leq \|F(x) - F(y)\| \leq K \cdot d(x, y)$ for all $x, y \in T$. Assume that X is n-dimensional. If an (ε, δ)-regular metric probability space as above is K-Lipschitz embedded into X, then $k(X) \geq c\left(\frac{\varepsilon\delta}{K}\right)^4 n$. In other words, X must have Euclidean subspaces of proportional dimension. This fact underlines the importance of the concentration of measure phenomenon on the sphere: if some metric probability space with a normal concentration function well embeds into a normed space, this must be also true for the Euclidean space.

2.3 The ℓ-position and Pisier's inequality

One of the fundamental facts in the local theory of normed spaces is Pisier's estimate on the K-convexity constant. Combined with important results of Lewis, Figiel and Tomczak-Jaegermann, it leads to the following geometric statement: every convex body K in $\mathbb{R}^n$ has an affine image TK of volume 1 whose mean width satisfies the "reverse Urysohn inequality"

$$w(TK) \leq c\sqrt{n} \log n, \tag{2.3.1}$$

where $c > 0$ is an absolute constant. In this subsection we give a very concise description of this circle of ideas.

Let X be an n-dimensional normed space, and let α be a norm on $L(\ell_2^n, X)$. The dual norm is defined on $L(X^*, \ell_2^n)$ by

$$\alpha^*(v) = \sup\{\mathrm{tr}(vu) : \alpha(u) \leq 1\}. \tag{2.3.2}$$

The lemma of Lewis [83] applies to any pair of trace dual norms.

Theorem 2.8 *For any norm α on $L(\ell_2^n, X)$, there exists $u : \ell_2^n \to X$ such that $\alpha(u) = 1$ and $\alpha^*(u^{-1}) = n$.* □

The ℓ-norm on $L(\ell_2^n, X)$ was defined by Figiel and Tomczak-Jaegermann in [45]: Let $\{g_1, \ldots, g_n\}$ be a sequence of independent standard Gaussian random variables on some probability space, and let $\{e_1, \dots, e_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. If $u : \ell_2^n \to X$, the ℓ-norm of u is defined by

$$\ell(u) = \left(\mathbb{E} \| \sum_{i=1}^{n} g_i u(e_i) \|^2 \right)^{1/2} . \tag{2.3.3}$$

A standard computation gives

$$\ell(u) \simeq \sqrt{n} w((u^{-1})^*(K^\circ)), \tag{2.3.4}$$

where K is the unit ball of X. This formula connects the ℓ-norm to the mean width. It is more instructive to replace the Gaussians by the Rademacher functions $r_i : E_2^n \to \{-1, 1\}$ defined by $r_i(\varepsilon) = \varepsilon_i$, where $E_2^n = \{-1, 1\}^n$ is viewed as a probability space with the uniform measure. An inequality of Maurey and Pisier (see [113] or [147]) shows that

$$\ell(u) \simeq \left(\int_{E_2^n} \| \sum_{i=1}^{n} r_i(\epsilon) u(e_i) \|^2 d\epsilon \right)^{1/2} \tag{2.3.5}$$

up to a $\sqrt{\log n}$-term.

Consider the Walsh functions $w_A(\varepsilon) = \prod_{i \in A} r_i(\varepsilon)$, where $A \subseteq \{1, \dots, n\}$. It is not hard to see that every function $f : E_2^n \to X$ is uniquely represented in the form

$$f(\epsilon) = \sum_A w_A(\epsilon) x_A, \tag{2.3.6}$$

for some $x_A \in X$. The space of all functions $f : E_2^n \to X$ becomes a Banach space with the norm

$$\|f\|_{L_2(X)} = \left(\int_{E_2^n} \|f(\epsilon)\|^2 d\epsilon \right)^{1/2} \tag{2.3.7}$$

The Rademacher projection $R_n : L_2(X) \to L_2(X)$ is the operator sending $f = \sum w_A x_A$ to the function $R_n f := \sum_{i=1}^{n} r_i x_{\{i\}}$. Denote by $\mathrm{Rad}(X)$ the norm of this projection. Pisier [119] gave a sharp estimate in terms of the Banach-Mazur distance $d(X, \ell_2^n)$.

Theorem 2.9 *Let X be an n-dimensional normed space. Then,*

$$\mathrm{Rad}(X) \leq c \log[d(X, \ell_2^n) + 1], \tag{2.3.8}$$

where $c > 0$ is an absolute constant. □

Figiel and Tomczak-Jaegermann [45] had previously shown the relevance of this estimate to the study of the ℓ-norm.

Theorem 2.10 *Let X be an n-dimensional normed space. There exists $u : \ell_2^n \to X$ such that*

$$\ell(u)\ell((u^{-1})^*) \leq n\mathrm{Rad}(X). \tag{2.3.9}$$

Let us briefly sketch the proof. From Theorem 2.8, we can find an isomorphism $u : \ell_2^n \to X$ such that $\ell(u)\ell^*(u^{-1}) = n$. On the other hand,

$$\ell\left((u^{-1})^*\right) = \left(\int_{E_2^n} \|\sum_{i=1}^n r_i(\epsilon)(u^{-1})^*(e_i)\|_*^2 d\epsilon\right)^{1/2}. \tag{2.3.10}$$

There exists a function $\phi : E_2^n \to X$, which can be represented in the form $\phi = \sum_A w_A x_A$ and has norm $\|\phi\|_{L_2(X)} = 1$, such that

$$\ell((u^{-1})^*) = \langle \sum_{i=1}^n r_i(u^{-1})^*(e_i), \phi\rangle = \sum_{i=1}^n \langle (u^{-1})^*(e_i), x_{\{i\}}\rangle. \tag{2.3.11}$$

If we define $v : \ell_2^n \to X$ by $v(e_i) = x_{\{i\}}$, we easily check that

$$\ell((u^{-1})^*) = \mathrm{tr}(u^{-1}v) \leq \ell^*(u^{-1})\ell(v). \tag{2.3.12}$$

On observing that

$$\ell(v) = \|R_n(\phi)\|_{L_2(X)} \leq \mathrm{Rad}(X)\|\phi\|_{L_2(X)} = \mathrm{Rad}(X), \tag{2.3.13}$$

we get

$$\ell(u)\ell((u^{-1})^*) \leq \ell(u)\ell^*(u^{-1})\mathrm{Rad}(X) = n\mathrm{Rad}(X). \tag{2.3.14}$$

This concludes the proof. □

Combining the above with John's estimate $d(X, \ell_2^n) \leq \sqrt{n}$ [73], we can give an upper bound for the "minimal mean width" of a symmetric convex body (see §4.1 for a discussion on different "positions" of convex bodies).

Theorem 2.11 *If K is a symmetric convex body in $\mathbb{R}^n$, there exists a linear image $\tilde{K}$ of K with volume $|\tilde{K}| = 1$ and mean width*

$$w(\tilde{K}) \leq c\sqrt{n}\log n, \tag{2.3.15}$$

where $c > 0$ is an absolute constant.

For the proof, consider the operator $u : \ell_2^n \to X_K$ in Theorem 2.10 and set $\tilde{K} = (u^{-1})^*(K)$. In view of (2.3.4), John's theorem and Theorem 2.9, we have

$$w(\tilde{K})w(\tilde{K}^\circ) \leq c_1 \log n. \tag{2.3.16}$$

Computing the volume of $\tilde{K}$ in polar coordinates and using Hölder's inequality, we check that $w(\tilde{K}^\circ)^{-1} \leq c_2\sqrt{n}|\tilde{K}|^{1/n}$. It follows that

$$w(\tilde{K}) \leq c_3\sqrt{n}\log n|\tilde{K}|^{1/n}. \tag{2.3.17}$$

Normalizing the volume we obtain the assertion of the theorem. A simple argument based on the Rogers-Shephard inequality [125] shows that the symmetry of K is not necessary.

2.4 Low M^*-estimate and the quotient of subspace theorem

The Low M^*-estimate is the first step towards a general theory of sections and projections of symmetric convex bodies in $\mathbb{R}^n$ with dimension proportional to n. In geometric terms, it says that for fixed $\lambda \in (0,1)$, the diameter of a random $[\lambda n]$-dimensional section of the body K is controlled by its mean width

$$M^* := M(X) = \int_{S^{n-1}} \|x\|_* \sigma(dx) \tag{2.4.1}$$

up to a function depending only on λ.

Theorem 2.12 (Milman, [98], [99]) *There exists a function $f : (0,1) \to \mathbb{R}^+$ with the following property: for every $\lambda \in (0,1)$ and every n-dimensional normed space X, a random subspace $H \in G_{n,[\lambda n]}$ satisfies*

$$\frac{f(\lambda)}{M^*}|x| \leq \|x\| \tag{2.4.2}$$

for every $x \in H$.

The precise dependence on λ was established in a series of papers. Theorem 2.12 was originally proved in [98] and a second proof using the isoperimetric inequality on S^{n-1} was given in [99], with a bound of the form $f(\lambda) \geq c(1-\lambda)$. Pajor and Tomczak-Jaegermann [123] later showed that one can take $f(\lambda) \geq c\sqrt{1-\lambda}$ (see also [106] for a different proof with this dependence on λ). Finally, Gordon [64] proved that the theorem holds true with

$$f(\lambda) \geq \sqrt{1-\lambda}\left(1 + O\big(\frac{1}{(1-\lambda)n}\big)\right). \tag{2.4.3}$$

If we dualize the statement of the theorem, we get that a random $[\lambda n]$-dimensional projection of K_X contains a ball whose radius is of the order of $1/M$. For a random $H \in G_{n,[\lambda n]}$ we have

$$P_H(K_X) \supseteq \frac{f(\lambda)}{M} B_2^n \cap H. \tag{2.4.4}$$

The next step is the quotient of subspace theorem (Milman, [100]). In geometric terms, it says that for every symmetric convex body K in $\mathbb{R}^n$ and any $\alpha \in [1/2, 1)$, we can find subspaces $G \subset H$ with $\mathrm{dim}G \geq \alpha n$ and an ellipsoid $\mathcal{E}$ in G such that

$$\mathcal{E} \subset P_G(K \cap H) \subset c(1-\alpha)^{-1}|\log(1-\alpha)|\mathcal{E}. \tag{2.4.5}$$

Theorem 2.13 [100] *Let X be an n-dimensional normed space and let $\alpha \in [1/2, 1)$. Then, there exist subspaces $H \supset G$ of X such that $k = \dim(H/G) \geq \alpha n$ and*

$$d(H/G, \ell_2^k) \leq c(1-\alpha)^{-1}|\log(1-\alpha)|. \tag{2.4.6}$$

The proof of the theorem is based on the Low M^*-estimate and an iteration procedure in which Pisier's inequality plays a crucial role. We show the idea by describing the first step. We may assume that K_X satisfies the assertion of Theorem 2.10: because of (2.3.4) this can be written in the form $M(X)M^*(X) \leq c\log[d(X,\ell_2^n)+1]$.

Let $\lambda \in (0,1)$. Theorem 2.12 shows that on a random $[\lambda n]$-dimensional subspace H of X we have

$$\frac{c_1\sqrt{1-\lambda}}{M^*(X)}|x| \leq \|x\| \leq b|x|. \tag{2.4.7}$$

It is easy to check that for most $H \in G_{n,[\lambda n]}$ we have

$$M(H) \leq c_2 M(X). \tag{2.4.8}$$

If H satisfies both conditions, repeating the same argument for H^*, we may find a subspace G of H^* with $\mathrm{dim}G = k \geq \lambda^2 n$ and

$$\frac{c_3\sqrt{1-\lambda}}{M(X)}|x| \leq \frac{c_1\sqrt{1-\lambda}}{M^*(H^*)}|x| \leq \|x\|_{H^*} \leq \frac{M^*(X)}{c_1\sqrt{1-\lambda}}|x| \tag{2.4.9}$$

for every $x \in G$. In other words, $F := H/G$ satisfies

$$d(F,\ell_2^k) \leq c_4(1-\lambda)^{-1}M(X)M^*(X) \leq c(1-\lambda)^{-1}\log[d(X,\ell_2^n)+1]. \tag{2.4.10}$$

To set up the iteration, we write $QS(X)$ for the class of all quotient spaces of a subspace of X, and define a function $f:(0,1)\to\mathbb{R}^+$ by

$$f(\alpha) = \inf\{d(F,\ell_2^k) : F \in QS(X), \mathrm{dim}F \geq \alpha n\}. \tag{2.4.11}$$

The argument we have just described is equivalent to the statement

$$f(\lambda^2\alpha) \leq c(1-\lambda)^{-1}\log f(\alpha). \tag{2.4.12}$$

This is enough to estimate the function f as in Theorem 2.13. $\square$

It is natural to ask whether the estimate on the diameter of proportional dimensional sections given by Theorem 2.12 is precise in some sense. From the computational geometry point of view it would be desirable to have a simple way to determine the diameter of a random section of fixed proportion. One can easily rephrase the Low M^*-estimate as follows [108]: If r_1 is the solution of the equation

$$M^*(K\cap sB_2^n) = f(\lambda)s, \tag{2.4.13}$$

then for a random $[\lambda n]$-dimensional section $K\cap H$ of K we have

$$\mathrm{diam}(K\cap H) \leq 2r_1. \tag{2.4.14}$$

In view of Gordon's proof of Theorem 2.12, we can choose $f(\lambda) = (1-\varepsilon)\sqrt{1-\lambda}$ for any $\varepsilon \in (0,1)$, and then (2.4.14) is satisfied for all H in a subset of $G_{n,[\lambda n]}$ of measure greater than $1 - c_1 \exp(-c_2\varepsilon^2(1-\lambda)n)$. It turns out that the function $s \mapsto M^*(K \cap sB_2^n)$ can be used for a dual estimate [52]. There exists a second function $g : (0,1) \to \mathbb{R}$ with the following property: if $\lambda \in (1/2, 1)$ and if r_2 is the solution of the equation $M^*(K \cap sB_2^n) = g(\lambda)s$, then a random $[\lambda n]$-dimensional section $K \cap H$ of K satisfies $\mathrm{diam}(K \cap H) \geq 2r_2$, This gives a "confidence interval" $[r_2, r_1]$ for $\mathrm{diam}(K \cap H)$, which may be viewed as an asymptotic formula. What is essential is of course that the functions f and g can be described analytically and they do not depend on the dimension n or on the body K.

Another consequence of the Low M^*-estimate is that very accurate linear relations hold true in full generality for the diameter of sections of a body and its polar. This fact can be made precise in the following way [107]. Let $t(r) = t(X_K; r)$ be the greatest integer k for which a random subspace $H \in G_{n,k}$ satisfies $\mathrm{diam}(K \cap H) \leq 2r$. If $t^*(r) = t(X^*; r)$, then for any $\zeta > 0$ and any $r > 0$ we have

$$t(r) + t^*\left(\frac{1}{\zeta r}\right) \geq (1-\zeta)n - C, \tag{2.4.15}$$

where $C > 0$ is an absolute constant.

2.5 Coordinate theory

We fix an orthonormal basis $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ and for every non empty $\sigma \subseteq \{1, \dots, n\}$ we consider the coordinate subspace $\mathbb{R}^\sigma = \mathrm{span}\{e_j : j \in \sigma\}$. The following coordinate version of the Low M^*-estimate was established by Giannopoulos and Milman in [51]: If K is an ellipsoid in $\mathbb{R}^n$, then for every $\lambda \in (0,1)$ we can find $\sigma \subseteq \{1, \dots, n\}$ of cardinality $|\sigma| \geq (1-\lambda)n$ such that

$$P_{\mathbb{R}^\sigma}(K) \supseteq \frac{[\lambda/\log(1/\lambda)]^{1/2}}{M(K)} B_2^n \cap \mathbb{R}^\sigma . \tag{2.5.1}$$

This observation (which has its origin in [48], [49]) has consequences for the question of the maximal Banach-Mazur distance to the cube (see also the proportional Dvoretzky-Rogers factorization theorem in §6.1). The proof has its roots in an isomorphic version of the Sauer-Shelah lemma from Combinatorics, which was proved by Szarek and Talagrand [141] (see also [3], [142]), and is close in spirit to the theory of restricted invertibility of operators which was developed by Bourgain and Tzafriri [30].

As the example of the cube shows, one cannot have a coordinate low M^*-estimate for an arbitrary convex body. Under assumptions which guarantee the existence of "large ellipsoids" of any proportional dimension inside the body, one can use the above ellipsoidal result and obtain analogues of (2.5.1). This is done in [51] for bodies whose volume ratio or cotype-2 constant is well-bounded. These results can be applied to give estimates on the number of points with "many" integer coordinates inside a given convex body.

Very recently, Rudelson and Vershynin [130] obtained a new family of coordinate results. Assume that K is a symmetric convex body in $\mathbb{R}^n$ such that the norm $\|\cdot\|$

induced by K satisfies the conditions $\|x\| \leq |x|$ for all x and $M = M(K) \geq \delta$ for some positive constant $\delta > 0$. Then, there exist two positive numbers s and t with $c\delta \leq t \leq 1$ and $st \geq \delta / \log^{3/2}(2/\delta)$ and a subset σ of $\{1, \dots, n\}$ with cardinality $|\sigma| \geq s^2 n$, such that

$$\Big\| \sum_{i\in\sigma} a_i e_i \Big\| \geq \frac{ct}{\sqrt{n}} \sum_{i\in\sigma} |a_i| \tag{2.5.2}$$

for all choices of reals a_i, $i \in \sigma$. From this statement one can recover Elton's theorem about spaces which contain large dimensional copies of ℓ_1's [43] in an optimal form.

Note that the space $X = (\mathbb{R}^n, \|\cdot\|)$ satisfies $k(X) \simeq n(M/b)^2 \geq \delta n$. In other words, the result concerns spaces which have Euclidean subspaces of some dimension proportional to n (depending on δ). The estimate in (2.5.2) shows that

$$K \cap \mathbb{R}^\sigma \subseteq c(\delta)\sqrt{n} B_1^\sigma. \tag{2.5.3}$$

This may be viewed as a coordinate version of the low M^*-estimate for this class of bodies. The formulation is dual to the one in (2.5.1): one now considers sections instead of projections. The condition $k(X) \simeq n$ is in some sense dual to the assumptions on the volume ratio or the cotype-2 constant in [51].

