Geometrization of Probability

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Dedicated to the memory of Alexander (Sasha) Reznikov, a remarkable mathematician with tragic fate, and who called me his advisor, of which I was always proud.

1. Introduction

1.a. A few historical remarks

The framework of the subject we will discuss in this survey involves very high dimensional spaces (normed spaces, convex bodies) and accompanying asymptotic (by increasing dimension) phenomena.

The starting point of this direction was the open problems of Geometric Functional Analysis (in the '60s and '70s). This development naturally led to the Asymptotic Theory of Finite Dimensional spaces (in '80s and '90s). See the books [MS86], [Pi89] and the survey [LM93] where this point of view still prevails.

During this period, the problems and methods of Classical Convexity were absorbed by the Asymptotic Theory (including geometric inequalites and many geometric, i.e. "isometric" as opposed to "isomorphic" problems).

As an outcome, we derived a new theory: Asymptotic Geometric Analysis. (Two surveys, [GM01] and [GM04] give a proper picture of this theory at this stage.)

One of the most important points of already the first stage of this development is a change in intuition about the behavior of high-dimensional spaces. Instead of the diversity expected in high dimensions and chaotic behavior, we observe a unified behavior with very little diversity. We analyze this change of intuition in [M98] and [M00]. We refer the reader to [M00] for some examples which illustrate this. Also in [M04], we attempt to describe the main principles and phenonema governing the asymptotic behavior of high-dimensional convex bodies and normed spaces.

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1.b. "Convergence" of geometric functional analysis and classical convexity, creating asymptotic geometric analysis

In this introduction, we will give only one result from the past, but will present it in two different forms: one which corresponds to the spirit of Functional Analysis, and the other in the spirit of Convexity Theory. We will meet this result in our main text later. I mean the result which is often called the "Quotient of a Subspace Theorem".

Theorem [M85]. *There is a universal constant $c > 0$ such that for any λ, $1/2 \le \lambda < 1$, and any n-dimensional normed space X, there exist subspaces $F \hookrightarrow E \hookrightarrow X$ with*

$$k = \dim E/F \ge \lambda n \,,$$

and

$$\mathrm{dist}(E/F, \ell_2^k) \le c\frac{|\log(1-\lambda)|}{1-\lambda}\,.$$

Here $\mathrm{dist}(X, Y)$ is the (multiplicative) distance between two normed spaces X and Y which is called the Banach–Mazur distance, and which is formally defined by

$$\mathrm{dist}(X,Y) = \inf\big\{\|T\|\cdot\|T^{-1}\| \;\big|\; T : X \to Y \text{ is an invertible linear operator between spaces } X \text{ and } Y\big\}\,.$$

This distance is defined as infinity if such an invertible operator does not exist.

Some additional remarks: Of course, we may consider the proportion $\lambda > 0$ to be below 1/2. In this case (i.e. for $0 < \lambda < 1/2$) there is another universal constant $C > 0$ such that

$$\mathrm{dist}\left(E/F, \ell_2^k\right) \le 1 + C\sqrt{\lambda}\,.$$

However, this is already an automatic consequence of the well-known and old results of the Asymptotic Theory (see [MS86]).

But the case of λ to be close to 1 is of very special importance. This is already a structural fact. One may start to feel how we can approach and deal with an arbitrary convex body and normed space.

We now present the above theorem in a geometric form. We often call it the global version of the QS-Theorem.

Theorem [M91]. *Let $K \subset \mathbb{R}^n$ be a convex compact body and 0 be its barycenter. There are two linear operators $u_1, u_2 \in SL_n$, such that if $T = K \cap u_1 K$ then $Q = \mathrm{Conv}(T \cup u_2 T)$ is c-isomorphic to an ellipsoid $\mathcal{E}$ (for a universal constant $c > 0$), i.e. $\frac{1}{c}\mathcal{E} \subset Q \subset c\mathcal{E}$. Also, the volume of $\mathcal{E}$ remains the same as the volume of the original body K.*

Note, that constant c doesn't depend on the dimension n or the body K. It is universal, and to feel the meaning of the theorem, one should think of n being very large. In this sense, both theorems above are *asymptotic* and their meaning and strength are revealed when dimension n increases to infinity.

2. Extension of the Category of Convex Bodies to the Category of Log-Concave Measures

Let us first define the class of log-concave measures and functions.

Definitions. A Borel measure μ on $\mathbb{R}^n$ is log-concave iff for any $0 < \lambda < 1$ and any $A, B \subset \mathbb{R}^n$ such that all involved sets $(A, B, \lambda A + (1-\lambda)B)$ are measurable

$$\mu\big(\lambda A + (1-\lambda)B\big) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

Here λA is a homothety and $+$ is the Minkowski sum, i.e. $\lambda A + (1-\lambda)B = \{\lambda x + (1-\lambda)y \mid x \in A \text{ and } y \in B\}$.

A few very important examples of log-concave measures:

(i) The standard volume on $\mathbb{R}^n$, $\mu(K) = \operatorname{Vol} K$ (by Brunn–Minkowski inequality).

(ii) The restriction of volume on a convex set K: $\mu_K(A) = \operatorname{Vol}(K \cap A)$, K-convex.

(iii) Marginals of volume restricted to a convex set.

Let μ be a measure on $\mathbb{R}^n$ with the density function $f(x)$, i.e. $d\mu = f(x)dx$. Let E be a subspace of $\mathbb{R}^n$. Then we define marginal $\operatorname{Proj}_E \mu$ of μ on E the measure on E with density

$$(\operatorname{Proj}_E f) = \int_{x+E^\perp} f(y)dy\,,$$

where $E^\perp$ is the orthogonal subspace of E.

Obviously, marginals of log-concave measures are log-concave measures. In particular, for a convex set K, we consider the measure $\mu_K = 1_K dx$ (where 1_K is the characteristic function of K) and the marginals of this measure are log-concave measures.

Function $f(x) \geq 0$ is called log-concave if $\log f$ is concave, i.e. $f(x) = e^{-\varphi(x)}$ and φ is convex.

The connection between log-concavity of measures and functions was established by C. Borell [Bo74]: Let the support of a measure μ, $\operatorname{Supp}\mu$, not belong to any affine hyperplane. Then μ is log-concave iff μ is absolutely continuous on $\operatorname{Supp}\mu$ and the density f is a log-concave function.

