GAFA, Geom. funct. anal.
Special Volume – GAFA2000, 792 – 815
1016-443X/00/S20792-24 $ 1.50+0.20/0

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GAFA Geometric And Functional Analysis

TOPICS IN ASYMPTOTIC GEOMETRIC ANALYSIS

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1 About the Subject

The term "Geometric Analysis" is a recent one but it has quickly become fashionable and is used too often and for very different mathematics. So, we added the adjective "asymptotic" to be more specific, and we will first explain the subject of this talk.

Some people use these words ("Geometric Analysis") to describe a mechanical use of Analysis in Geometry and vice versa. This is not what I mean. I mean the influence on the conceptual level: how the point of view of Analysis, or better to say, Functional Analysis, changes the questions we ask and the problems we consider in Geometry. This happens through the isomorphic study of geometric objects, instead of the isometric (or ε-isometric) as is usual in Geometry.

One of the main tools in Analysis is to reduce information (and complexity) through the use of inequalities, the main art being not to lose information during this process. We will show that an approach in this spirit may be applied also to some geometric problems.

Before I describe the subject let me mention that it was born inside Functional Analysis. In fact, during the 20th century, Functional Analysis was, perhaps, the leading force in the developing of Analysis and many different and rich directions in Mathematics germinated from Functional Analysis. This topic (Asymptotic Geometric Analysis) is just the last of them. Despite many achievements, and this article is about these achievements, I see it just as the start. It will "meld" with Complexity Theory, Statistical Physics and, even more than we see it already, with Geometry, and will become a significant part of the Mathematical Landscape.

In an article on such an occasion, it is difficult to avoid making a few brief historical remarks about Functional Analysis. If one were to ask what

Supported in part by a BSF grant, the Israel Science Foundation, and the Minkowski Minerva Center for Geometry at TAU.

Functional Analysis is, the short, and essentially correct answer should be: the study of infinite dimensional spaces ("dim $= \infty$"). But I am not sure it was so for some of the founders of Functional Analysis. The study of finite, but very high dimensional spaces and their asymptotic properties when dimension increases was, perhaps, one of the starting points for some of them. We see this in Minkowski, who for the purposes of Analytic Number Theory considered n-dimensional space from a geometric point of view (and before him, as well as long after him, geometry had to be 2 or 3-dimensional – see, e.g., the works of Blaschke); P. Lèvy [Lè]; von Neumann. I was very surprised when I discovered[1] the following paragraph, sounding so modern, in an old (1942) paper by von Neumann ([vN]; section 4 of the Introduction):

(Below H_n is an n-dimensional euclidean space and M_n is the space of $n \times n$ matrices.)

> "Our interest will be concentrated in this note on the conditions in H_n and M_n – mainly M_n – when n is *finite*, but *very great.* This is an approach to the study of the infinite dimensional, which differs essentially from the usual one. The usual approach consists in studying an actually infinite dimensional unitary space, i.e. the Hilbert space H, as done loc. cit. We wish to investigate instead the *asymptotic* behavior of H_n and M_n for finite n, when $n \to \infty$.
>
> We think that the latter approach has been unjustifiably neglected, as compared with the former one. It is certainly not contained in it,.....
>
> Since Hilbert space H was conceived as a limiting case of the H_n for $n \to \infty$, we feel that such a study is necessary in order to clarify to what extend H is or is not the only possible limiting case. Indeed we think that it is not, and that investigations on operator rings by F.J. Murray and the author show that other limiting cases exist, which under many aspects are more natural ones.
>
> Our present investigations originated in fact mainly from the desire to solve certain questions..... We hope, however, that the reader will find that they also have an interest of their own, mainly in the sense indicated above: as a study of the asymptotic behavior of H_n and M_n for finite n, when $n \to \infty$."