To feel the analogy even more, we state the following "condition-free" version of the result in [130]: Let K be a symmetric convex body in $\mathbb{R}^n$ with $B_2^n \subseteq K$. There exists a subset σ of $\{1, \dots, n\}$ with cardinality $|\sigma| \geq cf(M)n$, such that

$$M \cdot (K \cap \mathbb{R}^\sigma) \subseteq \sqrt{|\sigma|} B_1^\sigma, \tag{2.5.4}$$

where $f(x) = x \log^{-3/2}(2/x)$. Compare with the low M^*-estimate: one has sections of the body inside an appropriate ℓ_1-ball on coordinate subspaces (this is weaker, but the example of ℓ_1^n shows that it is natural). Also, the parameter $1/M^*$ is replaced by M (which is stronger). However, the estimates hold for some proportional dimensions and not for any proportion.

All these are still preliminary but interesting results which show that a coordinate theory may be further developed in the future. This would have several consequences for the theory.

2.6 Covering results

Let K_1 and K_2 be two convex bodies in $\mathbb{R}^n$. The covering number $N(K_1, K_2)$ is the minimal cardinality of a finite subset A of $\mathbb{R}^n$ with the property

$$K_1 \subseteq A + K_2 = \bigcup_{x\in A} (x + K_2). \tag{2.6.1}$$

Note the multiplicative inequality $N(K_1, stK_3) \leq N(K_1, sK_2)N(K_2, tK_3)$ for all $t, s > 0$.

If we require $A \subset K_1$ we get the variant $\tilde{N}(K_1, K_2)$. If K_2 is symmetric, it is easy to see that $\tilde{N}(K_1, 2sK_2) \leq N(K_1, sK_2) \leq \tilde{N}(K_1, sK_2)$ for every $s > 0$. The standard way to estimate $\tilde{N}(K_1, K_2)$ is to consider a maximal subset $\{x_1, \ldots, x_N\}$ of K_1 any two points of which are at distance greater than or equal to 1 with respect to $\|\cdot\|_{K_2}$. Then, $K_1 \subseteq \cup(x_i + K_2)$ and this shows that $\tilde{N}(K_1, K_2) \leq N$.

The most classical estimate on covering numbers is Sudakov's inequality which gives a bound on $N(K, tB_2^n)$ in terms of the mean width of K.

Theorem 2.14 *Let K be a convex body in $\mathbb{R}^n$. For every $t > 0$,*

$$\log N(K, tB_2^n) \leq cn\,(w(K)/t)^2\,, \tag{2.6.2}$$

where $c > 0$ is an absolute constant.

This fact is an immediate translation of an inequality of Sudakov [135] on the expectation of the supremum of a Gaussian process (this in turn follows from Slepian's lemma). Let $\mathcal{Y} = (Y_x)_{x\in A}$ be a Gaussian process and let ρ denote the induced semimetric on T. If $M(A, t)$ is the largest possible number of elements of A which are t-separated, then

$$\mathbb{E} \sup_{x\in A} Y_x \geq 2^{-1/2}\nu(M(A,t)) \log^{1/2}(M(A,t))t, \tag{2.6.3}$$

where $\nu(n) = 0.648$ for $1 \leq n \leq 23$ and $\nu(n) = 2^{1/2} - \log n^{-1/2}$ for $24 \leq n$ (see [85], Section 14). Actually, the inequality is true for the sequence $\nu(n) = 2^{1/2} - \log\log n/(2^{3/2}\log n) + O(1/\log n)$ as $n \to \infty$ (see [47]).

Let $g_1, \ldots, g_n$ be independent standard Gaussian random variables on some probability space and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis in $\mathbb{R}^n$. If we consider the Gaussian process $Y_x = \langle \sum g_i e_i, x\rangle$, $x \in K$, then the induced metric on K is the Euclidean one and the estimates above show that, asymptotically,

$$\log^{1/2}(N(K, tB_2^n))t \leq \mathbb{E}\|\sum g_i e_i\|_*, \tag{2.6.4}$$

which gives (2.6.2) with a constant $c = c_n \to 1$ as $n \to \infty$.
A dual inequality was proved by Pajor and Tomczak-Jaegermann [123].

Theorem 2.15 *Let K be a symmetric convex body in $\mathbb{R}^n$. For every $t > 0$,*

$$\log N(B_2^n, tK) \leq cn\,(w(K^\circ)/t)^2\,, \tag{2.6.5}$$

where $c > 0$ is an absolute constant.

A simple proof of this fact was given by Talagrand (see [81] or [54]). From Theorem 2.15 one can deduce Sudakov's inequality with a duality argument of Tomczak-Jaegermann [146].

We close this subsection with some information on the duality conjecture for the entropy numbers of operators. The conjecture, which was stated by Pietsch

[118], asserts that if X, Y are Banach spaces, if $T : X \to Y$ is a compact operator and if $N(T, \varepsilon)$ denotes the covering number $N(T(B_X), \varepsilon B_Y)$, then

$$b^{-1} \log N(T, a^{-1}\varepsilon) \leq \log N(T^*, \varepsilon) \leq b \log N(T, a\varepsilon) \tag{2.6.6}$$

for every $\varepsilon > 0$, where $a, b > 0$ are absolute constants, and T^* is the adjoint operator of T. Until recently, this conjecture had been verified only under strong assumptions for both spaces X and Y (see [65] and [123]). In the case where one of the two spaces is a Hilbert space, the conjecture is equivalent to the following statement about covering numbers of convex bodies: There exist two constants $a, b > 0$ such that

$$\frac{1}{b} \log N(B_2^n, a^{-1}K^\circ) \leq \log N(K, B_2^n) \leq b \log N(B_2^n, aK^\circ) \tag{2.6.7}$$

for every symmetric convex body K in $\mathbb{R}^n$.

A weaker but general duality inequality was proved by König and Milman [79]. Using the reverse Santaló and Brunn-Minkowski inequalities (see §5.2) they showed that

$$c^{-1} N(K_2^\circ, K_1^\circ)^{1/n} \leq N(K_1, K_2)^{1/n} \leq c N(K_2^\circ, K_1^\circ)^{1/n} \tag{2.6.8}$$

for every pair of symmetric convex bodies K_1 and K_2 in $\mathbb{R}^n$. Note that this inequality proves the duality conjecture in the case where the logarithm of the covering numbers is large enough with respect to the dimension n.

Very recently, Artstein, Milman and Szarek [7], [8] proved (2.6.7) in full generality. This settles the duality conjecture in the case either X or $Y = H$ (a Hilbert space). The proof consists of three steps: Given a symmetric convex body K in $\mathbb{R}^n$, in the first step one shows that there exists a parameter γ depending on K such that $N(K, B_2^n) \leq N(B_2^n, \gamma^{-1}K^\circ)^3$ and $N(B_2^n, \gamma K^\circ) \leq N(K, B_2^n)^2$, which is "the conjecture up to γ". The idea is to project onto a random k-dimensional subspace: one knows that c-separated sets of points are mapped onto $c\sqrt{k/n}$-separated sets under such random projections, so the information on covering numbers is kept during this process (with the cost of γ). The dimension k is chosen so that the result of [79] will be enough to give duality for the projected bodies.

This step can be iterated, each time applied to an intersection of some multiple of K with a ball of suitable radius (here, a variant of Tomczak's duality argument is used). As a result, $N(K, B_2^n)$ and $N(B_2^n, K^\circ)$ are bounded by products of covering numbers of polar bodies. In the last step, each product can be "telescoped" to a product of only two or three terms, which establishes duality.

2.7 Global theory and asymptotic formulas

Let K be a (symmetric) convex body in $\mathbb{R}^n$. For a fixed dimension $1 \leq l \leq n$ consider the expected value

$$D_l(K) = \int_{G_{n,l}} \mathrm{diam}(P_E(K)) \nu_{n,l}(dE) \tag{2.7.1}$$

of the diameter of the orthogonal projection $P_E(K)$ onto $E \in G_{n,l}$. Theorem 2.5 shows that there is a critical value $k^* = n\big(w(K)/\mathrm{diam}(K)\big)^2$ such that: if $1 \le l \le k^*$ then

$$cw(K) \le D_l(K) \le Cw(K), \tag{2.7.2}$$

while if $k^* \le l \le n$, then

$$c\sqrt{l/n}\,\mathrm{diam}(K) \le D_l(K) \le C\sqrt{l/n}\,\mathrm{diam}(K). \tag{2.7.3}$$

Observe the phase transition at k^*: the random diameter of l-dimensional projections is stabilized since below the critical dimension k^* maximal symmetry has been achieved: most projections of the body have become isomorphic Euclidean balls of radius $w(K)/2$.

The same situation appears if one considers a dual "global problem". We want to approximate a Euclidean ball by Minkowski averages of rotations

$$K_t = \frac{1}{t}\big(u_1(K) + \cdots + u_t(K)\big) \tag{2.7.4}$$

of the body K. One way is to fix an integer $t \ge 2$ and ask for the infimum of $\mathrm{diam}(K_t)$ or the expected value $\mathbb{E}\mathrm{diam}(K_t)$ over all choices of $u_1, \dots, u_t \in O(n)$. It turns out (see [115]) that both quantities are of the same order, and

$$\mathbb{E}\,\mathrm{diam}(K_t) \simeq \frac{\mathrm{diam}(K)}{\sqrt{t}} \tag{2.7.5}$$

if $1 \le t \le t^* = \big[(\mathrm{diam}(K)/w(K))^2\big]$, while

$$\mathbb{E}\,\mathrm{diam}(K_t) \simeq w(K) \tag{2.7.6}$$

if $t^* \le t \le n$. Again, observe the phase transition at t^*. Stabilization occurs at $t \simeq t^*$ because above this integer $K_t \simeq w(K)B_2^n$ with very high probability: the norm of a random K_t has already become roughly Euclidean. Note also that, in this global process of forming averages of rotations, the "best possibility" (infimum of the diameter) coincides with the random one (expectation of the diameter).

The fact that the "asymptotic formula" $k^*t^* \simeq n$ holds true for every convex body K is only one instance of a remarkable duality. Local statements can be translated to global ones, and a very useful intuition can be developed through their comparison. However, the proofs of dual statements are not "direct translations" of each other, and they should often be invented from the start.

We proceed to another example of phase transition in which the stabilized behaviour is of a different nature. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$, and let a, b be the smallest positive constants for which $(1/a)|x| \le \|x\| \le b|x|$ is satisfied for every $x \in \mathbb{R}^n$. For every $q \ge 1$ consider the parameter

$$M_q = \left(\int_{S^{n-1}} \|x\|^q \sigma(dx)\right)^{1/q}. \tag{2.7.7}$$

Then, if $k(X) = n(M_1/b)^2$ one has the following behaviour of M_q (see [89]):

(a) $M_q \simeq M_1$ if $1 \leq q \leq k(X)$.

(b) $M_q \simeq b\sqrt{q/n}$ if $k(X) \leq q \leq n$.

(c) $M_q \simeq b$ if $q > n$.

The global q-approximation results are as follows: write

$$\|x\|_{q,t} = \left(\frac{1}{t}\sum_{i=1}^{t} \|u_i x\|^q\right)^{1/q}, \tag{2.7.8}$$

where $u_1, \ldots, u_t \in O(n)$, and let t_q be the smallest integer for which there exist $u_1, \ldots, u_t \in O(n)$ such that

$$(M_q/2)|x| \leq \|x\|_{q,t} \leq (2M_q)|x|. \tag{2.7.9}$$

Then, for the optimal value of t_q a random choice of $u_1, \ldots, u_t$ satisfies (2.7.9) up to some universal constants, and $t_q \simeq t_1$ for $1 \leq q \leq 2$, while $t_q^{2/q} \simeq t_1(M_1/M_q)^2$ for $q \geq 2$. If we insert the formulas for M_q in the above relations, we check that there are two phase transitions which occur on the interval $(1, n)$ at the values $q = k(X)$ and $q = 2$.

In this more complicated example of process, the initial "constant behaviour" of M_q may be viewed as a concentration phenomenon: the norm is almost constant on the sphere and this creates "inertia" in the behaviour of M_q for small values of q.

Our next example is a problem of approximation: write $I = [-x, x]$ for an interval, where $x \in S^{n-1}$. We would like to approximate the Euclidean ball B_2^n by zonotopes $K_N = \frac{1}{N}\sum_{i=1}^N I_i$. If we fix the cardinality N of summands and ask for the best approximation $A(N,n) := \inf\{d(K_N, B_2^n) : x_1, \ldots, x_N \in S^{n-1}\}$, then we have $A(N,n) = \infty$ if $N < n$, $A(N,n) = \sqrt{n}$ if $N = n$, and $A(N,n) = C(\lambda)$ if $N = [\lambda n]$ for some $\lambda > 1$ [74]. The behaviour of $C(\lambda)$ (say, for $\lambda < 2$) was determined by Gluskin [62]:

$$C(\lambda) \simeq \min\left\{\sqrt{n}, \sqrt{\big(\log(1/(\lambda-1))\big)/(\lambda-1)}\right\}.$$

Observe that we have a sharp threshold at the value $N = n$.

The same problem can be generalized as follows: let $\|\cdot\|$ be the norm defined by a symmetric convex body K on $\mathbb{R}^n$. Consider bodies of the form $K_N = \frac{1}{N}\sum_{i=1}^N u_i(K)$, where $u_i \in O(n)$. The question is what is the minimal value of N for which there exist $u_1, \ldots, u_N \in O(n)$ such that e.g. $d(K_N, B_2^n) \leq 4$. The answer is $N_0 \simeq t^* = (\mathrm{diam}(K)/w(K))^2$, and typically we have a sharp threshold for $\inf d(K_N, B_2^n)$ at this point. So, changing our parameter of study from "minimal diameter of K_N" to "geometric distance from the Euclidean ball", we often observe a phase transition behaviour being replaced by a threshold type one. Again, optimal and random behaviours are equivalent: if $N \geq ct^*/\varepsilon^2$ then a random choice of $u_1, \ldots, u_N \in O(n)$ satisfies $d(K_N, B_2^n) \leq 1 + \varepsilon$.

3 Classical convexity connected to the asymptotic theory

3.1 Brunn-Minkowski inequality: classical proofs and functional forms

The fundamental Brunn-Minkowski inequality states that if K and T are two non-empty compact subsets of $\mathbb{R}^n$, then

$$|K+T|^{1/n} \geq |K|^{1/n} + |T|^{1/n}. \tag{3.1.1}$$

If we make the additional hypothesis that K and T are convex bodies, then we can have equality only if K and T are homothetical.

The inequality expresses in a sense the fact that volume is an "n-concave" function with respect to Minkowski addition. For this reason, it is often written in the following form: If K, T are non-empty compact subsets of $\mathbb{R}^n$ and $\lambda \in (0,1)$, then

$$|\lambda K + (1-\lambda)T|^{1/n} \geq \lambda |K|^{1/n} + (1-\lambda)|T|^{1/n}. \tag{3.1.2}$$

Using (3.1.2) and the arithmetic-geometric means inequality we can also write

$$|\lambda K + (1-\lambda)T| \geq |K|^{\lambda} |T|^{1-\lambda}. \tag{3.1.3}$$

This weaker, but actually equivalent, form of the Brunn-Minkowski inequality has the advantage (or disadvantage) of being dimension free.

There are many interesting proofs of the Brunn-Minkowski inequality, all of them related to important ideas. Historically, the first proof of the Brunn-Minkowski inequality was based on Brunn's concavity principle:

Theorem 3.1 *Let K be a convex body in $\mathbb{R}^n$ and let F be a k-dimensional subspace of $\mathbb{R}^n$, $1 \leq k \leq n$. Then, the function $f : F^{\perp} \to \mathbb{R}$ defined by $f(x) = |K \cap (F+x)|^{1/k}$ is concave on its support.*

The proof goes by symmetrization. The Steiner symmetrization of K in the direction of $\theta \in S^{n-1}$ is the set $S_\theta(K)$ consisting of all points of the form $x + \lambda\theta$, where x is in the projection $P_{\theta^\perp}(K)$ of K onto $\theta^\perp$ and $|\lambda| \leq \frac{1}{2} \times \text{length}(x+\mathbb{R}\theta) \cap K$. Steiner symmetrization preserves convexity and volume: if K is a convex body then $S_\theta(K)$ is also a convex body, and $|S_\theta(K)| = |K|$. A well known fact which goes back to Steiner and Schwarz is that for every convex body K one can find a sequence of successive Steiner symmetrizations in directions $\theta \in F$ so that the limiting convex body $\tilde{K}$ has the following property:

> For every $x \in F^{\perp}$, $\tilde{K} \cap (F+x)$ is a ball with center at x and radius $r(x)$ such that $|\tilde{K} \cap (F+x)| = |K \cap (F+x)|$.

Now, the proof of the theorem is immediate. Convexity of $\tilde{K}$ implies that r is concave on its support, and this shows that f is also concave.

Brunn's concavity principle implies the Brunn-Minkowski inequality as follows. If K and T are convex bodies in $\mathbb{R}^n$, we define $K_1 = K \times \{0\}$ and $T_1 = T \times \{1\}$ in $\mathbb{R}^{n+1}$ and consider their convex hull L. If we set $L(t) = \{x \in \mathbb{R}^n : (x,t) \in L\}$ for all $t \in [0,1]$, we easily check that $L(0) = K$, $L(1) = T$ and $L(1/2) = \frac{K+T}{2}$. Then, Brunn's concavity principle for $F = \mathbb{R}^n$ shows that

$$\Big|\frac{K+T}{2}\Big|^{1/n} \geq \frac{1}{2}|K|^{1/n} + \frac{1}{2}|T|^{1/n}. \tag{3.1.4}$$

A functional form of the Brunn-Minkowski inequality is an integral inequality which reduces to (3.1.1) by appropriate choice of the functions involved. The advantage of such functional inequalities is that they can be applied in many other contexts: an example is given by the Prékopa-Leindler inequality (see [121] or [14]) which is stated below: it can be applied to yield the logarithmic Sobolev inequality and several important concentration results in Gauss space.