Now we have many more examples of log-concave densities: Let $|x|$ define the standard euclidean norm on $\mathbb{R}^n$ and $\|x\|$ be any norm on $\mathbb{R}^n$. Then any of the following functions is the density of a log-concave measure:

(i) $e^{-|x|}$ (exponential distribution);

(ii) $\frac{1}{(\sqrt{2\pi})^n} e^{-|x|^2/2}$ (the gaussian distribution);

(iii) $e^{-\|x\|^p/p}$, for any norm and $1 \leq p < \infty$.

Log-concavity was used in Convexity Theory already from the '50s (Henstock–MacBeath) and later, say, Prékopa–Leindler extension of Brunn–Minkowsky inequality (see [Pi89]), or the use of log-concave functions to study volume of sections of ℓ_p^n by Meyer–Pajor [MP88]. But a purely geometric study of log-concavity waited until the end of the '80s, and was initiated by K. Ball [Ba86], who extended the study of some geometric problems of convexity to a larger category of log-concave measures. In particular, he studied isotropicity of such measures and connected it with isotropicity of convex bodies. He also considered some important geometric inequalities in the extended framework of log-concave measures ("functional versions" of geometric inequalities). However, just recently it was observed that such an extension is much broader than we thought, and is needed to understand and to solve some problems of asymptotic theory of high dimensional convexity proper.

Three features characterize this extension.

(i) On the one hand, important geometric inequalities (and other kinds of geometric statements) are interpreted, extended and proved for log-concave measures.
(ii) On the other hand, some typical probabilistic results (and thinking) are interpreted and proved in a geometric framework.
(iii) And most importantly, an extension of the geometric approach to the log-concave category is needed to solve some central problems of a purely geometric nature.

The goal of this article is to demonstrate examples of results to confirm this picture.

We consider only finite measures, and only normalization distinguishes them from probability measures. This is the reason I call this extension "Geometrization of Probability". In this extension we identify K with the measure

$$\mu_K := \mathrm{Vol}_{|_K} \quad (\text{i.e. } \mu(A) = \mathrm{Vol}(A \cap K))\,.$$

3. Functional form of some geometric inequalities

3.a. Prékopa–Leindler inequality (functional version of Brunn–Minkowski inequality).

We introduce first sup-convolution which we call, following [AKM04], the Asplund product:

$$(f \star g)(x) = \mathrm{Sup}_{x_1+x_2=x}\, f(x_1)g(x_2).$$

EXAMPLE. $1_K \star 1_T = 1_{K+T}$.
Also λ-homothety for function is defined by

$$(\lambda \cdot f)(x) := f^\lambda \left(\frac{x}{\lambda}\right), \quad \lambda > 0$$

(So $f \star f = 2 \cdot f$)

In this language, the Prékopa–Leindler inequality stated that, for $f, g : \mathbb{R}^n \to [0, \infty)$, $0 < \lambda < 1$,

$$\int (\lambda \cdot f) \star \big((1-\lambda) \cdot g\big) \geq \left(\int f\right)^\lambda \cdot \left(\int g\right)^{1-\lambda}.$$

In this formulation, Prékopa-Leindler is a functional analogue of the multiplicative, dimensional free, form of Brunn–Minkowski inequality:

$$\operatorname{Vol}\big(\lambda A + (1-\lambda)B\big) \geq (\operatorname{Vol} A)^{\lambda} \cdot (\operatorname{Vol} B)^{1-\lambda}$$

(for any subsets A and B of $\mathbb{R}^n$ and $0 < \lambda < 1$ such that all sets involved are measurable). Also "isomorphic" inequalities have their functional form. E.g. geometric statement:

Reverse Brunn–Minkowski inequality (Milman [M86])*:*
$\exists C$ such that for any convex, symmetric $K, P \subset \mathbb{R}^n$, there are linear transforms $T_K, T_P \in SL_n$ (where T_K depends solely on K, and T_P depends solely on P), such that if $\tilde{K} = T_K(K)$, $\tilde{P} = T_P(P)$, then

$$\operatorname{Vol}(\tilde{K} + \tilde{P})^{1/n} < C\big[\operatorname{Vol}(\tilde{K})^{1/n} + \operatorname{Vol}(\tilde{P})^{1/n}\big]\,.$$

Its *functional analogue* is the following statement (Klartag–Milman, [KM05]): For any even log-concave $f, g : \mathbb{R}^n \to (0, \infty)$ there are $T_f, T_g \in SL_n$, such that $\tilde{f} = f \circ T_f$, $\tilde{g} = g \circ T_g$ satisfy

$$\left[\int \tilde{f} \star \tilde{g}\right]^{1/n} < C\left[\left(\int \tilde{f}\right)^{1/n} + \left(\int \tilde{g}\right)^{1/n}\right]$$

where T_f depends solely on f and T_g solely on g (and C is, as before, a universal constant).

3.b. Notion of polarity for log-concave measures; functional version of Santaló inequality.

Let $K \subset \mathbb{R}^n$, convex, $0 \in K$. The polar set K° is define by

$$K^\circ := \big\{x \in \mathbb{R}^n : (x, y) \leq 1 \ \ \forall y \in K\big\}\,.$$

[Functional Analysis interpretation: If $K = -K$, $\|x\|_K$ – Minkowski functional of K, i.e. K is the unit ball of $X = (\mathbb{R}^n, \|\cdot\|_K)$. Then $X^* = (\mathbb{R}^n, \|\cdot\|_K^*)$ has K° its unit ball.]

Let D be the unit euclidean ball.

The following well-known geometric fact is called *Blaschke–Santaló inequality*:

Let $K = -K$, then

$$|K| \cdot |K^\circ| \leq |D|^2$$

(i.e. maximum is achieved on $K := D$).

Let us recall a well-known problem: What is $\min |K| \cdot |K^\circ|$ (Mahler, $\sim$'39)?

The asymptotic answer to this problem is given in Bourgain–Milman [BM85;87]: $\exists c > 0$ universal such that

$$c \leq \left(\frac{|K| \cdot |K^\circ|}{|D|^2}\right)^{1/n}.$$

Very recently, G. Kuperberg [Ku07] gave a different proof of this inequality which does not use the standard technique of the Asymptotic Theory.