We know now how far the asymptotic approach, the theory of finite, but infinitely increasing dimension, can go, and how deep it is. However, we also know that Asymptotic Theory is very different from Infinite Dimensional Theory and a lot of the phenomena of infinite dimensional spaces cannot be

[1] In fact, I. Halperin gave me this reference after my colloquium talk in Toronto at the beginning of the '90s

understood through finite dimensional asymptotics (see the recent surveys [O] and [M]).

But, at the beginning of the century, the study of infinite dimensional spaces seemed to be a good "approximation" for asymptotic finite dimensional problems. The achievements and success of the theory were indeed great and the asymptotic approach was ignored and forgotten. Only in the 60s, and more intensively in the 70s, in a search for new approaches to the accumulation of unsolved problems of purely infinite dimensional nature, did we return to the study of finite dimensional spaces, with clear emphasis on *asymptotic* behavior when dimension increases to infinity.

But, before this was understood, we had two classical theories. One was the study of finite dimensional spaces, and I will add, of small dimension. And "small" is any but fixed dimension (it could be 3 or 10 or n, but fixed). This is the subject of geometry, if problems are geometric (or Linear Algebra, say). And another is infinite dimensional theory which is Functional Analysis.

However, as I already hinted, there is a third possibility, a theory "in between".

My first goal is to show that such a theory exists, not just by presenting this or that result, but by demonstrating a new and very different intuition it represents and the new structures it helps to discover. These are, I believe, the most valuable tests of new directions and theories. In the main body of the paper we will avoid terminology and notion born inside the theory to avoid unnecessary terminological "pollution". It will, hopefully, aid the reader in building his/her own intuition on the subject.

I will also describe and discuss one extremely powerful tool, a concept, the so called Concentration Phenomenon, which is used throughout the theory. My choice of this subject is twofold.

First, I believe it is the right tool, not only for Geometric Analysis, but also in all other problems where large parametric families are involved and under study (be this Complexity Theory or Statistical Physics). In fact, it is already intensively used in Asymptotic problems of discrete mathematics including Complexity (see, e.g. [AS], [MoR]).

But secondly, we will see that it is a typical example in the spirit of the theory and a consequence of an isomorphic view of classical geometric problems (more precisely – isoperimetric type problems). So, a change of point of view, a change of concept, made possible the discovery of the new phenomenon.

2 Essay on Asymptotic Theory

2.1 Entropy and volume behavior in high dimension. What is the expected diversity in high dimensional spaces and what happens when dimension increases to infinity? Do we expect some form of order with increasing dimension? Our intuition says "No", we do not expect ordered behavior and we expect exponential increase in diversity. What is the source of this intuition, which, as we will see, is not exactly correct? Most probably, exponential expansion (by dimension) of volume which leads to exponential expansion of entropy (coverings) is the source of this intuition. However, there is a compensation factor, a concentration of measure around "thin" sets, which always accompanies high parametric spaces, and this phenomenon is not taken into account by our intuition. At least it was not taken into account till very recently. We will discuss this phenomenon later, but now let us go through an outcome of the compensation effect.

Some notation. For a set $K \subset \mathbb{R}^n$ we write $|K|$ for its volume and let $N(K,T) = \min\big\{N \text{ such that } \exists \{x_i\}_1^N \subset \mathbb{R}^n \text{ and } K \subset \bigcup_{i=1}^N (x_i + T)\big\}$ be the covering number of K by T. Also, $K + T = \{x + y \mid x \in K,\, y \in T\}$ is the Minkowski sum of sets K and T and $f \sim \varphi$ means universal equivalence: there is an absolute number C, independent of dimension or anything else, such that $\varphi/C \le f \le C \cdot \varphi$. We use $|\cdot|$ for the standard euclidean norm in $\mathbb{R}^n$ and $D = \{x \mid |x| \le 1\}$ is the standard euclidean ball in $\mathbb{R}^n$.

To compare the geometry around the same volume level, we introduce the *volume radius*, $\text{v.r.}\, K = (|K|/|D|)^{1/n}$. Also $d(K,T) = \inf\{a \cdot b \mid K \subset aT$ and $T \subset bK\}$ is a multiplicative geometric distance.