Theorem 3.2 *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ be measurable functions, and let $\lambda \in (0,1)$. We assume that f and g are integrable, and for every $x, y \in \mathbb{R}^n$*

$$h(\lambda x + (1-\lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}. \tag{3.1.5}$$

Then,

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^\lambda \left(\int_{\mathbb{R}^n} g\right)^{1-\lambda}. \tag{3.1.6}$$

We shall only sketch the case $n = 1$. We may assume that f and g are continuous and strictly positive and then define $x, y : (0,1) \to \mathbb{R}$ by the equations

$$\int_{-\infty}^{x(t)} f = t\int f \quad \text{and} \quad \int_{-\infty}^{y(t)} g = t\int g. \tag{3.1.7}$$

Then, x and y are differentiable, and for every $t \in (0,1)$ we have

$$x'(t)f(x(t)) = \int f \quad \text{and} \quad y'(t)g(y(t)) = \int g. \tag{3.1.8}$$

We now define $z : (0,1) \to \mathbb{R}$ by $z(t) = \lambda x(t) + (1-\lambda)y(t)$. Since x and y are strictly increasing, z is also strictly increasing, and the arithmetic-geometric means inequality shows that

$$z'(t) = \lambda x'(t) + (1-\lambda)y'(t) \geq (x'(t))^\lambda (y'(t))^{1-\lambda}. \tag{3.1.9}$$

Hence, we can estimate the integral of h making the change of variables $s = z(t)$:

$$\begin{aligned}
\int h &= \int_0^1 h(z(t))z'(t)dt \\
&\geq \int_0^1 h(\lambda x(t) + (1-\lambda)y(t))(x'(t))^{\lambda}(y'(t))^{1-\lambda}dt \\
&\geq \int_0^1 f^{\lambda}(x(t))g^{1-\lambda}(y(t)) \left(\frac{\int f}{f(x(t))}\right)^{\lambda} \left(\frac{\int g}{g(y(t))}\right)^{1-\lambda} dt \\
&= \left(\int f\right)^{\lambda} \left(\int g\right)^{1-\lambda}.
\end{aligned}$$

Induction on the dimension completes the proof.

The Brunn-Minkowski inequality is a simple consequence of Theorem 3.2. Let K and T be non-empty compact subsets of $\mathbb{R}^n$, and let $\lambda \in (0,1)$. We define $f = \chi_K$, $g = \chi_T$, and $h = \chi_{\lambda K + (1-\lambda)T}$. It is easily checked that the assumptions of Theorem 3.2 are satisfied, therefore

$$|\lambda K + (1-\lambda)T| = \int h \geq \left(\int f\right)^{\lambda} \left(\int g\right)^{1-\lambda} = |K|^{\lambda}|T|^{1-\lambda}. \tag{3.1.10}$$

There are many variants of the Prékopa-Leindler inequality. All of them can be proved by a "transportation of measure" argument similar to the one used above. We shall state one of them and use it to give a functional version of a proof of Brunn's principle which was given by Gromov and Milman [71].

We first introduce some notation: If $p > 0$ and $\lambda \in (0,1)$, for all $x, y > 0$ we set

$$M_p^{\lambda}(x,y) = (\lambda x^p + (1-\lambda)y^p)^{1/p}.$$

If $x, y \geq 0$ and $xy = 0$, we set $M_p^{\lambda}(x,y) = 0$. Observe that $\lim_{p\to 0^+} M_p^{\lambda}(x,y) = x^{\lambda}y^{1-\lambda}$.

Statement: *Suppose that $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ are measurable functions, and let $p > 0$, $\lambda \in (0,1)$. We assume that f and g are integrable, and for every $x, y \in \mathbb{R}^n$*

$$h(\lambda x + (1-\lambda)y) \geq M_p^{\lambda}(f(x), g(y)). \tag{3.1.11}$$

Then,

$$\int_{\mathbb{R}^n} h \geq M_{p/(pn+1)}^{\lambda}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right). \tag{3.1.12}$$

The proof of the statement is quite similar to the proof of the Prékopa-Leindler inequality given above.

We need a few more definitions: Let K be a convex set in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}^+$. We say that f is α-concave for some $\alpha > 0$, if $f^{1/\alpha}$ is concave on K. It is easy to see that if $f, g : K \to \mathbb{R}^+$ and if f is α-concave and g is β-concave, then fg is $(\alpha+\beta)$-concave.

Let now K be a convex body in $\mathbb{R}^n$ and let $\theta \in S^{n-1}$. For every $y \in P_{\theta^\perp}(K)$ we write I_y for the interval $\{t \in \mathbb{R} : y + t\theta \in K\}$. For every continuous function $f : K \to \mathbb{R}^+$ we define the projection $P_\theta f$ of f with respect to θ by

$$(P_\theta f)(y) := \int_{I_y} f(y+t\theta)dt, \quad y \in P_{\theta^\perp}(K). \tag{3.1.13}$$

If we define $F_y(t) = \chi_K(y+t\theta)f(y+t\theta)$ for $y \in P_{\theta^\perp}(K)$, then by the α-concavity of f and the convexity of K we easily check that

$$F_{\lambda y+(1-\lambda)w}(\lambda t + (1-\lambda)s) \geq M^\lambda_{1/\alpha}(F_y(t), F_w(s)) \tag{3.1.14}$$

for all $y, w \in P_{\theta^\perp}(K)$. Applying the statement, we immediately get:

Claim: *If f is α-concave, then $P_\theta f$ is $(1+\alpha)$-concave.*

We now finish the proof of Brunn's principle as follows. Let F be a k-dimensional subspace of $\mathbb{R}^n$. The indicator function of K is constant on K, and hence it is α-concave for every $\alpha > 0$. We choose an orthonormal basis $\{\theta_1, \ldots, \theta_k\}$ of F and perform successive projections in the directions of θ_i. The claim shows that the function $x \mapsto |K \cap (F+x)|$ is $(\alpha+k)$-concave on $P_{F^\perp}(K)$, for every $\alpha > 0$. It follows that $f(x) = |K \cap (F+x)|^{1/k}$ is concave.

The Prékopa–Leindler inequality and the statement above, have recently been extended to Riemannian manifolds [37]. There, the curvature plays an essential role (through the Ricci curvature, in particular) and a *distortion* coefficient has to be added to the condition (3.1.5). We will state the spherical extension of the Prékopa–Leindler inequality obtained in [36]. Let ρ denote the (geodesic) distance on the sphere S^n and σ the usual rotationally invariant measure on S^n. For $x, y \in S^n$ with $x \neq -y$, introduce the geodesic analogue of the point $tx + (1-t)y$, namely the point $z = \gamma_t(x,y) \in S^n$ verifying

$$\rho(x,z) = (1-t)\rho(x,y) \quad \text{and} \quad \rho(z,y) = t\rho(x,y). \tag{3.1.15}$$

If $x = \cos(\theta)y + \sin(\theta)v$ with $\theta \in [0,\pi)$ and $v \in S^n$ orthogonal to y, then $\gamma_t(x,y) = \cos(t\theta)y + \sin(t\theta)v$. For $t \in (0,1)$ and $d \in [0,\pi]$, set $S(d) := d^{-1}\sin d$ and

$$L_t(d) := \big(S(d)/S(td)\big)^t \big(S(d)/S((1-t)d)\big)^{1-t}. \tag{3.1.16}$$

Theorem 3.3 *Let $f, g, h : S^n \to R^+$ be Borel functions and $t \in (0,1)$. We assume that for every $x \neq -y \in S^n$,*

$$h(\gamma_t(x,y)) \geq L_t(\rho(x,y))^{n-1} f(x)^t g(y)^{1-t}. \tag{3.1.17}$$

Then

$$\int h\, d\sigma \geq \left(\int f\, d\sigma\right)^t \left(\int g\, d\sigma\right)^{1-t}. \tag{3.1.18}$$

Since $L_t(\pi) = 0$, the condition (3.1.17) is always satisfied when $x = -y$. From $L_t(d) \leq 1$, we deduce in particular that the Brunn-Minkowski inequality holds on the sphere for the geodesic midsum of two sets, say. It is known that $L_t(d) \leq e^{-t(1-t)d^2/2}$ and thus the coefficient $L_t(\rho(x,y))^{n-1}$ in (3.1.17) can be replaced by the coefficient

$$e^{-(n-1)t(1-t)\rho^2(x,y)/2}.$$

With this form, one can recover, as in [95], the classical concentration results for the sphere.

3.2 Geometric inequalities of hyperbolic type

We write $\mathcal{K}_n$ for the class of non-empty, compact convex subsets of $\mathbb{R}^n$. Minkowski's fundamental theorem states that if $K_1, \dots, K_m \in \mathcal{K}_n$, $m \in \mathbb{N}$, there exist coefficients $V(K_{i_1}, \dots, K_{i_n})$, $1 \leq i_1, \dots, i_n \leq m$ which are invariant under permutations of their arguments, such that

$$|t_1 K_1 + \cdots + t_m K_m| = \sum_{1 \leq i_1, \dots, i_n \leq m} V(K_{i_1}, \dots, K_{i_n}) t_{i_1} \dots t_{i_n} \tag{3.2.1}$$

for every choice of non-negative real numbers t_i (see [134] or [34]). The coefficient $V(A_1, \dots, A_n)$ is called the mixed volume of the compact convex sets $A_1, \dots, A_n$. A special case of Minkowski's theorem is Steiner's formula. If $K \in \mathcal{K}_n$, then

$$|K + tB_2^n| = \sum_{i=0}^{n} \binom{n}{i} V_{n-i}(K) t^i \tag{3.2.2}$$

for all $t > 0$, where $V_{n-i}(K) = V(K; n-i, B_2^n; i)$ is the i-th quermassintegral of K.

A very deep and strong generalization of the Brunn-Minkowski inequality is the Alexandrov-Fenchel inequality [1], [2] (see [134]): If $K, T, A_3, \dots, A_n \in \mathcal{K}_n$, then

$$V(K, T, A_3, \dots, A_n)^2 \geq V(K, K, A_3, \dots, A_n) V(T, T, A_3, \dots, A_n). \tag{3.2.3}$$

Among many consequences of (3.2.3), one should mention the inequalities

$$V_i(K + L)^{1/i} \geq V_i(K)^{1/i} + V_i(L)^{1/i} \tag{3.2.4}$$

which hold true for all convex bodies K, T in $\mathbb{R}^n$ and all $i \in \{1, \dots, n\}$, and the Alexandrov inequalities

$$\left(\frac{V_i(K)}{|B_2^n|}\right)^{1/i} \geq \left(\frac{V_j(K)}{|B_2^n|}\right)^{1/j}, \tag{3.2.5}$$

where $1 \leq i < j \leq n$. Note that the Brunn-Minkowski inequality and the isoperimetric inequality are special cases of (3.2.4) and (3.2.5) respectively.

Going back in time, we locate numerical inequalities which are surprisingly similar to the ones above (see [18]). Let $\overline{x} = (x_1, \dots, x_n)$ be an n-tuple of positive real numbers, and consider the normalized elementary symmetric functions $E_0(\overline{x}) \equiv 1$ and

$$E_i(x_1, \dots, x_n) = \frac{1}{\binom{n}{i}} \sum_{1 \le j_1 < \dots < j_i \le n} x_{j_1} x_{j_2} \dots x_{j_i} \tag{3.2.6}$$

for $i = 1, \dots, n$. With this definition, $E_1(\overline{x})$ and $E_n^{1/n}(\overline{x})$ correspond to the arithmetic and geometric means of $x_1, \dots, x_n$. Newton proved that

$$E_k^2(\overline{x}) \ge E_{k-1}(\overline{x}) E_{k+1}(\overline{x}) \tag{3.2.7}$$

for all $k = 1, \dots, n-1$, with equality if and only if all the x_i's are equal. Maclaurin observed that

$$E_1(\overline{x}) \ge E_2^{1/2}(\overline{x}) \ge \dots \ge E_n^{1/n}(\overline{x}). \tag{3.2.8}$$

These inequalities follow immediately from Newton's inequality (3.2.7) and they generalize the arithmetic-geometric means inequality.

One can feel the analogy with the Alexandrov-Fenchel inequalities even more, by considering the more recent Marcus-Lopes inequality

$$\frac{E_k(\overline{x} + \overline{y})}{E_{k-1}(\overline{x} + \overline{y})} \ge \frac{E_k(\overline{x})}{E_{k-1}(\overline{x})} + \frac{E_k(\overline{y})}{E_{k-1}(\overline{y})}, \tag{3.2.9}$$

which holds true for all $k = 1, \dots, n$. As a formal consequence one gets

$$[E_k(\overline{x} + \overline{y})]^{1/k} \ge [E_k(\overline{x})]^{1/k} + [E_k(\overline{y})]^{1/k}. \tag{3.2.10}$$

We now pass to the multidimensional case: let S_n^+ be the space of real positive symmetric $n \times n$ matrices. If $t_1, \dots, t_m > 0$ and $A_1, \dots, A_m \in S_n^+$, then

$$\det(t_1 A_1 + \dots + t_m A_m) = \sum_{1 \le i_1 \le \dots \le i_n \le m} n! D(A_{i_1}, \dots, A_{i_n}) t_{i_1} \dots t_{i_n}, \tag{3.2.11}$$

where the coefficient $D(B_1, \dots, B_n)$ is invariant under permutations of its arguments and is called the mixed discriminant of $B_1, \dots, B_n$. Based on the fact that $P(t) = \det(A + tI)$ has only real roots for any $A \in S_n^+$ leads to some very interesting inequalities about mixed discriminants, which are completely analogous to Newton's inequalities, and were discovered by Alexandrov. Examples are the inequalities

$$D(A, B, C_3, \dots, C_n)^2 \ge D(A, A, C_3, \dots, C_n) D(B, B, C_3, \dots, C_n) \tag{3.2.12}$$

for all $A, B, C_3, \dots, C_n \in S_n^+$ and

$$D(A_1, A_2, \dots, A_n) \ge \prod_{i=1}^{n} [\det A_i]^{1/n}. \tag{3.2.13}$$

There are many other inequalities on positive symmetric matrices, and one is tempted to look for their analogues in the setting of convex geometry. An inequality of Bergstrom (see [18]), which is the matrix analogue of (3.2.9), states that if A and B are symmetric positive definite matrices and if A_i, B_i denote the submatrices obtained by deleting the i-th row and column, then

$$\frac{\det(A+B)}{\det(A_i+B_i)} \geq \frac{\det(A)}{\det(A_i)} + \frac{\det(B)}{\det(B_i)}. \tag{3.2.14}$$

This is generalized by Ky Fan in the form

$$\left(\frac{\det(A+B)}{\det(A_k+B_k)}\right)^{1/k} \geq \left(\frac{\det(A)}{\det(A_k)}\right)^{1/k} + \left(\frac{\det(B)}{\det(B_k)}\right)^{1/k}, \tag{3.2.15}$$

where A_k is the submatrix of A we obtain if we delete k rows and the corresponding columns of A. When $k=n$, this reduces to Minkowski's inequality $[\det(A+B)]^{1/n} \geq [\det A]^{1/n} + [\det B]^{1/n}$. For related inequalities about mixed volumes see [50], [46].

One last comment is that behind all these numerical or convex geometric inequalities there is a unified principle: "the minimum of certain functionals is achieved on equal objects". Statements like the Brunn-Minkowski or the Alexandrov-Fenchel inequality may be equivalently expressed in the form

$$f(A,B) \geq \min\{f(A,A), f(B,B)\}. \tag{3.2.16}$$

The Brunn-Minkowski inequality can be rederived from its simple consequence $|aK+bT| \geq \min\{|(a+b)K|, |(a+b)T|\}$. Likewise, the Alexandrov-Fenchel inequality is equivalent to the inequality

$$V(K,T,A_3,\dots,A_n)^2 \geq \min\{V(K,K,A_3,\dots,A_n), V(T,T,A_3,\dots,A_n)\}. \tag{3.2.17}$$

The same principle applies to all the hyperbolic type inequalities we discussed in this subsection. In contrast, "elliptic type" inequalities like the triangle inequality and the Cauchy-Schwarz inequality obey a "maximum principle": for example, the latter unequality is equivalent to the statement

$$\int |f\cdot g| d\mu \leq \max\left\{ \int |f|^2 d\mu, \int |g|^2 d\mu \right\}. \tag{3.2.18}$$

The maximum of the functional $(f,g) \mapsto \int |f\cdot g| d\mu$ is "achieved on equal objects". Hölder's inequality is also a consequence of such an "elliptic" principle, which should however be correctly applied so that the functions f and g involved stay in "correct" spaces. If p and q are conjugate exponents, then the inequality

$$\int |f\cdot g| d\mu \leq \max\left\{ \left(\int |f|^p d\mu\right)^{1/(p-1)}, \left(\int |g|^q d\mu\right)^{1/(q-1)} \right\} \tag{3.2.19}$$

for all $f \in L^p$ and $g \in L^q$, is equivalent to the classical Hölder's inequality.

3.3 Volume preserving transformations

Let K and T be two open convex bodies in $\mathbb{R}^n$. A volume preserving transformation from K onto T is a map $\phi : K \to T$ which is one to one, onto and has a Jacobian with costant determinant equal to $|K|/|T|$. In this section we describe two such maps, the Knöthe map and the Brenier map. Applying each one of them we may obtain alternative proofs of the Brunn-Minkowski inequality.

The Knöthe map: We fix a coordinate system in $\mathbb{R}^n$. The properties of the Knöthe map [78] from K to T with respect to the given coordinate system are described in the following theorem.

Theorem 3.4 *Let K and T be open convex bodies in $\mathbb{R}^n$. There exists a map $\phi : K \to T$ with the following properties (for a proof see [113]):*

(a) ϕ is triangular: the i-th coordinate function of ϕ depends only on $x_1, \dots, x_i$. That is,

$$\phi(x_1, \dots, x_n) = (\phi_1(x_1), \phi_2(x_1, x_2), \dots, \phi_n(x_1, \dots, x_n)). \tag{3.3.1}$$

(b) The partial derivatives $\frac{\partial \phi_i}{\partial x_i}$ exist and they are positive on K, and the determinant of the Jacobian of ϕ is constant. More precisely, for every $x \in K$

$$|\det J_\phi(x)| = \prod_{i=1}^{n} \frac{\partial \phi_i}{\partial x_i}(x) = \frac{|T|}{|K|}. \tag{3.3.2}$$

The Brenier map: For any two open convex bodies K and T there exists a volume preserving transformation from K onto T, called the Brenier map [33], which is the gradient of a C^2 convex function. The existence of this remarkable map is a consequence of a more general transportation of measure result which we briefly describe.