For a general not necessarily centrally-symmetric convex body K, the Blaschke–Santaló inequality is also correct for a suitable shift of K: There exists x_0 such that, for $\widehat{K} = K - x_0$,

$$|\widehat{K}| \cdot |\widehat{K}^\circ| \le |D|^2$$

($\min_x |K| \cdot |(K-x)^\circ|$ is achieved for x_0 called the Santaló point of K; then 0 is the barycenter of $(K - x_0)^\circ$.)

Now the functional version of these inequalities:

We start with *Legendre transform*

$$\mathcal{L}\varphi(x) = \sup_{y \in \mathbb{R}^n} \left[(x,y) - \varphi(y) \right].$$

If φ is convex and low semi-continuous, then $\mathcal{L}\mathcal{L}\varphi = \varphi$.

We define polarity for non-negative functions by [AKM04]

$$f^\circ = e^{-\mathcal{L}(-\log f)}, \quad \text{i.e. } -\log f^\circ = \mathcal{L}(-\log f),$$

or

$$f^\circ(x) = \inf_{y \in \mathbb{R}^n} \frac{e^{-(x,y)}}{f(y)}.$$

If f is log-concave upper semi-continuous then $(f^\circ)^\circ = f$.

Examples. For any convex body K, such that $0 \in \mathring{K}$,

$$1_K^\circ = e^{-\|x\|_{K^\circ}}, \quad \left(e^{-\|x\|_K^2/2}\right)^\circ = e^{-\|x\|_{K^\circ}^2/2}.$$

So, the following triple is associated with K:

$$\left(1_K; e^{-\|x\|_K^2/2}; e^{-\|x\|_K}\right)$$

and its polar

$$\left(1_{K^\circ}; e^{-\|x\|_{K^\circ}^2/2}; e^{-\|x\|_{K^\circ}}\right).$$

The only f such that $f^\circ = f$ is the standard Gaussian density, which plays the role of Euclidean ball D, in the "functional" extension of convexity theory we are discussing.

Some elementary properties of *polarity*:

$$(f \star g)^\circ = f^\circ \cdot g^\circ$$

(and therefore, for log-concave functions $(f \cdot g)^\circ = f^\circ \star g^\circ$);

$$(\lambda \cdot f)^\circ = (f^\circ)^\lambda$$

(note, that the dot-product $\lambda \cdot f$ here is the λ-homothety defined in 3a).

Theorem (Artstein, Klartag, Milman [AKM04]). *Let $f : \mathbb{R}^n \to \mathbb{R}^+$, $\int f < \infty$. Then*

(i) *for some x_0 and $\widetilde{f}(x) = f(x - x_0)$,*

$$\int \widetilde{f} \cdot \int \widetilde{f}^\circ \le (2\pi)^n. \tag{1}$$

For log-concave f, we may take $x_0 = \int xf / \int f$.

In the case of f-even, obviously $x_0 = 0$, and the inequality (1) was proved by K. Ball in his thesis [Ba86].

(ii) $\min_{x_0} \int \widetilde{f} \cdot \int \widetilde{f}^\circ = (2\pi)^n$ *iff f is a.e. a gaussian density.*

The standard geometric Santaló inquality for convex bodies follows from (1): apply (1) to $f = e^{-\|x\|_K^2/2}$. Then $\int_{\mathbb{R}^n} f\,dx = c_n|K|$ where $c_n = (2\pi)^{n/2}/|D|$, and similarly for f° which implies Blaschke–Santaló's inequality.

Let us repeat the statement without using the polarity notion:

Theorem [AKM04]. *Let $f : \mathbb{R}^n \to \mathbb{R}^+$, $\int f < \infty$. Then, for some x_0,*

$$\int f \cdot \int \left[\inf_{y \in \mathbb{R}^n} \frac{e^{-(x,y)}}{f(y)} \right] e^{-(x,x_0)} dx \leq (2\pi)^n. \tag{2}$$

For log-concave f, we may take $x_o = \int xf / \int f$. Also, $\min_{x_0}$ of that expression is equal to $(2\pi)^n$ iff f is a.e. a gaussian density function $f(x) = \exp[(Ax,x)+(x,z)+a]$ for some vector z, and $a \in \mathbb{R}$ and an operator $A \geq 0$.

Also the reverse inequality is true in the functional form.

Theorem (Klartag–Milman [KM05]). $\exists c > 0$, *such that for every log-concave $f : \mathbb{R}^n \to \mathbb{R}^+$, $\int f < \infty$, we have*

$$c < \left(\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \right)^{1/n}.$$

We call a function $f \geq 0$ α-concave ($0 < \alpha < \infty$) if $f^{1/\alpha}$ is concave on $\operatorname{Supp} f$.

Important Example: Let $K \subset \mathbb{R}^{n+\alpha}$ be a convex set and E be a subspace, $\dim E = n$. Then, $f := \operatorname{Proj}_E \mathbb{1}_K$ is α-concave. Obviously, an α-concave function is log-concave.

Fact. *Any log-concave function $f : \mathbb{R}^n \to [0,\infty)$ is locally uniform on $\mathbb{R}^n$ approximated by α-concave functions f_α, $f_\alpha(x) \to f(x)$ ($\alpha \to \infty$), for*

$$f_\alpha(x) = \left(1 + \frac{\log f(x)}{\alpha}\right)_+^\alpha \leq f(x)\,,$$

Here, $\varphi(x)_+ = \max\{\varphi(x); 0\}$.

Define "α-duality" by

$$\mathcal{L}_\alpha f(x) = \inf_{y; f(y)>0} \frac{\left(1 - \frac{(x,y)}{\alpha}\right)_+^\alpha}{f(y)} \leq f^\circ(x)\,,$$

for $\alpha \geq 1$, and

$$\mathcal{L}_\alpha f(x) = \inf_{y; f(y)>0} \frac{(1-(x,y))_+^\alpha}{f(y)}\,,$$

for $0 < \alpha < 1$. Clearly, $\mathcal{L}_\alpha f$ is α-concave. Note also that $\mathcal{L}_\alpha f \to f^\circ$ for $\alpha \to \infty$ and $\mathcal{L}_\alpha 1_K \to 1_{K^\circ}$ for $\alpha \to 0$ and 1_T is the characteristic function of the set T.