Let us now measure differences in the geometry of two convex bodies K and T (in $\mathbb{R}^n$), of around the same v.r., by measuring $N(K,T)$ and $N(T,K)$. Of course, both numbers may be extremely, unboundedly large, not because of different geometry but for "incidental" reasons, because of a wrong *position* of comparison; consider, for example, D and a very "long" almost degenerated ellipsoid $\mathcal{E} = uD$, $u \in \mathrm{SL}_n$. So, we will measure

$$M(K,T) = \inf\left\{N(uK,T) \cdot N(T,uK) \mid u \in \mathrm{SL}_n\right\}.$$

Naturally, this number must behave exponentially by n for some bodies K and T and even for geometrically very similar bodies. Say, $M(2D,D)$ is C^n for some $C > 1$, and it is easy to build K of volume radius 1, $d(K,D) \le 2$, and covers $M(K,D)$ of the order C^n for $C > 1$. However, $\inf\{d(uK,T) \mid u \in \mathrm{SL}_n\}$ may be as large as $\sim n$ (as E. Gluskin showed in the 80's) for two centrally symmetric bodies (and, perhaps, significantly larger for not symmetric; this question is still under study now). So, the expected number $M(K,T)$ for some K and T of the same v.r. is of the order $\exp(cn\log n)$. However the actual result is the best imaginary:

There is a universal constant C, s.t. $M(K,T) \le e^{C\cdot n}$ for any two convex bodies of the same volume radius (i.e. $|K| = |T|$) in $\mathbb{R}^n$ (see [Mi3], [Pi2], [MiP2], for references, more details and extensions).

It follows from this fact that a family of ellipsoids $\{uD\}$, $u \in \mathrm{GL}_n$ is rich enough to represent (on a rough scale) ANY convex body in volume computations.

Precisely:

Let 0 be the barycenter of K. There is an ellipsoid $\mathcal{E}_K$ such that for any other convex body T with 0 being its barycenter

$$\text{v.r.}(K+T) \sim \text{v.r.}(\mathcal{E}_K + T)\,, \quad \text{v.r.}\, K\cap T \sim \text{v.r.}\, \mathcal{E}_K \cap T\,,$$
$$N(K,T)^{1/n} \sim N(\mathcal{E}_K,T)^{1/n} \quad \text{and} \quad N(T,K)^{1/n} \sim N(T,\mathcal{E}_K)^{1/n}\,.$$

(See the same references as above.)

2.1.1 Technical remarks. Convexity is not the main reason and source of the entropy and volume behavior we mentioned above. One may consider sets which are very far from being convex (but we will concentrate on centrally symmetric and star-shape bodies; the reason will be explained in 2.2.1). The natural notion here is quasi-convexity or p-convexity and the meaning is that our set is far from a "hedgehog-type" shape. See [BBP] for such an extension. We return to these extensions in the next subsection.

2.2 "Isomorphic" geometry. The previous discussion (and the results accompanying it) belongs to a broad direction and a large body of results which are called Geometric Inequalities. Brunn-Minkowski type inequalities, Alexandrov-Fenchel inequalities, and the many inequalities accompanying them, involving volumes, volume radius and so on, are today the classical examples of Geometric Inequalities which mostly "group" around Convexity Theory (see [BuZ], [S] on this subject). However, the difference which one immediately observes between the subject as described in these monographs and the direction outlined above is that the inequalities (equivalences) we described involved some universal constants. These are, of course, geometric type inequalities, but in "isomorphic" form.