Consider the space $\mathcal{P}(\mathbb{R}^n)$ of Borel probability measures on $\mathbb{R}^n$ as a subset of the unit ball of $C_\infty(\mathbb{R}^n)^*$ (the dual of the space of continuous functions which vanish uniformly at infinity). Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function which is defined μ-almost everywhere and satisfies $\nu(B) = \mu(T^{-1}(B))$ for every Borel subset B of $\mathbb{R}^n$, we say that T pushes forward μ to ν and write $T\mu = \nu$. It is easy to see that $T\mu = \nu$ if and only if for every bounded Borel measurable $g : \mathbb{R}^n \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^n} g(y) d\nu(y) = \int_{\mathbb{R}^n} g(T(x)) d\mu(x). \tag{3.3.3}$$

Generalizing work of Brenier, McCann [96] proved the following.

Theorem 3.5 *Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ and assume that μ is absolutely continuous with respect to Lebesgue measure. Then, there exists a convex function $f : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f : \mathbb{R}^n \to \mathbb{R}^n$ is defined μ-almost everywhere, and $(\nabla f)\mu = \nu$.*

The proof of Theorem 3.5 is based on the notion of cyclical monotonicity from convex analysis: A subset G of $\mathbb{R}^n \times \mathbb{R}^n$ is called cyclically monotone if for every $m \geq 2$ and $(x_i, y_i) \in G$, $i \leq m$, we have

$$\langle y_1, x_2 - x_1\rangle + \langle y_2, x_3 - x_2\rangle + \cdots + \langle y_m, x_1 - x_m\rangle \leq 0. \tag{3.3.4}$$

Fact 1: *Let μ and ν be Borel probability measures on $\mathbb{R}^n$. There exists a joint probability measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ which has cyclically monotone support and marginals μ, ν i.e. for all bounded Borel measurable $f, g : \mathbb{R}^n \to \mathbb{R}$ we have*

$$\int_{\mathbb{R}^n} f(x)d\mu(x) = \int_{\mathbb{R}^n\times\mathbb{R}^n} f(x)d\gamma(x,y) \tag{3.3.5}$$

and

$$\int_{\mathbb{R}^n} g(y)d\nu(y) = \int_{\mathbb{R}^n\times\mathbb{R}^n} g(y)d\gamma(x,y). \tag{3.3.6}$$

The second ingredient is the connection of cyclically monotone sets with convex functions (see [124]). For every proper convex function $f : \mathbb{R}^n \to \mathbb{R}$ we consider the subdifferential of f

$$\partial(f) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : f(z) \geq f(x) + \langle y, z - x\rangle,\ z \in \mathbb{R}^n\}. \tag{3.3.7}$$

The subdifferential parametrizes the supporting hyperplanes of f: the set $\partial(f)(x) = \{y : (x,y) \in \partial(f)\}$ is a closed and bounded convex set, and differentiability of f at x is equivalent to the existence of a unique $y \in \partial f(x)$, in which case $\nabla f(x) = y$.

Fact 2: *Let $G \subset \mathbb{R}^n \times \mathbb{R}^n$. Then, G is contained in the subdifferential of a proper convex function $f : \mathbb{R}^n \to \mathbb{R}$ if and only if G is cyclically monotone.*

We can now sketch the proof of Theorem 3.5. From Fact 1 there exists a probability measure γ on $\mathbb{R}^n \times \mathbb{R}^n$ which has cyclically monotone support and marginals μ, ν. Fact 2 shows that the support of γ is contained in the subdifferential of a proper convex function $f : \mathbb{R}^n \to \mathbb{R}$. Since f is convex and μ is absolutely continuous with respect to Lebesgue measure, f is differentiable μ-almost everywhere. Since $\mathrm{supp}(\gamma) \subset \partial(f)$, by the definition of the subdifferential we have $y = \nabla f(x)$ for almost all pairs (x,y) with respect to γ. Then, for every bounded Borel measurable $g : \mathbb{R}^n \to \mathbb{R}$ we see that

$$\int g(y)d\nu(y) = \int g(y)d\gamma(x,y) = \int g(\nabla f(x))d\gamma(x,y) = \int g(\nabla f(x))d\mu(x), \tag{3.3.8}$$

which shows that $(\nabla f)\mu = \nu$.

Assume that μ and ν are the normalized Lebesgue measures on some convex bodies K and T. Regularity results of Caffarelli show that in this case f may be assumed twice continuously differentiable. This proves the following.

Theorem 3.6 *Let K and T be open convex bodies in $\mathbb{R}^n$. There is a convex function $f \in C^2(K)$ such that $\phi = \nabla f : K \to T$ is one to one, onto and volume preserving.*

We can now show the Brunn-Minkowski inequality using either the Knöthe or the Brenier map. In each case we have $(I + \phi)(K) \subseteq K + T$. If ϕ denotes the Knöthe map, $J_{I+\phi}(x)$ is triangular and this implies

(3.3.9)
$$|\det J_{I+\phi}(x)|^{1/n} = \prod_{i=1}^{n} \left(1 + \frac{\partial \phi_i(x)}{\partial x_i}\right)^{1/n} \geq 1 + |\det J_\phi(x))^{1/n} = 1 + \left(\frac{|T|}{|K|}\right)^{1/n}.$$

If ϕ is the Brenier map, it is clear that the Jacobian $J_\phi = \mathrm{Hess}f$ is a symmetric positive definite matrix for every $x \in K$. Therefore,

$$|\det J_{I+\phi}(x)| = |\det (I + \mathrm{Hess}f)(x)| = \prod_{i=1}^{n}(1 + \lambda_i(x)) \tag{3.3.10}$$

where $\lambda_i(x)$ are the non negative eigenvalues of $\mathrm{Hess}f$. Moreover, by the volume preserving property of ϕ, we have $\prod_{i=1}^{n} \lambda_i(x) = |T|/|K|$ for every $x \in K$. Therefore, the arithmetic-geometric means inequality gives

$$|\det J_{I+\phi}(x)|^{1/n} \geq 1 + \left(\frac{|T|}{|K|}\right)^{1/n}. \tag{3.3.11}$$

In both cases,

$$|K + T| \geq \int_{(I+\phi)K} dx = \int_K |\det J_{I+\phi}(x)| dx \geq |K| \left(1 + (|T|/|K|)^{1/n}\right)^n, \tag{3.3.12}$$

which is the Brunn-Minkowski inequality.

For an arbitrary pair of open convex bodies K_1 and K_2 it would be desirable to achieve a volume preserving transformation $\psi : K_1 \to K_2$ for which $(I + \psi)(K_1) = K_1 + K_2$. This was recently done in [4]. There are two ingredients in the construction: the first one is a regularity result of Caffarelli [35] (see also [4]):

Fact 3: *If T is an open convex body in $\mathbb{R}^n$, f is a probability density on $\mathbb{R}^n$, and g is a probability density on T such that f is locally bounded and bounded away from zero on compact sets, and there exist $c_1, c_2 > 0$ such that $c_1 \leq g(y) \leq c_2$ for every $y \in T$, then the Brenier map $\nabla f : (\mathbb{R}^n, f dx) \to (\mathbb{R}^n, g dx)$ is continuous and belongs locally to the Hölder class C^α for some $\alpha > 0$.*

The second is a theorem of Gromov [67] (see also [4]):

Fact 4: *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2-smooth convex function with strictly positive Hessian. Then, the image of the gradient map $\mathrm{Im}\nabla f$ is an open convex set. Also, if f_1, f_2 are two such functions, then*

$$\mathrm{Im}(\nabla f_1 + \nabla f_2) = \mathrm{Im}(\nabla f_1) + \mathrm{Im}(\nabla f_2). \tag{3.3.13}$$

Having these tools in hand and given two open convex bodies K_1 and K_2 of volume 1 in $\mathbb{R}^n$, we choose a smooth strictly positive density ρ on $\mathbb{R}^n$ and consider the Brenier maps

$$\psi_i = \nabla f_i : (\mathbb{R}^n, \rho dx) \to (K_i, dx) \quad , \quad i = 1, 2. \tag{3.3.14}$$

Fact 3 shows that ψ_1 and ψ_2 are C^1-smooth. Applying Fact 4, we see that, for every $\lambda > 0$,

$$K_1 + \lambda K_2 = \{\nabla f_1(x) + \lambda \nabla f_2(x) : x \in \mathbb{R}^n\}. \tag{3.3.15}$$

Then, the map $\psi = \psi_2 \circ (\psi_1)^{-1} : K_1 \to K_2$ is a volume preserving C^1-diffeomorphism and satisfies $K_1 + \lambda K_2 = (I + \lambda\psi)(K_1)$ for all $\lambda > 0$.

This construction reveals the close relation between mixed volumes and mixed discriminants. Let $K_1, \dots, K_n$ be open convex bodies K_i with normalized volume $|K_i| = 1$, and consider the Brenier maps

$$\phi_i : (\mathbb{R}^n, \gamma_n) \to K_i, \tag{3.3.16}$$

where γ_n is the standard Gaussian probability density on $\mathbb{R}^n$. We have $\phi_i = \nabla f_i$, where f_i are convex functions on $\mathbb{R}^n$. By Caffarelli's regularity result, all the ϕ_i's are smooth maps. Then, the image of $(\mathbb{R}^n, \gamma_n)$ by $\sum t_i \phi_i$ is the interior of $\sum t_i K_i$. Since each ϕ_i is a measure preserving map, we have

$$\det\left(\frac{\partial^2 f_i}{\partial x_k \partial x_l}\right)(x) = \gamma_n(x) \quad , \quad i = 1, \dots, n. \tag{3.3.17}$$

It follows that

$$\begin{aligned} \Big|\sum_{i=1}^n t_i K_i\Big| &= \int_{\mathbb{R}^n} \det\left(\sum_{i=1}^n t_i \Big(\frac{\partial^2 f_i}{\partial x_k \partial x_l}\Big)\right) dx \\ &= \sum_{i_1, \dots, i_n = 1}^n t_{i_1} \dots t_{i_n} \int_{\mathbb{R}^n} D\left(\frac{\partial^2 f_{i_1}(x)}{\partial x_k \partial x_l}, \dots, \frac{\partial^2 f_{i_n}(x)}{\partial x_k \partial x_l}\right) dx. \end{aligned} \tag{3.3.18}$$

In this way, we recover Minkowski's theorem on $|\sum t_i K_i|$, and see the connection between the mixed discriminants $D(\mathrm{Hess} f_{i_1}, \dots, \mathrm{Hess} f_{i_n})$ and the mixed volumes

$$V(K_{i_1}, \dots, K_{i_n}) = \int_{\mathbb{R}^n} D(\mathrm{Hess} f_{i_1}(x), \dots, \mathrm{Hess} f_{i_n}(x)) dx. \tag{3.3.19}$$

The Alexandrov-Fenchel inequalities do not follow from the corresponding mixed discriminant inequalities, but the deep connection between the two theories is obvious. Also, some particular cases are indeed simple consequences. For example (see [4]), as a consequence of a similar inequality for mixed discriminants one can prove that

$$V(K_1, \dots, K_n) \geq \prod_{i=1}^n |K_i|^{1/n}. \tag{3.3.20}$$

4 Extremal problems and isotropic positions

4.1 Classical positions of convex bodies

The family of positions of a convex body K in $\mathbb{R}^n$ is the class $\{T(K) \mid T \in GL(n)\}$. The right choice of a position is often quite important for the study of geometric quantities. For example, let K be a symmetric convex body in $\mathbb{R}^n$ and consider the volume product $s(K) = \big(|K| \cdot |K^\circ|\big)^{1/n}$. The Blaschke-Santaló inequality asserts that $s(K)$ is maximized if and only if K is an ellipsoid (note that $s(K)$ is invariant under $GL(n)$). On the other hand, a simple application of Hölder's inequality shows that

$$\frac{|A|}{|B_2^n|} = \int_{S^{n-1}} \|\theta\|_A^{-n} \sigma(d\theta) \geq w(A^\circ)^{-n} \tag{4.1.1}$$

for every symmetric convex body A in $\mathbb{R}^n$. This implies that

$$\frac{s(B_2^n)}{s(K)} \leq \min_{T \in GL(n)} w(TK) w((TK)^\circ). \tag{4.1.2}$$

Therefore, in order to obtain a reverse Blaschke-Santaló inequality it is useful to study the quantity

$$\max_K \min_{T \in GL(n)} w(TK) w((TK)^\circ). \tag{4.1.3}$$

One way to estimate this minimum is using the ℓ-position of K, and Pisier's inequality shows that the above quantity is bounded by $C \log n$. Thus, the ℓ-position provides a first quite non-trivial reverse inequality for the volume product $s(K)$.

All classical positions of convex bodies arise as solutions of such extremal problems. We often normalize the volume of K to be 1 and ask for the maximum or minimum of $f(TK)$ over all $T \in SL(n)$, where f is some functional on convex bodies (in the example above, f is the product of the mean widths of a body and its polar). Another useful normalization is $|K| = |B_2^n|$: we then say that the volume radius of K is equal to 1. Below we describe some classical positions of a given convex body K which solve natural extremal problems. An interesting feature of this procedure is that a simple variational method leads to a geometric description of the extremal position, and that in many cases this position satisfies an isotropic condition for an appropriate measure on S^{n-1}. We say that a Borel measure μ on S^{n-1} is isotropic if

$$\int_{S^{n-1}} \langle x, \theta \rangle^2 \mu(d\theta) = \frac{\|\mu\|}{n} |x|^2 \tag{4.1.4}$$

for all $x \in \mathbb{R}^n$.

John's position: A symmetric convex body K is in John's position if the maximal volume ellipsoid of K is the Euclidean unit ball. John's theorem [73] asserts that, in this case, there exist contact points $u_1, \dots, u_m$ of K and B_2^n (common points of their boundaries) and positive real numbers $c_1, \dots, c_m$ such that

$$I = \sum_{j=1}^{m} c_j u_j \otimes u_j. \tag{4.1.5}$$

In particular, this decomposition of the identity implies that

$$|x|^2 = \sum_{j=1}^{m} c_j \langle x, u_j \rangle^2 \tag{4.1.6}$$

for every $x \in \mathbb{R}^n$. A direct consequence of (4.1.6) is the fact that $K \subset \sqrt{n} B_2^n$ (in other words, $d(X_K, \ell_2^n) \leq \sqrt{n}$). The condition in (4.1.6) may be viewed as an isotropic one: the measure μ supported by $\{u_1, \dots, u_m\}$ which gives mass c_j to u_j is isotropic. Moreover, Ball observed that this condition is also sufficient in the following sense.

Theorem 4.1 *Let K be a symmetric convex body in $\mathbb{R}^n$ such that $B_2^n \subseteq K$. Then, K is in John's position if and only if there exists an isotropic measure μ on S^{n-1} which is supported by the set of contact points of K and B_2^n.*

There exists an analogue of this fact for the not necessarily symmetric case (see e.g. [54]). From John's decomposition of the identity one can recover all the available information about John's position: for example, the Dvoretzky-Rogers lemma is a simple consequence of (4.1.5).

John's decomposition of the identity holds in a much more general context: If K and L are (not necessarily symmetric) convex bodies in $\mathbb{R}^n$, we say that L *is of maximal volume in* K if $L \subseteq K$ and, for every $w \in \mathbb{R}^n$ and $T \in SL_n$, the affine image $w + T(L)$ of L is not contained in the interior of K. If L is of maximal volume in K then for every $z \in \mathrm{int}(L)$, one can find contact points $v_1, \dots, v_m$ of $K - z$ and $L - z$, contact points $u_1, \dots, u_m$ of $(K - z)^\circ$ and $(L - z)^\circ$, and positive reals $c_1, \dots, c_m$, such that $\sum c_j u_j = 0$, $\langle u_j, v_j \rangle = 1$, and

$$I = \sum_{j=1}^{m} c_j u_j \otimes v_j.$$

Moreover, there exists an optimal choice of the "center" z so that, setting $z = 0$, we simultaneously have $\sum c_j u_j = \sum c_j v_j = 0$. This fact was proved in [57] under some conditions on K and L (in the symmetric case it had been observed by Milman, see [147]). A second proof was recently given in [66], where the decomposition is also used to establish that for *any* symmetric convex body K in $\mathbb{R}^n$ the Banach-Mazur distance (see §6.1) $d(K, T)$ is less than or equal to n for every convex body T in $\mathbb{R}^n$ and the distance $d(K, S_n)$ to the simplex S_n is *equal* to n.

Minimal mean width position: Recall that the mean width of a convex body K in $\mathbb{R}^n$ is the quantity $w(K) = \int_{S^{n-1}} h_K(\theta) \sigma(d\theta)$, where h_K is the support function of K (the mean width is clearly invariant under translation). We fix the volume of K to be equal to 1 and say that K has minimal mean width if $w(K) \leq w(TK)$ for all $T \in SL(n)$.

Let ν_K be the Borel measure on S^{n-1} with density h_K with respect to σ. An isotropic characterization of the minimal mean width position is proved in [53].

Theorem 4.2 *Let K be a convex body in $\mathbb{R}^n$. Then, K has minimal mean width if and only if the measure ν_K is isotropic. That is, if*

$$w(K) = n \int_{S^{n-1}} h_K(\theta)\langle \theta, x\rangle^2 \sigma(d\theta) \tag{4.1.7}$$

for all $x \in S^{n-1}$. Moreover, this position is uniquely determined up to $O(n)$.