Fact. *If f is upper semicontinuous and α-concave, $f(0) > 0$, then $\mathcal{L}_\alpha \mathcal{L}_\alpha f = f$.*

Theorem [AKM04]. *Let f be α-concave on $\mathbb{R}^n$, α is an integer, $\mathbb{E}f < \infty$ and $\int x f(x) = 0$. Then*

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} \mathcal{L}_\alpha(f) \le \frac{\alpha^n \kappa_{n+\alpha}^2}{\kappa_\alpha^2} \quad \Big(\underset{\alpha\to\infty}{\longrightarrow} (2\pi)^n \Big) , \tag{3}$$

where $\kappa_k = \operatorname{Vol} D_k$, and the inequality is exact.

Historical remark: The origin of the transform $\mathcal{L}_\alpha$ is from the 1960s. I searched for duality for new moduli, I worked with. Today they are called "asymptotic moduli".

The necessary transform was [M71a]

$$K\varphi = \operatorname{Sup}_y \frac{(x,y)+1}{\varphi} .$$

To deal with this transform we consider the following substitutions. We consider the function $f = \varphi - 1$ and the transform $L_1 f = K\varphi - 1$ to come to

$$L_1(f) = \operatorname{Sup}_y \frac{(x,y) - f(y)}{1 + f(y)} .$$

Consider it as a part of the family L_μ:

$$L_\mu(f) = \operatorname{Sup}_{y\in\mathbb{R}^{n-1}} \frac{(x,y) - f(y)}{1 + \mu f(y)} ,$$

where f is convex on $\mathbb{R}^{n-1}$. Of course, $\mu = 0$ gives the Legendre transform.

To understand the meaning and inversion formula introduce a norm on $\mathbb{R}^n$:

$$\left\| \left(y; \frac{1}{\sqrt{\mu}} \right) \right\| = \frac{1 + \mu f(y)}{\sqrt{\mu}} .$$

Then

$$\left\| \left(x; \frac{1}{\sqrt{\mu}} \right) \right\|^* = \frac{1 + \mu L_\mu f(y)}{\sqrt{\mu}} .$$

Reflexivity of finite dimenional space implies

$$L_\mu L_\mu f = f .$$

Interestingly, only $\mu = 0$, i.e., the case of the Legendre transform proper, lacks this geometric intepretation.

The inequality (3) was written in [AKM04] only for integer value of α. We take later $\alpha \to \infty$ to derive the inequality (2). However, a natural tensoration argument provides a similar inequality for any rational $\alpha > 0$ and, taking the limit, also any $\alpha > 0$. Such tensoration arguments were used by Klartag for proving Theorem 2.1 in [K07a]. At the same time, it is also a particular case of the result by Fradelizi–Meyer [FM07]. They prove the following fact.

Theorem [FM07]. *Let $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be a log-concave nonincreasing function and let φ be a convex function such that $0 < \int_{\mathbb{R}^n} \rho(\varphi(x))dx < \infty$. Define a shifted Legendre transform $\mathcal{L}^z$ by*

$$\mathcal{L}^z \varphi(y) = \mathrm{Sup}_x \big((x-z, y-z) - \varphi(x)\big)$$

for any $y \in \mathbb{R}^n$. Then, for some $z \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} \rho(\varphi(x))dx \int \rho\big(\mathcal{L}^z(\varphi(y))\big)dy \leq \left(\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{2}\right) dx \right)^2.$$

([FM07] *also provides equality conditions under the condition that ρ is a decreasing function). The particular cases corresponding to functions $\rho(t) = e^{-t}$ and $\rho(t) = (1-t)_+^\alpha$ lead to the previous results from* [AKM04].

There are many inequalities in the spirit of the above theorems. Some of them may be developed by the original approach of Ball [Ba86], and also by the method of [AKM04] or using the correspondence between log-concave functions and convex bodies as was put forward by Ball in [Ba86], [Ba88] and used in [KM05]. For other inequalities in this style, see [FM07]. However, we will concentrate our attention on some surprizing extensions which appeared in attempts to answer a question raised by D. Cordero-Erausquin.

He conjectured the following (very unusual) inequality:

Let K and T be any convex centrally symmetric bodies and D be the Euclidean ball. Is it true that

$$Vol(K \cap T) \cdot (K^\circ \cap T) \leq \mathrm{Vol}(D \cap T)^2? \tag{4}$$

He proved this conjecture [C02] for the case where K and $T \subset \mathbb{R}^{2n}$ could be realized as unit balls of complex Banach norms and, in addition T is invariant under complex conjugation. One may see (4) as a " localization" of the standard Blaschke–Santaló inequality.

The surprizing fact is that the functional version of (4) has been proved by Klartag [K07a] and Barthe–Cordero-Erausquin (unpublished) but the geometric conjecture (4) does not follow from it (or, at least, we can't see how it may follow). So, the proved theorem is

Theorem (Klartag [K07a]; Barthe–Cordero-Erausquin). *Let $f : \mathbb{R}^n \to (-\infty, \infty]$ be an even measurable function, and assume that μ is an even log-concave measure on $\mathbb{R}^n$. Then,*

$$\int_{\mathbb{R}^n} e^{-f} d\mu \int_{\mathbb{R}^n} e^{-\mathcal{L}f} d\mu \leq \left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} d\mu \right)^2,$$

whenever at least one of the integrals on the left-hand side is both finite and non-zero.

To describe one geometric consequence, we need the following:

Definition. If A is the unit ball of the norm $\|\cdot\|_A$ and B is the unit ball of the norm $\|\cdot\|_B$, then $A \cap_2 B$ is defined as the unit ball of the norm $\|x\|_{A\cap_2 B} = \sqrt{\|x\|_A^2 + \|x\|_B^2}$.