Looking back, I see three periods in the development of what one may call "classical quantitative" convexity theory and what some experts, Rolf Schneider for one, rightly call "The Brunn-Minkowski Theory". The first was in the 1840s through Cauchy's and Steiner's works; then about fifty years later Brunn-Minkowski's inequality was discovered, and Minkowski shaped our view of convexity theory for most of the twentieth century. However, again about fifty years later, in 1936, inequalities proved by Alexandrov and also announced by Fenchel marked a whole new period of very deep finite dimensional convexity theory through the study of (exact) geometric inequalities of many different types, description of extremal cases, approximation theory and so on. I recommend R. Schneider's monograph [S] for a very detailed account on these three periods of the theory.

It took another fifty years after Alexandrov-Fenchel's works, in the middle of the 1980s, for the next step in the development of quantitative convexity theory to crystallize.

We now consider geometric problems from a functional analytic point of view. Consequently, typical for geometry "isometric" problems and views are substituted by "isomorphic" ones. This became possible with the *asymptotic* approach (with respect to dimension increasing to infinity) to the study of high dimensional convex bodies.

In particular, the meaning of geometric inequalities was extended. *Isomorphic* geometric inequalities which involved exact dependence of constants on dimension naturally accompany the asymptotic theory of convex bodies.

However, Isomorphic Geometry is not limited to what we called Isomorphic Geometric Inequalities, i.e. to some numerical inequalities.

Also the shape of any convex body is quickly regularized to an ellipsoid, but again, in an isomorphic sense. Let us first see some fact in its exact form:

Theorem. *There is a universal number C such that, for every convex compact body $K \subset \mathbb{R}^n$ with 0 being its barycenter and $|K| = |D|$, one may choose a* position *$\hat{K} = uK$, $u \in SL_n$, such that*

$$\exists u, v \in O(n) \text{ and } P := \hat{K} \cap u\hat{K},$$
$$Q := P + vP \underset{C}{\sim} D$$

(i.e. $\frac{1}{C} D \subset Q \subset CD$).

ε-version: $\exists v_i$, $v_i \in O(n)$, $i = 1, \ldots, [C/\varepsilon^2] = t$

$$\frac{1}{t} \sum_{1}^{t} v_i P \underset{1+\varepsilon}{\sim} rD$$

(dimension free estimate).

Actually, this theorem represents a reformulation in geometric language of a fact first formulated in the language of Functional Analysis and which is proved using methods of Functional Analysis (see the surveys [Mi3] or [LiM] for further references, and [MiP2] for a recent extension to the not centrally symmetric case). It is very natural from the point of view of Functional Analysis to consider a family of spaces uniformly isomorphic to euclidean spaces. There is a corresponding geometric notion of an *Isomorphic ellipsoid* which sounds less natural, but we regularly meet it in Asymptotic Theory. A family of convex bodies $\{K_\alpha\}$ of infinitely increasing dimension represents an "isomorphic ellipsoid" if there is a constant C and a family of ellipsoids $\{\mathcal{E}_\alpha\}$ such that $\mathcal{E}_\alpha \subset K_\alpha \subset C\mathcal{E}_\alpha$ for every K_α in the family, i.e. $d(K_\alpha, \mathcal{E}_\alpha) \le C$.

So, we naturally derive *isomorphic* geometric objects as a family of objects in different spaces of increasing dimension. By *isomorphic* geometric properties, we mean a common property for such a family, and by *isomorphic* study we mean asymptotic behavior determined by some parameter (tending to infinite dimension, or another parameter having the meaning of dimension).

2.2.1 Remarks. 1. In many of the results we have described till now, convexity is not a crucial assumption (but *high* dimension is). One may substitute convexity by another, very weak condition. Define a convolution of two bodies $K \square T = \bigcup_{\substack{x\in K\\ y\in T}} [x,y]$, were $[x,y]$ is the interval joining points x and y.

Let K be a *centrally symmetric quasi-convex* star body, i.e.

(i) $tK \subset K$, $0 \le t \le 1$ (star-body condition); and $K = -K$;

(ii) $K \square K \subset C \cdot K$.