An interesting question is to determine the order of growth of the quantity

$$w(n) = \max_K \min_{T \in SL(n)} w(TK) \tag{4.1.8}$$

as n tends to infinity, where the maximum is over all convex bodies of volume 1 in $\mathbb{R}^n$. If $|K| = 1$, Urysohn's inequality implies that $w(K) \geq c\sqrt{n}$ where $c > 0$ is an absolute constant. Pisier's inequality shows that $w(n) \leq c_1\sqrt{n}\log n$, and the example of the ℓ_1^n ball shows that $w(n) \geq c_2\sqrt{n \log n}$.

Minimal surface area position: Recall that the area measure of a convex body K is the Borel measure σ_K on S^{n-1} with

$$\sigma_K(A) = \nu(\{x \in \mathrm{bd}(K) : \text{ the outer normal to } K \text{ at } x \text{ is in } A\})$$

where ν is the $(n-1)$-dimensional surface measure on K. The surface area of K is $\partial(K) = \|\sigma_K\|$. Again, we fix the volume of K to be equal to 1 and say that K has minimal surface area if $\partial(K) \leq \partial(TK)$ for all $T \in SL(n)$.

An isotropic characterization of the minimal surface area position was proved by Petty [117] (see also [56]).

Theorem 4.3 *Let K be a convex body in $\mathbb{R}^n$. Then, K has minimal surface area if and only if the measure σ_K is isotropic. That is, if*

$$\partial(K) = n \int_{S^{n-1}} \langle \theta, x\rangle^2 \sigma_K(d\theta) \tag{4.1.9}$$

for all $x \in S^{n-1}$. Moreover, this position is uniquely determined up to $O(n)$.

As in the case of the mean width, it is natural to study the quantity

$$\partial(n) = \max_K \min_{T \in SL(n)} \partial(TK) \tag{4.1.10}$$

and its behaviour as n tends to infinity, where the maximum is over all convex bodies of volume 1 in $\mathbb{R}^n$. If $|K| = 1$, the isoperimetric inequality implies that $\partial(K) \geq c\sqrt{n}$ where $c > 0$ is an absolute constant. A sharp upper bound for $\partial(n)$ was given by Ball ([12], see §4.4). The extremal bodies are: the cube in the symmetric case and the simplex in the general case.

4.2 Isotropic position and the slicing problem

The slicing problem asks if there exists an absolute constant $c > 0$ with the following property: for every convex body K of volume 1 in $\mathbb{R}^n$, with centre of mass at the origin, there exists $\theta \in S^{n-1}$ such that $|K \cap \theta^{\perp}| \geq c$. This is an important question in modern convex geometry, which is deeply connected with the asymptotic versions of several classical geometric problems.

The question is in a sense equivalent to the study of linear functionals on convex bodies. Indeed, by Brunn's principle, for any $\theta \in S^{n-1}$ the function $f_{K,\theta}(t) = |K \cap (\theta^{\perp} + t\theta)|$ is log-concave, and this implies that

$$\frac{c_1}{|K \cap \theta^{\perp}|^2} \leq \int_K \langle x, \theta\rangle^2 dx \leq \frac{c_2}{|K \cap \theta^{\perp}|^2}, \tag{4.2.1}$$

where $c_1, c_2 > 0$ are absolute constants. In this way, the volume of sections is measured by the moments of inertia of the body.

This brings into play the Binet ellipsoid $E_B(K)$ of K, a notion coming from classical mechanics. The norm of the Binet ellipsoid is defined by

$$\|y\|^2_{E_B(K)} = \frac{1}{|K|}\int_K \langle x, y\rangle^2 dx \tag{4.2.2}$$

and a suitable homothet of its polar (the Legendre ellipsoid $E_L(K)$ of K) satisfies the equation

$$\int_{E_L(K)} \langle x, y\rangle^2 dx = \int_K \langle x, y\rangle^2 dx \tag{4.2.3}$$

for every $y \in \mathbb{R}^n$ (it has the same moments of inertia as K).

We say that a convex body K of volume 1 with centre of mass at the origin is isotropic if the Legendre ellipsoid $E_L(K)$ is a multiple of B_2^n. Equivalently, if there exists a constant $L_K > 0$ such that

$$\int_K \langle y, \theta\rangle^2 dy = L_K^2 \tag{4.2.4}$$

for every $\theta \in S^{n-1}$. Every convex body (in fact, every compact set) has an isotropic position, which is unique up to orthogonal transformations. This position may again be described as the solution of an extremal problem of the type we discussed in the previous section (see [111] for an extensive survey of all these facts).

Theorem 4.4 *Let K be a convex body of volume 1 in $\mathbb{R}^n$, with centre of mass at the origin. Then,*

$$\int_K |x|^2 dx \leq \int_{TK} |x|^2 dx \tag{4.2.5}$$

for every $T \in SL(n)$ if and only if there exists a constant $L_K > 0$ such that

$$\int_K \langle y, \theta\rangle^2 dy = L_K^2 \tag{4.2.6}$$

for every $\theta \in S^{n-1}$.

Uniqueness of the isotropic position up to $O(n)$ shows that this isotropic constant L_K is invariant for the class of K. It is easily proved that $L_K \geq L_{B_2^n} \geq c > 0$ for every convex body K in $\mathbb{R}^n$, where $c > 0$ is an absolute constant. For an isotropic convex body K, (4.2.1) shows that all $(n-1)$-dimensional sections through the origin are approximately equal to $1/L_K$. Therefore, the slicing problem becomes a question about the uniform boundedness of L_K. In fact, it is not hard to see that an affirmative answer to the slicing problem is in full generality equivalent to the following statement:

> There exists an absolute constant $C > 0$ such that $L_K \leq C$ for every convex body K of volume 1 with centre of mass at the origin.

One can easily obtain the estimate $L_K = O(\sqrt{n})$ for every convex body K. In the symmetric case, this is an immediate consequence of John's theorem, while in the general case it can be deduced from Blaschke's identity which connects the matrix of inertia of K with the expected value of the volume of a random simplex inside K. Uniform boundedness of L_K is known for some classes of bodies: unit balls of spaces with a 1-unconditional basis, zonoids and their polars, etc. For partial answers to the question, see [111], [9]. The best known general upper estimate is due to Bourgain [23]: $L_K \leq c\sqrt[4]{n}\log n$ for every convex body K in $\mathbb{R}^n$. For a sketch of the proof, see [54] (the argument follows the presentation of [38], see also [116] for the not-necessarily symmetric case).

There is a renewed interest in the problem. We mention here a very recent result of Bourgain, Klartag and Milman [24] which reduces the question to convex bodies with bounded volume ratio. There exists a constant $A > 1$ with the following property: if for all n and all convex bodies K in $\mathbb{R}^n$ with $\mathrm{vr}(K) \leq A$ we have $L_K \leq \alpha$ for some constant α, then for all n and all convex bodies K in $\mathbb{R}^n$ we have $L_K \leq c(\alpha)$ for some constant $c(\alpha)$ depending only on α. Actually, the dependence of $c(\alpha)$ on α is almost linear. The proof of this fact uses two tools: Steiner symmetrization and the existence and properties of M-ellipsoids (see §5.2).

4.3 Brascamp-Lieb inequality and its reverse form

The Brascamp-Lieb inequality concerns the multilinear operator $I : L^{p_1}(\mathbb{R}) \times \cdots \times L^{p_m}(\mathbb{R}) \to \mathbb{R}$ defined by

$$I(f_1, \ldots, f_m) = \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(\langle u_j, x\rangle)\, dx, \tag{4.3.1}$$

where $m \geq n$, $p_1, \ldots, p_m \geq 1$ with $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = n$, and $u_1, \ldots, u_m \in \mathbb{R}^n$.

Brascamp and Lieb [31] proved that the norm of I is the supremum D of

$$\frac{I(g_1, \ldots, g_m)}{\prod_{j=1}^{m} \|g_j\|_{p_j}} \tag{4.3.2}$$

over all centered Gaussian functions $g_1, \dots, g_m$, i.e. over all functions of the form $g_j(t) = e^{-\lambda_j t^2}$, $\lambda_j > 0$. This fact is a generalization of Young's convolution inequality $\|f * g\|_r \leq C_{p,q}\|f\|_p\|g\|_q$ for all $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, where $p, q, r \geq 1$ and $1/p + 1/q = 1 + 1/r$. The best constants $C_{p,q} = A_p A_q A_{r'}$ (where $A_s = (s^{1/s}/(s')^{1/s'})^{1/2}$ and s' is the conjugate exponent of s) had been also obtained by Beckner [19] who showed that Gaussian functions play the role of maximizers.

The original proof of the Brascamp-Lieb inequality was based on a general rearrangement inequality of Brascamp, Lieb and Luttinger [32], who showed that if f^* is the symmetric decreasing rearrangement of a Borel measurable function f vanishing at infinity, then

$$I(f_1, \dots, f_m) \leq I(f_1^*, \dots, f_m^*). \tag{4.3.3}$$

A generalization of this fact to functions of several variables (based on Steiner symmetrization) and the fact that radial functions in high dimensions behave like Gaussian functions were the key ingredients of the original proof. Setting $c_j = 1/p_j$ and replacing f_j by $f_j^{c_j}$ one can reformulate the Brascamp-Lieb inequality as follows.

Theorem 4.5 *If $m \geq n$, $u_1, \dots, u_m \in \mathbb{R}^n$ and $c_1, \dots, c_m > 0$ with $c_1 + \cdots + c_m = n$, then*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{c_j}(\langle x, u_j\rangle)dx \leq D \cdot \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j\right)^{c_j} \tag{4.3.4}$$

for all integrable functions $f_j : \mathbb{R} \to \mathbb{R}^+$.

Testing on the Gaussians, one can see that $D = 1/\sqrt{F}$ where

$$F = \inf\left\{\frac{\det\left(\sum_{j=1}^m c_j\lambda_j u_j \otimes u_j\right)}{\prod_{j=1}^m \lambda_j^{c_j}} \mid \lambda_j > 0\right\}. \tag{4.3.5}$$

Barthe [16] proved the following reverse form of Theorem 4.5 which was conjectured by Ball.

Theorem 4.6 *Let $m \geq n$, $c_1, \dots, c_m > 0$ with $c_1 + \cdots + c_m = n$, and $u_1, \dots, u_m \in \mathbb{R}^n$. If $h_1, \dots, h_m : \mathbb{R} \to \mathbb{R}^+$ are measurable functions, we set*

$$K(h_1, \dots, h_m) = \int_{\mathbb{R}^n}^* \sup\left\{\prod_{j=1}^m h_j^{c_j}(\theta_j) \mid \theta_j \in \mathbb{R}\ ,\ x = \sum_{j=1}^m \theta_j c_j u_j\right\}dx, \tag{4.3.6}$$

where $\int^*$ denotes the outer integral. Then,

$$\inf\left\{K(h_1, \dots, h_m) \mid \int_{\mathbb{R}} h_j = 1\ ,\ j = 1, \dots, m\right\} = \sqrt{F}. \tag{4.3.7}$$

The proof is remarkably elegant and, at the same time, it gives a new direct proof of the Brascamp-Lieb inequality. We will briefly discuss the argument. Again, first testing on centered Gaussian functions, one observes that

$$\inf\left\{K(h_1,\dots,h_m)\mid \int_{\mathbb{R}} h_j=1\ ,\ j=1,\dots,m\right\}\le\sqrt{F}. \tag{4.3.8}$$

The main step in Barthe's argument is the following proposition.

Proposition 4.1 *Let $f_1,\dots,f_m:\mathbb{R}\to\mathbb{R}^+$ and $h_1,\dots,h_m:\mathbb{R}\to\mathbb{R}^+$ be integrable functions with*

$$\int_{\mathbb{R}} f_j(t)dt=\int_{\mathbb{R}} h_j(t)dt=1,\quad j=1,\dots,m.$$

Then,

$$F\cdot I(f_1,\dots,f_m)\le K(h_1,\dots,h_m). \tag{4.3.9}$$

Proof: We may assume that f_j,h_j are continuous and strictly positive. We may also assume that $0<F<+\infty$ (F is not degenerated). We use the transportation of measure idea that was used for the proof of the Prékopa-Leindler inequality: For every $j=1,\dots,m$ we define $T_j:\mathbb{R}\to\mathbb{R}$ by the equation

$$\int_{-\infty}^{T_j(t)} h_j(s)ds=\int_{-\infty}^{t} f_j(s)ds. \tag{4.3.10}$$

Then, each T_j is strictly increasing, 1-1 and onto, and

$$T_j'(t)h_j(T_j(t))=f_j(t),\quad t\in\mathbb{R}. \tag{4.3.11}$$

We now define $W:\mathbb{R}^n\to\mathbb{R}^n$ by

$$W(y)=\sum_{j=1}^m c_jT_j(\langle y,u_j\rangle)u_j. \tag{4.3.12}$$

A simple computation shows that $J(W)(y)=\sum_{j=1}^m c_jT_j'(\langle y,u_j\rangle)u_j\otimes u_j$. This impliess that $\langle [J(W)(y)](v),v\rangle>0$ if $v\neq 0$ and hence, W is injective. Consider the function

$$m(x)=\sup\left\{\prod_{j=1}^m h_j^{c_j}(\theta_j)\mid x=\sum_{j=1}^m\theta_jc_ju_j\right\}.$$

Then, (4.3.12) shows that

$$m(W(y))\ge\prod_{j=1}^m h_j^{c_j}(T_j(\langle y,u_j\rangle)) \tag{4.3.13}$$

for every $y \in \mathbb{R}^n$. It follows that

$$\begin{aligned}
\int_{\mathbb{R}^n} m(x)dx &\geq \int_{W(\mathbb{R}^n)} m(x)dx \\
&= \int_{\mathbb{R}^n} m(W(y)) \cdot |J(W)(y)|\, dy \\
&\geq \int_{\mathbb{R}^n} \prod_{j=1}^{m} h_j^{c_j}(T_j(\langle y, u_j\rangle)) \det\left(\sum_{j=1}^{m} c_j T_j'(\langle y, u_j\rangle) u_j \otimes u_j\right) dy.
\end{aligned}$$

By the definition of F we have

$$\det\left(\sum_{j=1}^{m} c_j T_j'(\langle y, u_j\rangle) u_j \otimes u_j\right) \geq F \cdot \prod_{j=1}^{m} \left(T_j'(\langle y, u_j\rangle)\right)^{c_j}. \tag{4.3.14}$$

Therefore, taking (4.3.11) into account we have

$$\begin{aligned}
\int_{\mathbb{R}^n} m(x)dx &\geq F \cdot \int_{\mathbb{R}^n} \prod_{j=1}^{m} h_j^{c_j}(T_j(\langle y, u_j\rangle)) \cdot \prod_{j=1}^{m} \left(T_j'(\langle y, u_j\rangle)\right)^{c_j} dy \\
&= F \cdot \int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j^{c_j}(\langle y, u_j\rangle) dy \\
&= F \cdot I(f_1, \dots, f_m).
\end{aligned}$$

In other words, $F \cdot I(f_1, \dots, f_m) \leq K(h_1, \dots, h_m)$. $\square$

One can now prove simultaneously Theorems 4.5 and 4.6. The computation leading to (4.3.5) shows that

$$\sup\left\{ I(f_1, \dots, f_m) \mid \int_{\mathbb{R}} f_j = 1\ ,\ j = 1, \dots, m \right\} \geq \frac{1}{\sqrt{F}}. \tag{4.3.15}$$

From Proposition 4.1, (4.3.8) and (4.3.15) we get

$$\begin{aligned}
\frac{1}{\sqrt{F}} &\leq \sup\left\{ I(f_1, \dots, f_m) \mid \int_{\mathbb{R}} f_j = 1 \right\} \\
&\leq \frac{1}{F} \cdot \inf\left\{ K(h_1, \dots, h_m) \mid \int_{\mathbb{R}} h_j = 1 \right\} \leq \frac{1}{\sqrt{F}}.
\end{aligned}$$

We must have equality everywhere, and this ends the proof(s).

There is a multidimensional generalization of both inequalities. Let $S^+(\mathbb{R}^k)$ be the set of $k \times k$ symmetric, positive definite matrices. If $A \in S^+(\mathbb{R}^k)$, we write G_A for the centered Gaussian function $G_A : \mathbb{R}^k \to \mathbb{R}$ defined by $G_A(x) = \exp(-\langle Ax, x\rangle)$, and $L_1^+(\mathbb{R}^k)$ for the class of integrable non-negative functions $f : \mathbb{R}^k \to \mathbb{R}$. Let $m \geq n$, and assume we are given real numbers $c_1, \dots, c_m > 0$ and integers $n_1, \dots, n_m$ less than or equal to n, such that $\sum_{j=1}^m c_j n_j = n$. We are also given linear maps $B_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ which are onto and satisfy $\bigcap_{j=1}^m \mathrm{Ker}(B_j) = \{0\}$.

Consider the operators $I, K : L_1^+(\mathbb{R}^{n_1}) \times \cdots \times L_1^+(\mathbb{R}^{n_m}) \to \mathbb{R}$ defined by

$$I(f_1, \dots, f_m) = \int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{c_j}(B_j x) dx \tag{4.3.16}$$

and

$$K(h_1, \dots, h_m) = \int_{\mathbb{R}^m}^* m(x) dx, \tag{4.3.17}$$

where

$$m(x) = \sup \left\{ \prod_{j=1}^m h_j^{c_j}(y_j) \mid y_j \in \mathbb{R}^{n_j} \quad \text{and} \quad \sum_{j=1}^m c_j B_j^* y_j = x \right\}. \tag{4.3.18}$$

Let E be the largest constant for which

$$K(h_1, \dots, h_m) \geq E \cdot \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} h_j \right)^{c_j} \tag{4.3.19}$$

holds true for all $h_j \in L_1^+(\mathbb{R}^{n_j})$, and let F be the smallest constant for which

$$I(f_1, \dots, f_m) \leq F \cdot \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{c_j} \tag{4.3.20}$$

holds true for all $f_j \in L_1^+(\mathbb{R}^{n_j})$. Then, the following holds true.