Corollary (Klartag [K07a]). *Let $K, T \subset \mathbb{R}^n$ be centrally-symmetric, convex bodies. Then,*

$$\operatorname{Vol}_n(K \cap_2 T)\operatorname{Vol}_n(K^\circ \cap_2 T) \le \operatorname{Vol}_n(D \cap_2 T)^2.$$

Note that $A \cap B \subset A \cap_2 B \subset \sqrt{2}(A \cap B)$ for any centrally-symmetric convex sets $A, B \subset \mathbb{R}^n$. Thus, the theorem immediately implies that

$$\operatorname{Vol}_n(K \cap T)\operatorname{Vol}_n(K^\circ \cap T) \le 2^n \operatorname{Vol}_n(D \cap T)^2.$$

Let us show one more fact in this spirit from [K07a].

Let $\psi : \mathbb{R}^n \to (-\infty, \infty]$ be a convex, even function, and let $\alpha > 0$ be a parameter. Let μ be a measure on $\mathbb{R}^n$ whose density $F = d\mu/dx$ is

$$F(x) = \int_0^\infty t^{n+1} e^{-\alpha t^2} e^{-\psi(tx)}\, dt\,.$$

Then, for any centrally-symmetric, convex body $K \subset \mathbb{R}^n$,

$$\mu(K)\mu(K^\circ) \le \mu(D)^2.$$

An example of a measure which is covered by this theorem is, e.g. the measure with density $\frac{1}{(1+\|x\|^2)^{n+2}}$ where $\|\cdot\|$ is a norm on $\mathbb{R}^n$. So, such measures may have "heavy tails" and not be log-concave.

3.c. Functional form of Urysohn inequality (Urysohn inequality for log-concave functions)**.**

Recall the classical Urysohn inequality:

$$\left(\frac{\operatorname{Vol} K}{\operatorname{Vol} D}\right)^{1/n} \le M^\star(K) := \int_{S^{n-1}} \sup_{y\in K}(x,y)\, d\sigma(x)$$

and, by Steiner formula,

$$\operatorname{Vol}(D + \varepsilon K) = \operatorname{Vol} D + \varepsilon n M^\star(K)\operatorname{Vol} D + O(\varepsilon^2)\,.$$

So, we may define the analogous quantity. Let $G(x) = e^{-|x|^2/2}$. Then define

$$V_G(f) = \lim_{\varepsilon\to O^+} \frac{\int G \star [\varepsilon\cdot f] - \int G}{\varepsilon}$$

(one may show that lim exists).

Denote $M^\star(f) = 2\frac{V_G(f)}{n\int G} = \frac{V_G(f)}{\frac{n}{2}(2\pi)^{n/2}}$. Then $M^\star(G) = 1$.

If $f = 1_K$ then (calculation)

$$V_G(1_K) = \frac{(2\pi)^{\frac{n-1}{2}} n\kappa_n}{\kappa_{n-1}} M^\star(K)$$

($\kappa_n = \operatorname{Vol} D_n$).

So $M^\star(K) = c_n M^\star(1_K)$ for $c_n \sim \sqrt{n}$.

The quantity $M^*(f)$ has the following properties:

(i) Linearity:: $M^\star(f \star g) = M^\star(f) + M^\star(g)$;

(ii) Homogenuity:: $M^\star(\lambda \cdot f) = \lambda M^\star(f)$, $\lambda > 0$.

Theorem [KM05]. *Let $f : \mathbb{R}^n \to [0,\infty]$ be an even log-concave function such that $\int f = \int G$ $(= (2\pi)^{n/2})$. Then*

$$M^\star(f) \geq M^\star(G) = 1\,.$$

3.d. Mixed measures – what are they?

Introducing $M^*(f)$ in the previous section creates a feeling that there is a natural and clear notion of mixed measures which extends the notion of mixed volumes. However, the situation is not so, and what mixed measures are is absolutely not yet clear to me. This stage of "geometrization of probability" is still ahead of us.

We see only some examples, mostly on the level of "experiments", which demonstrate, however, the high interest the theory should generate. I will describe below a couple of examples (from Klartag [K07a]).

For $f : \mathbb{R}^n \to [0,\infty)$, concave on $\operatorname{Supp} f$, define a variant of the Legendre transform

$$\mathcal{L}' f = \sup_{y; f(y)>0} \big[f(y) - (x,y)\big]$$

(note $\mathcal{L}'f$ is convex).

For $f_i : \mathbb{R}^n \to [0,\infty)$, $i = 0,1,\dots,n$, compactly supported, concave on their Supp, denote

$$V(f_0,\dots,f_n) = \int_{\mathbb{R}^n} [\mathcal{L}' f_o](x) D\big(\operatorname{Hess}[\mathcal{L}' f_1](x),\dots,\operatorname{Hess}[\mathcal{L}' f_n]\big)dx\,.$$

(See the Appendix for a definition and a few properties of mixed discriminants $D(A_1,\dots,A_n)$ of matrices $A_i \geq 0$.)

Then the following is true: The multilinear form V is

(i) fully symmetric with respect to permutations of $\{0,1,\dots,n\}$;
(ii) monotone; i.e. if f_i and g_i as above and $f_i \leq g_i$ then $V(f_0,\dots,f_n) \leq V(g_0,\dots,g_n)$.
(iii) satisfies "hyperbolic" Alexandrov–Fenchel type inequality

$$V(f_0,f_1,\dots,f_n)^2 \geq V(f_0,f_0,f_2\dots,f_n)\cdot V(f_1,f_1,f_2,\dots,f_n).$$

And now "the dual" statement: Let $K \subset \mathbb{R}^n$ be convex compact. Let $f_i : \mathbb{R}^n \to [0,\infty)$, $i = 0,1,\dots,n$, be concave, vanishing on ∂K, with bounded second derivatives in $\mathring{K}$. Denote:

$$I(f_0,\dots,f_n) = \int_K f_0(x) D(-\operatorname{Hess} f_1,\dots,-\operatorname{Hess} f_n)dx\,.$$

Then, the multilinear form I is:

(i) fully symmetric with respect to permutations;
(ii) monotone (in the above class of functions);
(iii) the following "elliptic-type" inequality is satisfied:

$$I(f_0,f_1,\dots,f_n)^2 \leq I(f_0,f_0,f_2\dots,f_n)\cdot I(f_1,f_1,f_2,\dots,f_n)\,.$$

So, the Legendre transform "transforms" elliptic type inequalities into hyperbolic type! Why? We could not observe this kind of phenomenon in the category of convex sets because the functional duality is not closed in this category.