(We say that K is C-quasi-convex. Example: $K_{\ell_p^n}$, the unit ball of ℓ_p^n space for $0 < p < 1$, is a $C(p)$-quasi-convex for some constant $C(p)$ independent of dimension n.) Clearly, $C = 1$ iff K is convex.

Then the results of 2.1 and the theorem above are true for quasi-convex bodies with constant C in these facts depending on the quasi-convexity constant *only* (see [Mi3]).

2. The meaning of the extension of the theory to the quasi-convex setting may be explained by the following example: Let $f \in C(S^{n-1})$, $f > 0$ and even. Consider its homogeneous extension to $\mathbb{R}^n$ by $\hat{f}(x) = |x| f(x/|x|)$, for $x \ne 0$. Let $K = \{x \in \mathbb{R}^n \mid \hat{f}(x) \le 1\}$ be a C-quasi-convex body. This means that the infimum convolution $(\hat{f} \square \hat{f})(x) = \inf\{\hat{f}(x_1) + \hat{f}(x_2) \mid x_1 + x_2 = x\}$ is $1/C$-equivalent to $\hat{f}$, i.e. $\hat{f} \square \hat{f} \ge \frac{1}{C}\hat{f}$. We say

that such an f belongs to the class $C(\alpha)$, for $\alpha = 1/C$. This condition should be seen as a different type (the global one) of "smoothness" type conditions. An application of the theorem in §2.2 states that, for $1 \geq \alpha > 0$, there is a constant K_α depending on $\alpha > 0$ only, such that for any n and $f(x)$ defined on S^{n-1} and from the class $C(\alpha)$, there are three operators $u_1, u_2, u_3 \subset SL_n$ such that the convolution $\psi(x) = \varphi(x) \,\square\, \varphi(u_3 x)$ for the function $\varphi(x) = \hat{f}(u_1 x) + \hat{f}(u_2 x)$ is K_α-equivalant to the euclidean norm x in $\mathbb{R}^n : r|x| \leq \psi(x) \leq K_\alpha r|x|$ (for some $r > 0$ and any $x \in \mathbb{R}^n$).

2.3 More asymptotic ideology (examples of *isomorphic* study). Next, we will see examples of asymptotic behavior which, in some cases, resemble very much phase transition type behavior in Statistical Physics while in some other cases look like thresholds in asymptotic combinatorial problems. We will also see that phase transition type behavior may be viewed in some problems as refining threshold type behavior. And I think this is a very promising observation. Needless to repeat that in our theory, we will have an isomorphic form of phase transitions. However, the thresholds are usually already described in Isomorphic form and, although we have different examples than, say, graph theory considers, there is no novelty in our point of view.

2.3.1 Example of phase transition; Local form. Now $A \subset \mathbb{R}^n$ is *any* bounded set, $d(A) = \text{diameter}\, A$, and

$$w(A) = \int_{S^{n-1}} w(x) d\sigma(x)$$

is the mean-width; here $w(x) = sup_{y \in A}(x,y) - inf_{y \in A}(x,y)$. In Functional Analysis we use half the mean-width, $M^* := M^*(A) = \frac{1}{2} w(A)$. Let

$$d_m(A) = \mathbb{E}\{d(P_E A) \mid P_E\text{-orthoproj. onto subspace } E \text{ and } \dim E = m\}.$$

Fact. *For* $m \geq k^* = n\left(\frac{w(A)}{d(A)}\right)^2$

$$d_m(A) \sim \sqrt{\tfrac{m}{n}}\, d(A)$$

and for $m \leq k^*$

$$d_m(A) \sim w(A) \quad (\textit{so } d_m(A) \textit{ is stabilized}).$$

We see a phase transition type behavior at $m = k^*$.

Moreover, for any $\theta > 0$ there is (small) $c(\theta) > 0$ such that for $m \leq c(\theta)k^*$ the set $P_E A$ is a (θM^*)-net for $M^* \cdot D(E)$ (and $P_E A \subset (1+\theta)M^* \cdot D(E)$).