Theorem 4.7 *The constants E and F can be computed using centered Gaussian functions. Moreover, if D is the largest real number for which*

$$\det \left(\sum_{j=1}^m c_j B_j^* A_j B_j \right) \geq D \cdot \prod_{j=1}^m (\det A_j)^{c_j}, \tag{4.3.21}$$

for all $A_j \in S^+(\mathbb{R}^{n_j})$, we have

$$E = \sqrt{D} \quad \text{and} \quad F = 1/\sqrt{D}. \tag{4.3.22}$$

The multidimensional version of the Brascamp-Lieb inequality was first established by Lieb in [84]. The simultaneous proof of both this inequality and its reverse form is due to Barthe [16] and follows the idea of the proof of the one-dimensional case. However, instead of the direct transportation of measure argument there, one now has to make essential use of the Brenier map.

4.4 Sharp geometric inequalities

As §4.1 shows, isotropic positions of convex bodies and the corresponding decompositions of the identity are typical in the asymptotic theory: isotropicity may be viewed as the ultimate form of non-degeneracy. Ball made the very important observation that the constants in the Brascamp-Lieb inequality and its reverse form take a surprisingly simple form in the presence of such a decomposition of the identity.

Theorem 4.8 *Assume that the vectors $u_1, \dots, u_m \in S^{n-1}$ and the positive weights $c_1, \dots, c_m$ satisfy the isotropic condition*

$$I = \sum_{j=1}^{m} c_j u_j \otimes u_j. \tag{4.4.1}$$

Then, the constant $F = F(\{u_j\}, \{c_j\})$ in Theorems 4.5 and 4.6 is equal to 1.

Ball applied the Brascamp-Lieb inequality in this context to solve purely geometric problems. A well-known example is his reverse isoperimetric inequality [12], which gives the exact value of the constant $\partial(n)$ in (4.1.4). We ask for the best constant $\partial(n)$ for which every symmetric convex body K in $\mathbb{R}^n$ has a position $\tilde{K}$ satisfying

$$\partial(\tilde{K}) \leq \partial(n)|\tilde{K}|^{(n-1)/n}. \tag{4.4.2}$$

The natural position of K is the minimal surface area position. However, Ball's solution of the problem employs John's position. Assume that B_2^n is the maximal volume ellipsoid of K. Then,

$$\partial(K) = \lim_{t \to 0^+} \frac{|K + tB_2^n| - |K|}{t} \leq \lim_{t \to 0^+} \frac{|K + tK| - |K|}{t} = n|K|. \tag{4.4.3}$$

We claim that among all bodies in John's position the cube has maximal volume.

Theorem 4.9 *Let $Q_n = [-1,1]^n$ be the unit cube in $\mathbb{R}^n$. If K is a symmetric convex body in John's position in $\mathbb{R}^n$, then $|K| \leq 2^n = |Q_n|$.*

For the proof we use John's representation of the identity (4.4.1), where the u_j's are contact points of K and B_2^n. Observe that

$$K \subseteq M := \{x : |\langle x, u_j \rangle| \leq 1,\ j = 1, \dots, m\}. \tag{4.4.4}$$

Therefore,

$$\begin{aligned} |K| &\leq |M| = \int_{\mathbb{R}^n} \prod_{j=1}^{m} \chi_{[-1,1]}^{c_j}(\langle x, u_j \rangle) dx \\ &\leq \prod_{j=1}^{m} \left(\int_{\mathbb{R}} \chi_{[-1,1]}(t) dt \right)^{c_j} = 2^{\sum_{j=1}^{m} c_j} = 2^n, \end{aligned}$$

where we used the Brascamp-Lieb inequality together with the observation of Theorem 4.8, and the fact that $\sum_{j=1}^m c_j = n$, which is a simple consequence of (4.4.1). $\square$

Now, (4.4.3) shows that $\partial(K) \leq n|K| \leq 2n|K|^{(n-1)/n}$, and since K was arbitrary, $\partial(n) \leq 2n$. There is equality in the case of the cube, and this shows that $\partial(n) = 2n$.

Theorem 4.9 shows that the cube has maximal volume ratio among all symmetric convex bodies. In the general case, one can show that the simplex Δ_n is the extremal convex body. The reverse Brascamp-Lieb inequality can be used for the dual statements: consider the external volume ratio $\mathrm{evr}(K) = \inf \left(|E|/|K|\right)^{1/n}$, where the infimum is taken over all ellipsoids containing K. Then, $\mathrm{evr}(K) \leq \mathrm{evr}(\Delta_n)$ for every convex body K in $\mathbb{R}^n$. In the symmetric case the extremal body is the cross-polytope (the unit ball of ℓ_1^n).

The Brascam-Lieb inequality and its reverse were also used for sharp estimates on the volume of sections and projections of the unit ball B_p^n of ℓ_p^n [10]. If $p > 0$ and H is a k-dimensional subspace of $\mathbb{R}^n$, then $|B_p^n \cap H| \leq |B_p^k|$ if $p \leq 2$, and

$$|B_p^n \cap H| \leq \left(\frac{n}{k}\right)^{k(1/2-1/p)} |B_p^k| \tag{4.4.5}$$

if $p \geq 2$. This last estimate is sharp if k divides n. On the other hand, $|P_H(B_p^n)| \geq |B_p^k|$ if $p \geq 2$, and

$$|P_H(B_p^n)| \geq \left(\frac{k}{n}\right)^{k(1/p-1/2)} |B_p^k| \tag{4.4.6}$$

if $p \leq 2$. This last estimate is sharp if $p > 1$ and k divides n. The proof of all these inequalities is based on the observation that if $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis in $\mathbb{R}^n$, then the obvious representation $I = \sum_{j=1}^n e_j \otimes e_j$ of the identity implies that

$$P_H = \sum_{j=1}^n a_j^2 u_j \otimes u_j, \tag{4.4.7}$$

where $a_j = |P_H(e_j)|$ and $u_j = P_H(e_j)/a_j$.

The multidimensional version of the reverse Brascamp-Lieb inequality is used in the proof of the following Brunn-Minkowski type inequality of Barthe [16]. Let m, n be integers. Let E_i, $i \leq m$ be linear subspaces of $\mathbb{R}^n$. Assume that there exist positive c_i's such that $I = \sum_{i \leq m} c_i P_i$ where P_i is the orthogonal projection onto E_i. Then, the inequality

$$\left|\sum c_i K_i\right| \geq \prod |K_i|^{c_i} \tag{4.4.8}$$

holds for any compact subsets K_i of E_i, where $|K_i|$ is the volume of K_i in E_i. In the case where each K_i is a line segment, this reduces to an inequality of Ball [11] which was proved by induction on the dimension.

Another extremal property of the simplex was proved by Barthe [17]. Assume that K is a convex body whose minimal volume ellipsoid is B_2^n. Then, $M(K) \leq M(\Delta_n)$, where Δ_n is the regular simplex inscribed in B_2^n. In the symmetric case one has $M(K) \leq M(B_1^n)$ (this is much simpler and was observed by Schechtman and Schmuckenschläger [133]). The proof of both inequalities makes use of the reverse Brascamp-Lieb inequality. In John's position, the simplex and the cube are the extremal bodies for $M(K)$.

For a different application, consider a polytope K with facets F_j and normals u_j, $j = 1, \ldots, m$. If K is in minimal surface area position, Petty's theorem 4.3 is equivalent to the statement

$$I = \sum_{j=1}^{m} \frac{n|F_j|}{\partial(K)} u_j \otimes u_j. \tag{4.4.9}$$

The projection body ΠK of K is defined by

$$h_{\Pi K}(x) = \frac{1}{2} \int_{S^{n-1}} |\langle x, z\rangle| \sigma_K(dz). \tag{4.4.10}$$

In our case, $\Pi K = \frac{\partial(K)}{2n} \sum_{j=1}^{m} c_j[-u_j, u_j]$, and using (4.4.8) one can give a lower bound of its volume [56]. Namely,

$$|\Pi K| \geq 2^n \left(\frac{\partial(K)}{2n}\right)^n. \tag{4.4.11}$$

The example of the cube shows that this inequality is sharp for bodies with minimal surface area.

Combined with Theorem 4.2 this volume estimate leads to a sharp reverse Urysohn inequality for zonoids [55]. If Z be a zonoid in $\mathbb{R}^n$ with volume 1 and minimal mean width, then

$$w(Z) \leq w(Q_n) = \frac{2\omega_{n-1}}{\omega_n}. \tag{4.4.12}$$

For the proof, recall that Z is the projection body ΠK of some convex body K. Using (4.4.10) and the characterizations of Theorems 4.2 and 4.3 we check that K has minimal surface area. We have

$$w(Z) = 2\int_{S^{n-1}} h_Z(x)\sigma(dx) = \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, z\rangle| \sigma_K(dz)\sigma(dx) = \frac{2\omega_{n-1}}{n\omega_n}\partial(K), \tag{4.4.13}$$

and (4.4.11) shows that $w(Z) \leq 2\omega_{n-1}/\omega_n$. We have equality when K is a cube, and this corresponds to the case $Z = (1/2)Q_n$.

4.5 Study of geometric probabilities

In this short subsection we describe some recent results from [63] on random properties of the uniform distribution over a convex body K in $\mathbb{R}^n$. To fix terminology, for any (measurable) set $A \subset \mathbb{R}^n$, the geometric probability of A is $P(A) := |A \cap K|/|K|$.

Theorem 4.10 *Let T_i be measurable sets in $\mathbb{R}^n$, $i = 1, \dots, m$, and K be a star-shaped body with $0 \in \mathrm{int}(K)$. Assume that $|K| = |T_1| = \cdots = |T_m|$. Consider the positively homogeneous function*

$$|||\tilde{\lambda}||| = \frac{1}{\prod_{i=1}^m |T_i|} \int_{T_1} \cdots \int_{T_m} \Big\| \sum_{i=1}^m \lambda_i x_i \Big\|_K dx_m \dots dx_1 \tag{4.5.1}$$

on $\mathbb{R}^m$. Then,

$$|||\tilde{\lambda}||| \geq c\sqrt{\sum \lambda_i^2} \tag{4.5.2}$$

for every $\tilde{\lambda} \in \mathbb{R}^m$, where $c > 0$ is an absolute constant ($c \geq c_n/\sqrt{2}$, where $c_n \to 1$ as $n \to \infty$).

The proof of Theorem 4.10 is a direct consequence of the following fact: If K and T_i are as above and if $|K| = |T_i| = |B_2^n|$ for every i, then, for any scalars λ_i, $i = 1, \dots, m$ and for any $t > 0$, we have

$$P\left\{(x_i \in T_i)_{i=1}^m : \Big\| \sum_1^m \lambda_i x_i \Big\|_K < t\right\} \leq P\left\{(x_i \in B_2^n)_{i=1}^m : \Big\| \sum_1^m \lambda_i x_i \Big\|_{B_2^n} < t\right\}. \tag{4.5.3}$$

One then knows that the extremal case is $K = T_1 = \cdots = T_m = B_2^n$ and a simple argument based on Kahane's inequality leads to the lower bound.

The proof of (4.5.3) uses the rearrangement inequality of Brascamp, Lieb and Luttinger [32] which was the starting point for the first proof of the Brascamp-Lieb inequality.

An interesting question is to give exact estimates for the probability in (4.5.3) in terms of $\{\lambda_i\}$ and t. This is done in [63] with a method which uses the sharp multivariable version of Young's inequality, proved by Brascamp and Lieb [31]. [This approach was first used by Arias-de-Reyna, Ball and Villa in [6] to establish the case $m = 2$, $\lambda_1 = -\lambda_2 = 1/\sqrt{2}$, $T_i = K$ (where K is a symmetric convex body)]:

Fact. Assume that $|K| = |T_1| = \cdots |T_m| = 1$. Then, for any scalars $\lambda_i \in \mathbb{R}$ and any $0 < t \leq 1$,

$$P\left\{(x_i \in T_i)_{i=1}^m : \Big\| \sum_1^m \lambda_i x_i \Big\|_K < t\sqrt{\sum_1^m \lambda_i^2}\right\} \leq t^n \exp\left[\frac{(1-t^2)}{2} n\right]. \tag{4.5.4}$$

A consequence of (4.5.4) is the fact that every n-dimensional normed space X has *random cotype 2* with constant bounded by an absolute constant $C > 0$ (see [63]). We say that X has random cotype 2 with constant $A > 0$ if with the probability greater than $1 - e^{-an}$ ($a > 0$ is a fixed universal number), n independent random vectors $\{x_i\}_1^n$ uniformly distributed over the unit ball K of X satisfy for every $\lambda_i \in \mathbb{R}$

$$\mathrm{Ave}_{\varepsilon_i = \pm 1} \Big\| \sum_1^n \varepsilon_i \lambda_i x_i \Big\| \geq \frac{1}{A}\sqrt{\sum_1^n |\lambda_i|^2}. \tag{4.5.5}$$

Note that the norms $\|x_i\|$ do not enter in the definition, since with probability exponentially close to 1 we have $1/2 \leq \|x_i\| \leq 1$ and hence the norms are absorbed in A.

5 Asymptotic results with a classical convexity flavor

5.1 Classical symetrizations

Symmetrization procedures play an important role in classical convexity. Until recently, the bounds on the number of successive symmetrizations of a certain type which are needed in order to obtain from a given body K a body $\tilde{K}$ which is close to a ball were at least exponential in the dimension. The methods of asymptotic convex geometry show that a linear in the dimension number of steps is enough.

Minkowski symmetrization. Consider a convex body K in $\mathbb{R}^n$ and a direction $u \in S^{n-1}$. The Minkowski symmetrization of K with respect to u is the convex body $\frac{1}{2}(K + \pi_u K)$, where π_u denotes the reflection with respect to $u^{\perp}$. This operation is linear and preserves mean width. A random Minkowski symmetrization of K is a body $\pi_u K$, where u is chosen randomly on S^{n-1} with respect to the probability measure σ. Bourgain, Lindenstrauss and Milman [25] proved that for every $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for every $n \geq n_0$ and every convex body K, if we perform $N = Cn\log n + c(\varepsilon)n$ independent random Minkowski symmetrizations on K we receive a convex body $\tilde{K}$ such that

$$(1-\varepsilon)w(K)B_2^n \subset \tilde{K} \subset (1+\varepsilon)w(K)B_2^n \tag{5.1.1}$$

with probability greater than $1 - \exp(-c_1(\varepsilon)n)$. The method of proof is closely related to the concentration phenomenon for $SO(n)$.

Recently, Klartag [75] showed that if we perform a specific non-random choice of $5n$ Minkowski symmetrizations we may transform any convex body into an approximate Euclidean ball. We briefly describe the process. We may clearly start with the normalization $w(K) = 1$. We fix an orthonormal basis $\{e_1, \dots, e_n\}$ and first symmetrize K with respect to the e_j's. In this way we obtain a 1-unconditional convex body K_1 with the property $K_1 \subseteq c\sqrt{n}B_1^n$.

Let $Q = \sqrt{n}B_1^n$ and consider a "Walsh basis" of $\mathbb{R}^n$. This is an orthonormal basis $\{u_1, \dots, u_n\}$ satisfying $|\langle u_i, e_j\rangle| \leq 2/\sqrt{n}$ for every $i, j \leq n$. If we symmetrize Q with respect to $u_1, \dots, u_{n-1}$, we obtain a new body $\tilde{Q}$ with $\mathrm{diam}(\tilde{Q}) \leq c\sqrt{\log n}$. Applying the same sequence of symmetrizations to K_1 we arrive at a convex body K_2 with $w(K_2) = 1$ and

$$K_2 \subset c\sqrt{n}B_1^n \cap tB_2^n \tag{5.1.2}$$

with respect to a new orthonormal basis, where $t = \mathrm{diam}(K_2) \leq c\sqrt{\log n}$.

The next step shows that one can achieve a logarithmic decay of the diameter under $2n$ additional symmetrizations with respect to two independent random orthonormal bases.

Claim: *Let $Q_t = \sqrt{n}B_1^n \cap tB_2^n$. If $\{v_j\}$, $\{w_j\}$ are independent random orthonormal bases of $\mathbb{R}^n$, then symmetrization of Q_t with respect to $v_1, \dots, v_{n-1}$ and $w_1, \dots, w_{n-1}$ produces with high probability a convex body $\tilde{Q}_t$ with $\tilde{Q}_t \subset (C \log t)B_2^n$.*

It follows that the same sequence of symmetrizations applied to K_2 produces a convex body K_3 with $\mathrm{diam}(K_3) \leq c \log\log n$. One may then iterate this step and arrive at a body for which $\mathrm{diam}(K_s)$ is bounded by a universal constant. Then, the proof of [25] shows that $O(n)$ symmetrizations of K_s bring it close to a ball. Instead of this, one can show by concentration techniques that a second application of the claim's symmetrization process to the body K_3 is enough.

Even more recently, using spherical harmonics, Klartag [76] showed that for every convex body K and any $0 < \varepsilon < 1/2$ there exist $cn \log(1/\varepsilon)$ successive Minkowski symmetrizations which transform K to a convex body $\tilde{K}$ satisfying $(1-\varepsilon)w(K)B_2^n \subseteq \tilde{K} \subseteq (1+\varepsilon)w(K)B_2^n$.

Steiner symmetrization. It is well-known that for any convex body K in $\mathbb{R}^n$ there exists a sequence of directions $\theta_j \in S^{n-1}$ such that $(S_{\theta_n} \circ \cdots \circ S_{\theta_1})(K)$ converges to a ball in the Hausdorff metric (S_θ is the Steiner symmetrization in the direction of θ). In fact, Mani [90] has proved that if we choose an infinite random sequence of directions $\theta_j \in S^{n-1}$ and apply successive Steiner symmetrizations S_{θ_j} of K in these directions, then we almost surely get a sequence of convex bodies converging to a ball.

Bourgain, Lindenstrauss and Milman [26] proved an isomorphic version of this fact. There exist absolute constants $c, c_1, c_2 > 0$ with the following property: if K is a convex body in $\mathbb{R}^n$, there exist $k \leq cn \log n$ unit vectors θ_j such that successive Steiner symmetrizations in the directions of θ_j transform K into a convex body K_1 with

$$c_1 \rho B_2^n \subseteq K_1 \subseteq c_2 \rho B_2^n, \tag{5.1.3}$$

where B_2^n is the Euclidean unit ball and $|K| = |\rho B_2^n|$. This was a dramatic improvement with respect to the previously known estimate $(cn)^{n/2}$ of Hadwiger (1955). An essentially best possible result was recently obtained by Klartag and Milman [77].