4. A Central Limit Theorem (CLT) for Convex Sets and Log-Concave Measures

In the classical geometric approach, we study a geometric shape of projections (or sections) of convex body K, and we know that they are, with high probability, close to euclidean balls for small enough rank of projections.

The exact old estimate stated [M71b] that, with high probability, a random projection P_E of a convex body K in $\mathbb{R}^n$ of rank $k^* < cn\big(\frac{M^*(K)}{\operatorname{diam} K}\big)^2$ is isomorphic upto a constant 2 to a euclidean k^*-dimensional ball. Here c is a universal constant, $M^*(K)$ was defined in 3.c and $\operatorname{diam} K$ is the diameter of K.

But what about measure projections (marginals) of convex bodies in place of geometric projections? This question was first asked by Gromov [Gr88]. He made some initial observations, but recently the structure of random marginals was understood completely. To describe the results we need some notions.

Normalize the convex body $K \subset \mathbb{R}^n$ such that

$$\operatorname{Vol} K = 1, \quad \int_K \vec{x} dx = 0, \quad \int_K \langle x, \theta \rangle^2 dx = |\theta|^2 L_K^2,$$

for any $\theta \in \mathbb{R}^n$. We say that K is in "isotropic" position and the constant L_K is called the isotropic constant of K.

Theorem (Klartag [K07b], [K07c]). *Suppose $K \subset \mathbb{R}^n$ is convex and isotropic, and X is distributed uniformly in K. Then $\exists \Theta \subset S^{n-1}$ with $\sigma_{n-1}(\Theta) \geq 1 - \delta_n$, such that for $\theta \in \Theta$,*

$$\sup_{A \subset \mathbb{R}} \left| \operatorname{Prob}\big\{ \langle X, \theta \rangle \in A \big\} - \frac{1}{L_K \sqrt{2\pi}} \int_A e^{-t^2/2L_K^2} dt \right| \leq \epsilon_n \,.$$

Here, say, $\delta_n < \exp(-cn^{0.9})$, $\epsilon_n < Cn^{-1/100}$.

Progress towards this goal was obtained earlier by Brehm–Voigt [BV00] and Anttila–Ball–Perissinaki [ABP03]. There is an analogue multi-dimensional version

Theorem (Klartag [K07b]). *Let $K \subset \mathbb{R}^n$ be convex and isotropic. The r.v. X is distributed uniformly in K. Suppose $\epsilon > 0$ and $k < c\epsilon^2 \frac{\log n}{\log \log n}$.*

Then $\exists \mathcal{E} \subset G_{n,k}$ with $\sigma_{n,k}(\mathcal{E}) \geq 1 - \exp(-cn^{0.9})$, such that for $E \in \mathcal{E}$,

$$\sup_{A \subset E} \left| \operatorname{Prob}\big\{ \operatorname{Proj}_E(X) \in A \big\} - \frac{1}{L_K^k} \int_A \frac{e^{-|x|^2/2L_K^2}}{(2\pi)^{k/2}} dx \right| \leq \epsilon \,.$$

Very recently, Klartag [K07c] improved all estimates in the two previous results: instead of log-type estimates in the previous result, he proved a polynomial type estimate. This means that there is a principle difference between the dimension k^* such that geometric shape of projections on subspaces of this dimension can be approximately euclidean and the dimension of marginals which are approximately gussian. In the first case, in some examples, say a cross-polytope – the unit ball of ℓ_1^n space, k^* cannot be above $\sim \log n$, but in the second case we have $\sim$ gaussian marginals in dimensions of the order of say $n^{1/20}$.

5. Isotropic Position and Isotropic Constant

We again recall that a convex body $K \subset \mathbb{R}^n$, with the barycenter of K at 0, is in isotropic position iff $\operatorname{Vol} K = 1$ and, $\forall i, j = 1, \dots, n$,

$$\int_K x_i x_i ds = \delta_{ij} L_K^2$$

($x = (x_1, \dots, x_n)$). We call L_K the isotropic constant of K. It is an old and famous problem of Bourgain if isotropic constants $\{L_K\}$ are uniformly bounded (by dim. n and convex bodies in $\mathbb{R}^n$). A well-known 20-year-old estimate of Bourgain's states that $L_K \le Cn^{1/4} \log n$. However, recently Klartag proved

Theorem (Klartag [K06]). *For any convex body $K \subset \mathbb{R}^n$ and $\epsilon > 0$ there exists a convex body $T \subset \mathbb{R}^n$, such that*

$$(1 - \epsilon)T \subset K - x_0 \subset (1 + \epsilon)T$$

and $L_T < c/\sqrt{\epsilon}$.

COROLLARY (Klartag [K06], relying on Paouris' recent theorem [P06]).

$$L_K < Cn^{1/4} \quad \textit{when } K \subset \mathbb{R}^n .$$

It is important to note that the proof of the last theorem requires the extension of Asymptotic Theory of Convexity to the category of log-concave measures.

In a very rough sketch of his proof, Klartag considered the 'momentum' map

$$F(x) = \log \int_K e^{\langle x,y \rangle} dy$$

(K is a convex body in the isotropic position) which produces (by considering gradient) the transportation of measure from $\mathbb{R}^n$ to K. This creates the family $\{f_x(y) = e^{\langle x,y \rangle} 1_K(y)\}_{x \in nK^\circ}$ of log-concave densities.

The boundedness of the isotropic constant for any of these measures (the isotropic constant of a measure should be defined) would imply the theorem (it would construct an approximation T). In the next step, this fact is proved in the average (which means the existence of one such measure). The proof uses the reverse Santaló inequality [BM87].

To give some details of the proof of the theorem, we need to establish a connection between log-concavity and convex bodies.

For any even log-concave $f : \mathbb{R}^n \to \mathbb{R}^+$ we associate a norm (K. Ball [Ba86])

$$\|x\|_f = \left(\int_0^\infty n f(rx) r^{n-1} dr \right)^{-1/n} .$$

Denote K_f be the unit ball of $\|\cdot\|_f$.