Theorem 5.1 *For every $\varepsilon > 0$ there exist constants $c_1(\varepsilon), c_2(\varepsilon) > 0$ such that: for every convex body K in $\mathbb{R}^n$ with $|K| = |B_2^n|$, there exist $k \leq (2+\varepsilon)n$ unit vectors θ_j such that successive Steiner symmetrizations in the directions of θ_j transform K into a convex body K' with*

$$c_1(\varepsilon) B_2^n \subseteq K' \subseteq c_2(\varepsilon) B_2^n. \tag{5.1.4}$$

The main steps of the argument are the following. Starting with a convex body of volume 1, we need $2n$ Steiner symmetrizations in order to obtain a convex body

K_2 which is 1-unconditional (symmetric with respect to the coordinate subspaces) and "almost isotropic" in the following sense: for every $\theta \in S^{n-1}$,

$$\int_{K_2} \langle x, \theta \rangle^2 dx \leq 2. \tag{5.1.5}$$

The first n symmetrizations lead to a 1-unconditional body K_1. If the polar of the Binet ellipsoid of K_1 is transformed into a ball by n additional symmetrizations, it is proved that the same sequence of symmetrizations, applied to K_1, produces K_2. By recent results of Bobkov and Nazarov [21], it follows that

$$P_n \subseteq K_2 \subseteq cnB_1^n, \tag{5.1.6}$$

where P_n is a box with respect to the same coordinate system, having volume $|P_n|^{1/n} \simeq 1$ (equivalently, one may use a classical result of Losanovskii and a modification of this argument). This implies that it is enough to symmetrize P_n and the cross-polytope B_1^n. The same sequence of symmetrizations will transform K_2 into an isomorphic ball.

The analysis for these two particular bodies already proves that $(4+\varepsilon)n$ Steiner symmetrizations are enough. Employing this fact and using the quotient of subspace theorem (Theorem 2.13), one can build an iteration scheme which reduces the number to $(2+\varepsilon)n$.

Floating bodies - centroid bodies. We close this subsection with some interesting observations on the connections of the Legendre ellipsoid with the centroid and floating bodies (for the proofs of these facts, see [111]). Let K be a symmetric convex body in $\mathbb{R}^n$ with $|K| = 1$. The centroid body of K is defined by $Z(K) = \int_K [0, x] dx$, where $[0, x]$ is the line segment from 0 to x. Equivalently, its dual norm is given by

$$\|y\|_{Z(K)^\circ} = \frac{1}{2} \int_K |\langle x, y \rangle| dx. \tag{5.1.7}$$

A consequence of the Brunn-Minkowski inequality is that $Z(K)$ is uniformly (i.e. up to an absolute constant) equivalent to the Legendre ellipsoid of K in the Hausdorff sense.

For every $0 < \delta < 1/2$, the floating body K_δ of K is defined to be the envelope of all hyperplanes that cut off a set of volume δ from K. It can be proved that K_δ is convex. Moreover, K_δ is $C(\delta)$ equivalent to the Legendre ellipsoid of K, where $C(\delta)$ is a constant depending only on δ.

The process of forming the floating body may be viewed as a "one step symmetrization". One arrives at an "isomorphic ellipsoid" although one would expect that K_δ will stay close to K for small values of $\delta > 0$.

5.2 Isomorphic symmetrization

In this subsection we describe isomorphic geometric inequalities which are proved by the method of isomorphic symmetrization. This is our second main example of a body of results which answer deep questions of the Brunn-Minkowski theory, at least in their asymptotic version. Here, the main ideas and methods we described in §2 find applications to classical convexity.

Our first example is the inverse Blaschke-Santaló inequality of Bourgain and Milman [28], which gives an "affirmative answer" to Mahler's conjecture (see §4.1).

Theorem 5.2 *There exists an absolute constant $c > 0$ such that*

$$0 < c \leq \frac{s(K)}{s(B_2^n)} \leq 1 \tag{5.2.1}$$

for every symmetric convex body in $\mathbb{R}^n$.

The inequality on the right is the Blaschke-Santaló inequality. The left hand-side inequality answers the question of Mahler in the asymptotic sense: For every symmetric convex body K, the quantity $s(K)$ is of the order of $1/n$.

The original proof of Theorem 5.1 used a dimension descending procedure which was based on the quotient of subspace theorem. We will describe a proof using the method of isomorphic symmetrization [105]. This is closer to classical convexity and much more geometric in nature since it preserves dimension: however, it is a symmetrization scheme which is in many ways different from the classical symmetrizations. In each step, none of the natural parameters of the body is being preserved, but the ones which are of interest remain under control. After a finite number of steps, the body has come close to an ellipsoid, but there is no natural notion of convergence to an ellipsoid.

Since $s(K)$ is an affine invariant, we may start from a position of K which satisfies the inequality $M(K)M^*(K) \leq c\log[d(X_K, \ell_2^n) + 1]$ (this is allowed by Theorems 2.9 and 2.10). We may also normalize so that $M(K) = 1$. We define

$$\lambda_1 = M^*(K)a_1 \quad , \quad \lambda_1' = M(K)a_1, \tag{5.2.2}$$

for some $a_1 > 1$, and consider the new body

$$K_1 = \mathrm{co}\left((K \cap \lambda_1 B_2^n) \cup \frac{1}{\lambda_1'} B_2^n\right). \tag{5.2.3}$$

Sudakov's inequality (Theorem 2.14) and elementary properties of the covering numbers show that

$$|K_1| \geq |K \cap \lambda_1 B_2^n| \geq |K|/N(K, \lambda_1 B_2^n) \geq |K|\exp(-cn/a_1^2). \tag{5.2.4}$$

In an analogous way, using the dual Sudakov inequality (Theorem 2.15) one can show that

$$|K_1| \leq |\mathrm{co}(K \cup (1/\lambda_1')B_2^n)| \leq \exp(cn/a_1^2). \tag{5.2.5}$$

By the definition of K_1 one can apply the same reasoning to K_1°, and this shows that

$$\exp(-c/a_1^2) \leq \frac{s(K_1)}{s(K)} \leq \exp(c/a_1^2). \tag{5.2.6}$$

By construction, for the new body K_1 we have $d(X_{K_1}, \ell_2^n) \leq M(K)M^*(K)a_1^2$ and, since $s(K_1)$ is an affine invariant, we may assume that $M(K_1)M^*(K_1) \leq c\log[d(X_{K_1}, \ell_2^n) + 1]$ and $M(K_1) = 1$. If we set $\lambda_2 = M^*(K_1)a_2$, $\lambda_2' = M(K_1)a_2$ and define $K_2 = \mathrm{co}\left((K_1 \cap \lambda_2 B_2^n) \cup \frac{1}{\lambda_2'}B_2^n\right)$, we obtain

$$\exp(-c/a_2^2) \leq \frac{s(K_2)}{s(K_1)} \leq \exp(c/a_2^2). \tag{5.2.7}$$

We now iterate this procedure, choosing $a_1 = \log n$, $a_2 = \log\log n, \ldots, a_t = \log^{(t)} n$ – the t-iterated logarithm of n, and stop the procedure at the first t for which $a_t < 2$. It is easy to check that $d(X_{K_t}, \ell_2^n) \leq C$, therefore

$$\frac{1}{C} \leq \frac{s(K_t)}{s(B_2^n)} \leq C. \tag{5.2.8}$$

On the other hand,

$$c_1 \leq \exp\left(-c\big(\frac{1}{a_1^2} + \cdots + \frac{1}{a_t^2}\big)\right) \leq \frac{s(K_t)}{s(K)} \leq \exp\left(c\big(\frac{1}{a_1^2} + \cdots + \frac{1}{a_t^2}\big)\right), \tag{5.2.9}$$

which proves the theorem (observe that the series $\frac{1}{a_1^2} + \cdots + \frac{1}{a_t^2} + \cdots$ remains bounded by an absolute constant). $\square$

As a second important application of the method we prove the existence of "M-ellipsoids" associated to any convex body.

Theorem 5.3 *There exists an absolute constant $c > 0$ with the following property: For every symmetric convex body K in $\mathbb{R}^n$ there exists an ellipsoid M_K such that $|K| = |M_K|$ and for every body T in $\mathbb{R}^n$*

$$\frac{1}{c}\big|M_K + T\big|^{1/n} \leq \big|K + T\big|^{1/n} \leq c\big|M_K + T\big|^{1/n} \tag{5.2.10}$$

and

$$\frac{1}{c}|M_K^\circ + T|^{1/n} \leq |K^\circ + T|^{1/n} \leq c|M_K^\circ + T|^{1/n}. \tag{5.2.11}$$

For the proof of Theorem 5.3 we define the same sequence of bodies as in Theorem 5.1. For every s, we check that

$$\exp(-cn/a_s^2) \leq \frac{|K_s + T|}{|K_{s-1} + T|} \leq \exp(cn/a_s^2), \tag{5.2.12}$$

for every convex body T, and the same holds true for K_s°. After t steps, we arrive at a body K_t which is c-isomorphic to an ellipsoid M. Our volume estimates show that $|K_t|^{1/n} \simeq |K|^{1/n}$ up to an absolute constant. If we define $M_K = \rho M$ where $\rho > 0$ is such that $|M_K| = |K|$, then $\rho \simeq 1$ and the result follows. $\square$

A consequence of Theorem 5.3 is that for every body K in $\mathbb{R}^n$ there exists a position $\tilde{K} = u_K(K)$ of volume $|\tilde{K}| = |K|$ such that for every pair of convex bodies K_1 and K_2 in $\mathbb{R}^n$,

$$|t_1\tilde{K_1} + t_2\tilde{K_2}|^{1/n} \leq c\left(t_1|\tilde{K_1}|^{1/n} + t_2|\tilde{K_2}|^{1/n}\right), \tag{5.2.13}$$

for all $t_1, t_2 > 0$, where $c > 0$ is an absolute constant. This statement is the "reverse Brunn-Minkowski inequality" (Milman, [101]).

The ellipsoid M_K in Theorem 5.3 is called an M-ellipsoid for K. The symmetry of K is not really needed (see e.g. [112]). It can be proved that the existence of an M-ellipsoid for K is equivalent to the following statement: There exists a constant $c > 0$ such that for every body K we can find an ellipsoid M_K with $|M_K| = |K|$ and $N(K, M_K) \leq \exp(cn)$.

Interchanging the roles of K and M_K, we say that a convex body K is in M-position (with constant c) if $|K| = |B_2^n|$ and $N(K, B_2^n) \leq \exp(cn)$. With this terminology, Theorem 5.3 is equivalent to the existence of a constant $c > 0$ such that in the affine class of any convex body there exists a representative which is in M-position with constant c. This condition on $N(K, B_2^n)$ implies that $\max\{N(B_2^n, K), N(K^\circ, B_2^n), N(B_2^n, K^\circ)\} \leq \exp(c_1 n)$ for some constant c_1 which depends only on c. If K_1 and K_2 are in M-position with constant c, using these estimates one can easily check that

$$|K_1+K_2|^{1/n} \leq C\left(|K_1|^{1/n} + |K_2|^{1/n}\right) \quad \text{and} \quad |K_1^\circ+K_2^\circ|^{1/n} \leq C\left(|K_1^\circ|^{1/n} + |K_2^\circ|^{1/n}\right) \tag{5.2.14}$$

where C is a constant depending only on c (one just uses the volume estimate $|A + B| \leq N(A, B) \cdot |2B|$). If K is in M-position with constant c, setting $K_1 = K$, $K_2 = B_2^n$ and using the reverse Santaló inequality (Theorem 5.2), we get

$$c^n|K| \cdot |K^\circ| \leq |K \cap B_2^n| \cdot |\mathrm{co}(K^\circ \cup B_2^n)| \leq |K \cap B_2^n| \cdot |K^\circ + B_2^n|, \tag{5.2.15}$$

which, combined with (5.2.14), gives

$$|K \cap B_2^n| \geq c^n|K|. \tag{5.2.16}$$

The next fact about the M-position which is used in many applications is the following statement: If K is in M-position with constant c, then for any $\lambda \in (0, 1)$ a random orthogonal projection $P_E(K)$ onto a $[\lambda n]$-dimensional subspace E has volume ratio bounded by a constant $C(c, \lambda)$. To see this, note that $|\mathrm{co}(K^\circ \cup B_2^n)|^{1/n} \leq C|B_2^n|^{1/n}$ where C depends on c (this is a consequence of (5.2.14)). In other words, $W = \mathrm{co}(K^\circ \cup B_2^n)$ has bounded volume ratio, and Theorem 2.7 shows that for a random $E \in G_{n,[\lambda n]}$,

$$K^\circ \cap E \subseteq W \cap E \subseteq C(c, \lambda)B_E. \tag{5.2.17}$$

By duality, this means that $P_E(K)$ contains a ball rB_E of radius $r \geq 1/C(c, \lambda)$. Since

$$|P_E(K)| \leq N(P_E(K), B_E)|B_E| \leq N(K, B_2^n)|B_E| \leq \exp(cn)|B_E|, \tag{5.2.18}$$

this implies a bound on $\big(|P_E(K)|/|rB_E|\big)^{1/n}$.

Pisier (see [121], Chapter 7) offers a different approach to these results, which provides a construction of special M-ellipsoids with regularity estimates on the covering numbers. The precise statement is as follows: for every $\alpha > 1/2$ and every body K there exists an affine image $\tilde{K}$ of K which satisfies $|\tilde{K}| = |B_2^n|$ and

(5.2.19)

$$\max\{N(K, tB_2^n), N(B_2^n, tK), N(K^\circ, tB_2^n), N(B_2^n, tK^\circ)\} \le \exp(c(\alpha)nt^{-1/\alpha})$$

for every $t \ge 1$, where $c(\alpha)$ is a constant depending only on α, with $c(\alpha) = O((\alpha - \frac{1}{2})^{-1/2})$ as $\alpha \to \frac{1}{2}$. We then say that K is in M-position of order α or α-regular M-position.

6 Additional information in the spirit of geometric functional analysis

6.1 Banach-Mazur distance estimates

Recall the definition of the Banach-Mazur distance: if X and Y are two n-dimensional normed spaces, then

$$d(X, Y) = \min\{\|T\|\,\|T^{-1}\| \mid T : X \to Y \text{ is an isomorphism}\}. \tag{6.1.1}$$

Let $\mathcal{B}_n$ be the collection of all equivalence classes of n-dimensional normed spaces, where $X_1 \sim X_2$ if X_1 and X_2 are isometrically isomorphic. The Banach-Mazur compactum (of order n) is the compact metric space $(\mathcal{B}_n, \log d)$.

The quantitative study of the geometry of the Banach-Mazur compactum essentially starts with John's theorem [73]. For every $X \in \mathcal{B}_n$ one has $d(X, \ell_2^n) \le \sqrt{n}$, and the multiplicative triangle inequality for d shows that $\mathrm{diam}(\mathcal{B}_n) \le n$. The right order of growth of $\mathrm{diam}(\mathcal{B}_n)$ as $n \to \infty$ was established by Gluskin [58] who showed that the Banach-Mazur distance of a typical pair of n-dimensional projections of the unit ball of ℓ_1^{2n} is asymptotically equivalent to n. Gluskin's theorem was the starting point for a deep study of "random spaces" and of random sections and projections of general convex bodies, which is briefly described in the next subsection.

In many interesting cases, the Banach-Mazur distance $d(X, Y)$ is significantly smaller than n. A first example is given by the classical estimates of Gurarii, Kadec and Macaev: $d(\ell_p^n, \ell_q^n) = n^{1/p-1/q}$ if $1 \le p \le q \le 2$ or $2 \le p \le q \le \infty$, and $c_1 n^\alpha \le d(\ell_p^n, \ell_q^n) \le c_2 n^\alpha$, where $c_1, c_2 > 0$ are absolute constants and $\alpha = \max\{1/p - 1/2, 1/2 - 1/q\}$, if $1 \le p < 2 < q \le \infty$. This suggests that the diameter $\mathrm{diam}(\mathcal{A}_n)$ of some important families $\mathcal{A}_n \subset \mathcal{B}_n$ may be of lower order. This has been proved to be true in two important cases: Let $\mathcal{S}_n$ be the family of all 1-symmetric spaces. Tomczak-Jaegermann [145] showed that $\mathrm{diam}(\mathcal{S}_n) \simeq \sqrt{n}$ (Gluskin [60] and Tomczak-Jaegermann had previously obtained the upper bound $c\sqrt{n}\log^b n$).

The same question remains open for the family $\mathcal{U}_n$ of 1-unconditional spaces. It is conjectured that the right order of $\mathrm{diam}(\mathcal{U}_n)$ is close to $\sqrt{n}$. Lindenstrauss and Szankowski [88] have shown that this quantity is bounded by cn^α for some $\alpha \leq 2/3$.

In many cases, the diameter of a subclass of $\mathcal{B}_n$ is estimated by probabilistic methods. The general idea is to estimate the distance $d(X,Y)$ by a suitable average of norm-products. The method of random orthogonal factorizations (which has its origin in work of Tomczak-Jaegermann, and was later developed and used by Benyamini and Gordon [20]) uses the integral

$$\int_{O(n)} \|T\|_{X\to Y}\|T^{-1}\|_{Y\to X}\, d\nu(T) \tag{6.1.2}$$

with respect to the probability Haar measure ν on $O(n)$ as an upper bound for $d(X,Y)$. An inequality of Marcus and Pisier allows one to pass from $O(n)$ to matrices whose entries are independent standard Gaussian variables and then use Chevet's inequality from the theory of Gaussian processes in order to controll this average (see [147]). Using this method one can prove a general inequality in terms of the type-2 constants of the spaces [39]:

$$d(X,Y) \leq c\sqrt{n}[T_2(X) + T_2(Y^*)] \tag{6.1.3}$$

for every $X, Y \in \mathcal{B}_n$. This was further improved by Bourgain and Milman [27] to

$$d(X,Y) \leq c\Big(d(Y,\ell_2^n)T_2(X) + d(X,\ell_2^n)T_2(Y^*)\Big). \tag{6.1.4}$$

A similar technique is used in [27] where it is shown that $d(X,X^*) \leq c(\log n)^\gamma n^{5/6}$ for every $X \in \mathcal{B}_n$. All these results indicate that the distance between spaces whose unit balls are "quite different" is not of the order of n.