Let us note a few properties of this correspondence:

1. $\operatorname{Vol} K_f = \int f$

2. Define $\overline{\overline{K}}_f = \{x \in \mathbb{R}^n : f(x) > e^{-n}\}$. Then, for a universal $c > 0$,

$$K_f \subset \overline{\overline{K}}_f \subset cK_f\,.$$

3. Let f and g be log-concave functions and $f(0) = g(0) = 1$. Then, for some universal constants c_1 and c_2

$$c_1 K_{f\star g} \subset K_f + K_g \subset c_2 K_{f\star g}$$

and

$$c_1 n K_f^\circ \subset K_{f^\circ} \subset c_2 n K_f^\circ\,.$$

Let us now define the isotropic constant of a log-concave measure. We say that f is in the *isotropic position* if

$$\operatorname{Sup}_{x\in\mathbb{R}^n} f(x) = 1 = \int f(x)dx \quad \text{and}$$

$$\int_{x\in\mathbb{R}^n} x_i x_j\, f dx = \delta_{ij} L_f^2$$

and the constant L_f is called the *isotropic constant* of the measure $f dx$.

One may write a formula for L_f without "putting" $f dx$ in the isotropic position,

$$L_f = \left(\frac{\operatorname{Sup}_{x\in\mathbb{R}^n} f(x)}{\int_{\mathbb{R}^n} f dx}\right)^{1/n} (\det \operatorname{Cov} f)^{1/2n}$$

where covariance matrix

$$\operatorname{Cov} f = \big(\operatorname{Cov}_f(x_i, x_j)\big)\,,$$

$$\operatorname{Cov}_f(x_i, x_j) = \frac{\int_{\mathbb{R}^n} x_i x_j f dx}{\int f dx} - \frac{\int x_i f}{\int f} \cdot \frac{\int x_j f}{\int f}\,.$$

Then, for any K convex, $L_K = L_{1_K}$.

A sketch of Klartag's proof of a solution of the "isomorphic" slicing problem. Let K be convex compact, $O \in K$, $\operatorname{Vol} K = 1$. We will divide the proof into a few steps, and we will refer to [K06] for the proofs which will not be presented.

1. Let $f : K \to [0, \infty)$ be a log-concave function. Assume

$$\left(\frac{\operatorname{Sup}_{x\in K} f}{\inf_{x\in K} f}\right)^{1/n} < C\,.$$

Then K_f isomorphic to K, i.e. $\exists\, c_1 := c_1(C)$ such that

$$\tfrac{1}{c_1} K_f \subset K \subset c_1 K_f$$

(here, as before,

$$K_f = \left\{x \in \mathbb{R}^n;\ \int_0^\infty n f(rx) r^{n-1} dr \geq 1\right\}$$

and is a convex set by K. Ball).

2. (K. Ball [Ba86]) $L_f \simeq L_{K_f}$. So, our goal is to find such an f that $L_f <$ const. (which implies that $L_{K_f} <$ const.).

3. Consider a (convex) function $F_K(x) := F(x)$

$$F(x) = \log \int_K e^{\langle x,y\rangle} dy\,.$$

(a) This function produces a transportation of measure

$$\nabla F := \psi : \mathbb{R}^n \longrightarrow \overset{\circ}{K}$$

(similar to the so called 'momentum' map). Recall the notation of transportation. Let μ_1 and μ_2 be two Borel measures in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ such that, for any measurable set $A \subset \mathbb{R}^n$,

$$\mu_2(A) = \mu_1(T^{-1}A)\,.$$

Then we say that T transports μ_1 to μ_2. Equivalently, $\forall \varphi \in C^+(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \varphi(x) d\mu_2(x) = \int_{\mathbb{R}^n} \varphi(Tx) d\mu_1(x)\,.$$

The following fact is straightforward.

Fact. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be C^2-smooth strictly convex and $K = \mathrm{Im}(\nabla F)$. Let measure μ have density $\frac{d\mu}{dx} = \det \mathrm{Hess}\, F(x)$. Then $\nabla F : \mathbb{R}^n \to \mathbb{R}^n$ transports μ to* $\mathrm{Vol}_{|K}$.

Applying this to our situation, we see that ∇F transports the measure μ to the uniform measure on K.

Using this, we see that, for any measurable set $A \subset \mathbb{R}^n$,

$$\int_A \det \mathrm{Hess}\, F = \mathrm{Vol}((\nabla F)A) \leq 1\,.$$

(b) Note that $\nabla F(x) = \int y\, d\mu_{K,x}(y)$ and the density of $\mu_{K,x}$ is

$$\frac{e^{\langle x,y\rangle} 1_K(y)}{\int_K e^{\langle x,z\rangle} dz}\,.$$

Also $\mathrm{Hess}(F)(x) = \mathrm{Cov}(\mu_{K,x})$. Therefore

$$\det \mathrm{Hess}\, F(x) = \left(\int f_x / \operatorname{Sup} f_x \right)^2 \cdot L_{f_x}^{2n}$$

where $f_x(y) = e^{\langle x,y\rangle} 1_K(y)$.

So, we consider the family of log-concave functions and we search for a function as in 1. and 2. inside this family.

4. Let $x \in nK^\circ$. (Note that the volume $|K| = 1$ implies $|nK^\circ|^{1/n} \sim 1$ by the Bourgain–Milman reverse Santaló inequality [BM87].)

(a) Then

$$\left(\frac{\operatorname{Sup}_{y\in K} f_x(y)}{\inf_{y\in K} f_x(y)}\right)^{1/n} < C.$$

Indeed, $\operatorname{Sup}_{y\in K} f_x(y) = \operatorname{Sup}_{y\in K} e^{\langle x,y\rangle} \le e^{\|x\|^*} \le e^n$. (Similarly for $\inf \ge e^{-n}$.) So we know that $K_{f_x} \sim K$ for any $x \in nK^\circ$.

(b) We want to find $x \in nK^\circ$ such that $L_{f_x} <$ Const., i.e. to estimate from above by some constant

$$\big(\det \operatorname{Hess} F(x)\big)^{1/2n} \left(\frac{\operatorname{Sup} f_x}{\int f_x}\right)^{1/n}.$$

Actually, it is enough to find $x \in nK^\circ$ such that

$$\det \operatorname{Hess} F(x) < \text{Const.}^n.$$

We prove this "on average":

$$\frac{1}{|nK^\circ|}\int_{nK^\circ} \det \operatorname{Hess} F(x) \le \frac{1}{|nK^\circ|} \operatorname{Vol}(\operatorname{Im}(\nabla F)) \le \frac{1}{|nK^\circ|} \le C^n$$

(this is the reverse Santaló inequality we already mentioned).

6. Is Further Extension Possible?

Does the family of log-concave measures (we discussed in sections 2 and 3) represent the largest class of probability measures where Geometry is extended so naturally?

This is not clear. But let us consider a much larger class of "convex measures" (I also like the terminology "hyperbolic measures").

In section 3b, we introduced the class of α-concave functions for $0 < \alpha < \infty$. We used there the terminology from [GrM87]. We now extend this class to negative α but also we will change the notation and follow C. Borell's approach. The new "s-concavity", for positive s, will correspond to $1/\alpha$-concavity above, i.e., $s = 1/\alpha$.

Definition (C. Borell, '74). Fix $-\infty \le s \le 1$; a measure μ on $\mathbb{R}^n$ is s-concave iff $\forall\, A, B \subset \mathbb{R}^n$ non-empty and measurable, $t \in (0,1)$,

$$\mu\big(tA + (1-t)B\big) \ge \big(t\mu(A)^s + (1-t)\mu(B)^s\big)^{1/s}.$$

Note, for $s = 0$, it is exactly the log-concavity condition:

$$\mu\big(tA + (1-t)B\big) \ge \mu(A)^t \mu(B)^{1-t},$$

and, for $s = -\infty$,

$$\mu\big(tA + (1-t)B\big) \ge \min\big(\mu(A), \mu(B)\big). \tag{5}$$

Denote $\mathcal{M}(s)$ the class of all finite s-concave measures. Clearly $\mathcal{M}(s_1) \supseteq \mathcal{M}(s_2)$ for $s_1 < s_2$, and so for any s an s-concave measure satisfies (5).

New example: Cauchy distribution with density

$$p(x) = \frac{c_n}{(1+|x|^2)^{\frac{n+1}{2}}};$$

in this case $s = -1$ (the so-called "heavy tails" distributions).

C. Borell: (i) $\forall \mu \in \mathcal{M}(-\infty)$ has a convex $\operatorname{supp} K \subset \mathbb{R}^n$ and μ is absolutely continuous (w.r.t Lebesgue measure) on K;

(ii) If μ is s-concave, then $s \le 1/\dim K$;

(iii) If $\dim K = n$, the density p of μ satisfies $\forall\, x, y \in K$

$$p\big(tx + (1-t)y\big) \ge \big(tp(x)^{s_n} + (1-t)p(y)^{s_n}\big)^{1/s_n}$$

for $s_n = \frac{s}{1-ns}$. (So, if μ is log-concave then also its density is a log-concave function; however, if $s = -\infty$ then its density is $(-1/n)$-concave.)

Also, levels of densities of convex measures are boundaries of convex sets. Recently, interest in s-concave measures for negative s has been revived; see [B06].

Connection with the Classical Convexity. The definition of convex measures corresponds to the unified principle behind most (or, perhaps, all) geometric inequalites, a principle of minimization:

$$f(A;B) \ge \min\big\{f(A;A), f(B;B)\big\}$$

["the minimum is achieved on equal objects"].

Examples. (i) Alexandrov–Fenchel inequality is equivalent to the above minimization principle

$$V(A;B;C_1,\dots) \ge \min\big(V(A;A;C_1,\dots);V(B;B;C_1,\dots)\big)\,.$$

(ii) Brunn–Minkowski inequality: $\forall\, t, \tau > 0$ and A, B convex:

$$|tA + \tau B|^{1/n} \ge t|A|^{1/n} + \tau|B|^{1/n}$$

is again equivalent to the minimization principle:

$$|tA + \tau B| \ge \min\big(|(t+\tau)A|, |(t+\tau)B|\big).$$

And so on (see, [GM04] for more examples and a discussion on this subject).

Is this an incidental similarity? Or does a deeper meaning lie behind it?

Appendix: Mixed Discriminants

Consider the space S_n of real symmetric $n \times n$ matrices. We polarize the function $A \to \det A$ to obtain the symmetric multilinear form

$$D(A_1,\dots,A_n) = \frac{1}{n!}\sum_{\varepsilon\in\{0,1\}^n} (-1)^{n+\sum \varepsilon_i} \det\Big(\sum \epsilon_i A_i\Big)\,,$$

where $A_i \in S_n$. Then, if $t_1, \dots, t_m > 0$ and $A_1, \dots, A_m \in S_n$, the determinant of $t_1 A_1 + \cdots + t_m A_m$ is a homogeneous polynomial of degree n in t_i, which we write in the form

$$\det(t_1 A_1 + \cdots + t_m A_m) = \sum_{1 \le i_1 \le \cdots \le i_n \le m} n! D(A_{i_1}, \dots, A_{i_n}) t_{i_n} \cdots t_{i_n} ,$$

where the coefficients $D(A_{i_1}, \dots, A_{i_n})$ are independent of permutations of variables A_i. The coefficient $D(A_1, \dots, A_n)$ is called the mixed discriminant of $A_1, \dots, A_n$. Note that $D(A, \dots, A) = \det A$. The fact that the polynomial $P(t) = \det(A + tI)$ has only real roots for any $A \in S_n$ plays the central role in the proof of a number of very interesting inequalities connecting mixed discriminants, which are quite similar to the classical Newton inequalities. They were first discovered by Alexandrov [Al38] in one of his approaches to what is now called Alexandrov–Fenchel inequalities. Today, they are part of a more general theory (see, e.g., [H94] or the Appendix in [K07a]). For example, if all matrices involved are positive, Alexandrov proved,

$$D(A_1, \dots, A_{n-2}, B, C)^2 \ge D(A_1, \dots, A_{n-2}, B, B) \cdot D(A_1, \dots, A_{n-2}, C, C) .$$

There are many interesting inequalities for matrices which are corollaries of this remarkable inequality. For example,

$$D(A_1, A_2, \dots, A_n) \ge \prod_{i=1}^{n} [\det A_i]^{1/n}.$$

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