The Banach-Mazur distance $d(K,L)$ between two not necessarily symmetric convex bodies K and L is the smallest $d > 0$ for which there exist $z_1, z_2 \in \mathbb{R}^n$ and $T \in GL(n)$ such that $K - z_1 \subseteq T(L - z_2) \subseteq d(K - z_1)$. The question of the maximal distance between non-symmetric bodies is open. John's theorem implies that $d(K,L) \leq n^2$. Better estimates were obtained with the method of random orthogonal factorizations and recent progress on the non-symmetric analogue of Theorem 2.11. In [15] it was proved that every convex body K has an affine image K_1 such that $w(K_1)w(K_1^\circ) \leq c\sqrt{n}$, a bound which was improved to $cn^{1/3}\log^9 n$ in [127]. Using this fact, Rudelson showed that $d(K,L) \leq cn^{4/3}\log^9 n$ for any $K, L \in \mathcal{K}_n$.

In another direction, for every $X \in \mathcal{B}_n$ let us consider the "radius" $R_n(X)$ of the Banach-Mazur compactum $\mathcal{B}_n$ with respect to X, defined by

$$R_n(X) = \max\{d(X,Y) : Y \in \mathcal{B}_n\}. \tag{6.1.5}$$

In this terminology, John's theorem states that $R_n(\ell_2^n) = n^{1/2}$. A natural question asked by Pelczynski is to determine the order of the radius $R_n(\ell_p^n)$ for other values of p. In the case of the cube, one has the estimates $n^{1/2} \leq R_n(\ell_\infty^n) \leq n$ as a consequence of John's theorem. Bourgain and Szarek [29] proved that $R_n(\ell_\infty^n) = o(n)$ and gave a proportional version of the Dvoretzky-Rogers lemma on the contact points of a body and its minimal volume ellipsoid: Assume that B_2^n is the ellipsoid of minimal volume containing K. For every $\delta \in (0,1)$ there exist $m \geq (1-\delta)n$ and contact points $x_1, \dots, x_m$ of K and B_2^n, such that

$$f(\delta)\left(\sum_{i=1}^m t_i^2\right)^{1/2} \leq |\sum_{i=1}^m t_i x_i| \leq \|\sum_{i=1}^m t_i x_i\|_K \leq \sum_{i=1}^m |t_i|. \tag{6.1.6}$$

for every choice of scalars $t_1, \dots, t_m$. This fact can be stated as a proportional factorization theorem [29].

Theorem 6.1 *Let X be an n-dimensional space. For every $\delta \in (0,1)$ one can find $m \geq (1-\delta)n$ and two operators $\alpha : \ell_2^m \to X$, $\beta : X \to \ell_\infty^m$, such that the identity $\mathrm{id}_{2,\infty} : \ell_2^m \to \ell_\infty^m$ is written as $\mathrm{id}_{2,\infty} = \beta \circ \alpha$ and $\|\alpha\| \, \|\beta\| \leq 1/f(\delta)$, where $f(\delta)$ is a function depending only on the proportion $\delta \in (0,1)$.* □

Using this result Bourgain and Szarek gave a final answer to the problem of the uniqueness up to constant of the center of the Banach–Mazur compactum. This can be made a precise question as follows: Does there exist a function $f(\lambda)$, $\lambda \geq 1$, such that for every $X \in \mathcal{B}_n$ with $R_n(X) \leq \lambda\sqrt{n}$ we must have $d(X, \ell_2^n) \leq f(\lambda)$? In other words, are all the "asymptotic centers" of the Banach–Mazur compactum close to the Euclidean space? The answer is negative and the main tool in the proof is Theorem 6.2: Let $X = \ell_2^s \oplus \ell_1^{n-s}$ where $s = [n/2]$. Then $R_n(X) \leq c\sqrt{n}$ for some absolute constant but $d(X, \ell_2^n) \geq (n/2)^{1/2}$. Therefore, there exist asymptotic centers of the Banach-Mazur compactum with distance to ℓ_2^n of the order of $R_n(\ell_2^n)$.

The same inequality allowed Bourgain and Szarek to show that $R_n(\ell_\infty^n) = o(n)$. It is now known (see [141], [49]) that (3) holds true with $f(\delta) = c\delta$, and this gives a better upper bound for $R_n(\ell_\infty^n)$, which however does not seem to give the right order of the quantity: There exists an absolute constant $c > 0$ such that $R_n(\ell_\infty^n) \leq cn^{5/6}$ (see [48]). On the other hand, Szarek [138] using random spaces (see the next subsection) proved that $R_n(\ell_\infty^n) \geq c\sqrt{n}\log n$.

6.2 Random spaces

The theory of random spaces started with Gluskin's theorem [58] on the diameter of the Banach-Mazur compactum. He considered a class $X_{n,m}$ of random n-dimensional normed spaces and showed that with high probability the Banach-Mazur distance of two spaces $X_1, X_2 \in X_{n,2n}$ exceeds cn, where $c > 0$ is an absolute constant.

The class $X_{n,m}$ is defined as follows: we consider a sequence $g_1, \dots, g_m$ of independent standard Gaussian random variables on some probability space $(\Omega, \mathcal{A}, P)$, and for each $\omega \in \Omega$ we define the space $X(\omega)$ whose unit ball is the symmetric convex body

$$B_m(\omega) = \mathrm{absconv}\{e_1, \dots, e_n, g_1(\omega), \dots, g_m(\omega)\}. \tag{6.2.1}$$

Alternatively, one can consider the class $Y_{n,m}$ of spaces $Y(\omega)$ with unit ball

$$\tilde{B}_m(\omega) = \mathrm{absconv}\{g_1(\omega), \dots, g_m(\omega)\}. \tag{6.2.2}$$

If $m \geq n$, then $\tilde{B}_m(\omega)$ has non-empty interior almost surely and defines a norm on $\mathbb{R}^n$. The random space $X(\omega)$ or $Y(\omega)$ can be identified with a quotient of ℓ_1^{n+m} or ℓ_1^m respectively.

Fix $m = 2n$. The basic geometric properties of $B_m(\omega)$ are the following:

1. $B_m(\omega) \supseteq (1/\sqrt{n})B_2^n$.
2. $|B_m(\omega)|^{1/n} \leq c_1 |(1/\sqrt{n})B_2^n|^{1/n}$, where $c_1 > 0$ is an absolute constant.

Consider the class of pairs $(X(\omega_1), X(\omega_2)) \in X_{n,m} \times X_{n,m}$. If we fix ω_2 and $T \in SL(n)$, using the above properties of $B_m(\omega_2)$ we see that

$$\mathrm{Prob}\left(\omega_1 : \|T : X(\omega_2) \to X(\omega_1)\| \leq c_2 \rho\sqrt{n}\right) < \rho^{2n^2} \tag{6.2.3}$$

for every $0 < \rho < 1$, where $c_2 > 0$ is an absolute constant. Our aim is to show that the probability $P_1 := \mathrm{Prob}(\omega_1 : X(\omega_1) \in \mathcal{L}(\omega_2))$ is small, where

$$\mathcal{L}(\omega_2) := \{X(\omega_1) : \exists T \in SL(n) : \|T : X(\omega_1) \to X(\omega_2)\| \leq \alpha\sqrt{n}\} \tag{6.2.4}$$

for some constant $0 < \alpha < 1$ to be determined. To this end, we define

$$\mathcal{M}(\omega_2) = \{T \in SL(n) : \|Te_j\|_{X(\omega_2)} \leq \sqrt{n}, j = 1, \dots, n\}, \tag{6.2.5}$$

and consider a ε-net $\mathcal{N}(\omega_2)$ of $\mathcal{M}(\omega_2)$ in the norm $\|\cdot : \ell_2^n \to \ell_2^n\|$. If $X(\omega_1) \in \mathcal{L}(\omega_2)$, then there exists $T \in \mathcal{M}(\omega_2)$ such that $\|T : X(\omega_1) \to X(\omega_2)\| \leq \alpha\sqrt{n}$. It follows that $\|S : X(\omega_1) \to X(\omega_2)\| \leq (\alpha + \varepsilon)\sqrt{n}$ for some $S \in \mathcal{N}(\omega_2)$. If we set $\alpha = \varepsilon = c\rho/2$, combining with (6.2.3) we see that

$$P_1 < |\mathcal{N}(\omega_2)| \cdot \rho^{2n^2}. \tag{6.2.6}$$

The cardinality of the net is smaller than $(c_3/\varepsilon)^{n^2} = (c_4/\rho)^{n^2}$, and this shows that $P_1 < (1/2)^{n^2}$ if ρ is chosen small enough.

It is now clear that with probability greater than $1 - 2(1/2)^{n^2}$ in $X_{n,m} \times X_{n,m}$ we have

$$\|T : X(\omega_1) \to X(\omega_2)\| \cdot \|T^{-1} : X(\omega_2) \to X(\omega_1)\| \geq \rho^2 n \tag{6.2.7}$$

for all $T \in SL(n)$, which implies $d(X(\omega_1), X(\omega_2)) \geq \rho^2 n$. This proves Gluskin's theorem:

Theorem 6.2 *There exists a constant $c > 0$ such that* $\mathrm{diam}(\mathcal{B}_n) \geq cn$ *for every* $n \in \mathbb{N}$. □

Let us mention the following recent result of Rudelson [128] which complements Gluskin's theorem. If K_1, K_2 are symmetric convex bodies in $\mathbb{R}^n$ and if $k < n$, write $d_k(K_1, K_2)$ for the smallest Banach-Mazur distance between k-dimensional subspaces of K_1 and K_2 respectively. If $D(n,k)$ is the supremum of $d_k(K_1, K_2)$ over all pairs of symmetric convex bodies in $\mathbb{R}^n$, then $D(n,k) \simeq \sqrt{k}$ if $k \leq n^{2/3}$ and $D(n,k) \simeq k^2/n$ if $k \geq n^{2/3}$ (in this statement, $\simeq$ means "up to a fixed power of $\log n$").

Theorem 6.2 was the starting point for a systematic study of random spaces. Random quotients of ℓ_1^m provided examples of the worst possible order for several parameters of the local theory. It turns out that a random space $X \in X_{n,m}$ has a rather "poor" family of bounded operators. It was observed by Gluskin [59], that a random space X_{n,n^2} has the following property: any projection P in X of rank $k \leq n/2$ satisfies

(6.2.8) $$\|T : X \to X\| \geq ck/\sqrt{n \log n}\ .$$

As a consequence such a space has basis constant $bc(X) \geq c'\sqrt{n/\log n}$. [Recall that the basis constant $bc(X)$ of an n-dimensional normed space X is the infimum of the basis constants $bc\{x_1, \dots, x_n\}$ over all bases of X.] This follows immediately from the fact that, by the definition of the basis constant, in any n-dimensional normed space X there exists a projection P of rank $k = [n/2]$ with $\|P : X \to X\| \leq bc(X)$.

Szarek [137] modified the random structure on $X_{n,m}$ and was able to construct an n-dimensional normed space X with $bc(X) \geq c\sqrt{n}$. Because of John's theorem this order is optimal. Mankiewicz [91] applied the random spaces method to construct finite dimensional spaces with the worst (in order) possible symmetric constant. In this work Mankiewicz used the "space mixing" property of the irreducible group of operators. Szarek [139] explicitly introduced the notion of the class $M(k, \alpha)$ of mixing operators which is the set of all linear operators T, satisfying

(6.2.9) $$dist(Tx, E) = |P_{E^\perp} Tx| \geq \alpha |x|$$

for some k-dimensional subspace E and every $x \in E$. It is not difficult to show that any projection P of rank $k \leq n/2$ is $(k, 1/2)$ mixing. Then, Szarek showed that the mixing property is sufficient for proving the results of [91], but also [59] and [137]. In particular, he proved that for a random space $X \in X_{n,n^2}$ one has

(6.2.10) $$\|T : X \to X\| \geq \alpha ck/\sqrt{n \log n}$$

for any $T \in Mix(k, \alpha)$ and that for some modified probability in $X_{n,m}$ the following result holds.

Theorem 6.3 *For every $0 < \alpha \leq 1/2$ and $\delta > 0$, a random space $X(\omega) \in X_{n,m}$ where $m = [\delta n]$, satisfies $\|T : X(\omega) \to X(\omega)\| \geq c(\alpha, \delta)\sqrt{n}$ for every $T \in \mathrm{Mix}(\alpha n, 1)$.* □

It should be mentioned that the random space method allows us to construct a sequence of finite dimensional normed spaces, which serve as blocks for the construction of examples of infinite dimensional spaces with some unexpected properties:

real isomorphic complex Banach spaces which are not complex isomorphic (Bourgain [22]), a Banach space without a basis which has the bounded approximation property (Szarek [140]) etc.

The class $Y_{n,m}$, $m \simeq n^{1+\delta}$ provides examples of random spaces with large Banach-Mazur distance to ℓ_1^n. The distribution of $Y(\omega)$ is the same with the distribution of ℓ_1^m/H where H is a random $(m-n)$-dimensional subspace of ℓ_1^m, and thus $Y_{n,m}$ reflects completely the geometry of quotients of ℓ_1^m. The following theorem of Szarek [138] gives the only known example of a pair of spaces with distance significantly larger than $\sqrt{n}$, in which one of the two spaces is concrete.

Theorem 6.4 *For every $\delta > 0$, a random space $Y(\omega) \in Y_{n,m}$ where $m = [n^{1+\delta}]$, satisfies $d(Y(\omega), \ell_1^n) \geq c(\delta)\sqrt{n}\log n$.*

The proof involves a precise distributional inequality on the singular numbers s_i of random Gaussian matrices, which is a quantitative finite version of Wigner's semicircle law: if $G(\omega)$ is an $n \times n$ matrix with independent $N(0, 1/n)$ Gaussian entries, then

$$\text{Prob}(\omega : c_1 k/n \leq s_{n-k}(G(\omega)) \leq c_2 k/n) > 1 - c_3 \exp(-c_4 k^2), \tag{6.2.11}$$

for all $k \leq n/2$, where the c_i's are absolute positive constants.

In the last years it was understood that the ideas and arguments used in the study of random quotients of ℓ_1^{n+m} could be transferred to a much more general setting. The idea of studying random projections of arbitrary high-dimensional convex bodies comes from Bourgain, and it was developed in a whole theory by Mankiewicz and Tomczak-Jaegermann (see the survey article [93]). The starting observation is that the main geometric properties of a random space in $X_{n,m}$ can be satisfied by projections of an arbitrary convex body if they are put in a suitable position. More precisely, for fixed $0 < \rho < 1$ and for every n-dimensional convex body K, there exist a $[\rho n]$-dimensional projection $T = P_E(K)$ and a Euclidean norm on E satisfying the following properties:

1. $\text{vr}(T) \leq C_1(\rho)$.
2. $d(X_K, \ell_2^n)^{-1} B_E \subseteq T \subseteq 2B_E$.
3. There is an orthonormal basis $\{x_j\}$ in X_T with $\max_j \|x_j\|_T \leq C_2(\rho)$.

The proof of this fact makes use of the M-ellipsoids. Properties 1 and 2 correspond to the two geometric properties of $X(\omega) \in X_{n,m}$. The third one, which was also clear by construction in our previous discussion, is allowed in the general setting because of the proportional Dvoretzky-Rogers factorization (Theorem 6.1).

An example of this line of thought is the following recent result from [92]: If K_1 and K_2 are two symmetric convex bodies in $\mathbb{R}^n$ whose minimal volume ellipsoid is the Euclidean unit ball, then for every proportional dimension $k = \lambda n$ the average distance between k-dimensional projections $P_{H_1}(K_1)$ and $P_{H_2}(K_2)$ of K_1 and K_2 is bounded from below by the product of the average distances

$$\int_{G_{n,s}} d(P_{L_i}(K_i), \ell_2^s) d\mu_{n,s}(L_i)$$

where s can be taken equal to $s = (1/2 - \varepsilon)k$ for any small $\varepsilon > 0$.

Random spaces were used very recently by Szarek and Tomczak-Jaegermann [144] to provide a strong negative answer to a series of questions raised in the mid-eighties (see [102]), which roughly speaking asked if the cotype properties of every n-dimensional normed space improve by passing to quotients of proportional dimension. A typical example is the following: Is it true that there is an absolute constant $C > 0$ such that every n-dimensional space X has a quotient X_1 of dimension $\dim(X_1) \geq n/2$ such that the cotype-2 constant of X_1 is bounded by C? Recall that this is true if we replace bounded cotype-2 constant by bounded volume ratio (and, by a result of Bourgain and Milman [28], the first property implies the second). A positive answer would be of obvious importance, since all the theory of type and cotype would enter decisively in the study of general convex bodies.

For any given finite dimensional space W, Szarek and Tomczak-Jaegermann construct a space X of an appropriately larger dimension, which is well saturated with W. The precise statement is the following: Let n and m_0 be positive integers with $\sqrt{n \log n} \leq m_0 \leq n$. If W is a normed space with $\dim(W) \leq c \min\{m_0/\sqrt{n}, m_0^2/(n \log n)\}$, there exists an n-dimensional normed space X such that: if $m_0 \leq m \leq n$, every m-dimensional quotient X_1 of X contains a 1-complemented subspace isometric to W.

Let us give a direct application of this fact: If we choose $W = \ell_\infty^k$ with $k \simeq \sqrt{n}$ and consider an n-dimensional space X as above, taking m_0 proportional to n we see that the cotype-2 constant of every m_0-dimensional quotient X_1 of X is at least of the order of $\sqrt[4]{n}$ (and the cotype-q constant of every such X_1 is at least of the order of $n^{1/(2q)}$).

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A.A. Giannopoulos: Department of Mathematics, University of Crete, Iraklion, Greece. *E-mail:* `giannop@fourier.math.uoc.gr`

V.D. Milman: Department of Mathematics, Tel Aviv University, Tel Aviv, Israel. *E-mail:* `vitali@math.tau.ac.il`