References/Remarks. The above result is an obvious extension of the corresponding fact for convex centrally symmetric bodies, which is in turn an interpretation of an old estimate [Mi5] in Dvoretsky-type theorem and its exactness shown in [MiS2]. These theorems are dealing with euclidean sections (or, by duality, projections) of convex bodies.

This is the traditional interest of Geometric Functional analysis. The influence of Classical Convexity was to turn these achievements (we call "Local") to the study of the properties of the original convex sets. This study is now called the "*Global Theory*". Many results of "Local Theory" have their precise analogue in the Global form. I think every local result should have its global analogue, and the dictionary between two theories is clear. However, proofs should be often invented from the start.

2.3.2 Let us demonstrate the *global form* of the phase transition we saw before.

Denote $A_t = \frac{1}{t}\sum_1^t u_i A$, $u_i \in O(n)$ (this is the Minkowski sum); of course,A_t depends on $\{u_i\}$ but we do not specify this in the notation,

$$\overline{d}_t(A) := \min\left\{ d\left(\frac{1}{t}\sum_1^t u_i A\right) \,\middle|\, u_i \in O(n)\right\}.$$

Fact. *For $t \le t_A = (d(A)/w(A))^2$*

$$\overline{d}_t(A) \sim \mathbb{E}\left\{ d\left(\frac{1}{t}\sum_1^t u_i A\right)\right\} \sim \frac{1}{\sqrt{t}}\, d(A)$$

and stabilizes for $t \ge t_A$:

$$\overline{d}_t(A) \sim w(A)\,.$$

(Again, we observe a phase transition type behavior at t_A.)

Moreover, for any $\theta > 0$ there is $C(\theta)$ (large) so that for $t \sim C(\theta)t_A$ and "random" $u_i \in O(n)$, A_t is (θM^*)-net in $M^* D$ (and $A_t \subseteq (1+\theta)M^* D$).

Note also that $t_A \le n$ which means that very few rotations of any set A are enough for A_t to become almost a euclidean ball (see [BoLM1]).

References. As in the Local case before, this is an easy extension of convex centrally symmetric case and the main fact we used is proved in [MiS2].

Two comments follow:

2.3.3 The first one is about *threshold view on phase transition* and we will demonstrate it in the following example:

Let the set A in the previous fact be an interval $I = [-1,1]$; then $M^*(I) \sim 1/\sqrt{n}$ and therefore

$$\min_{u_i \in O(n)} d\left(\frac{1}{t}\sum_1^t u_i I\right) \sim \frac{1}{\sqrt{t}} \quad \text{for} \quad t \leq n\,.$$

Defining $A_t = \frac{1}{t}\sum_1^t u_i I$, we see that

$$d(A_t, D) = \begin{cases} \infty & t < n \\ \sqrt{n} & t = n \\ C(\lambda) & t = \lambda n\,,\ 1 < \lambda < 2\,. \end{cases}$$

(One may also show that $C(\lambda) \sim \min\left\{\sqrt{n}, \sqrt{\log(1/(\lambda-1))/(\lambda-1)}\right\}$ (E. Gluskin).)

So, the function $\operatorname{dist}(A_t, D)$ has a sharp threshold, around $t \sim n$. At the same time, if one measures changes in the set A_t by studying a different parameter, the diameter $d(A_t)$, then this function behaves more regularly, but not completely so: at the interval of the threshold for the distance-function, we have a phase transition type behavior for the diameter. (See another example of such behavior in [Kl].)

2.3.4 To come to our second comment, we first compare the result in 2.3.1 with one of a "baby-model" in Spinglasses theory. (I am indebted to M. Talagrand whose Bourbaki talk in March '99 triggered the comparison between the two theories, and to L. Pastur who explained this and other points of the connections to me.) For this consider a special set $A = [-1,1]^n \subset \mathbb{R}^n$, the unit cube in $\mathbb{R}^n$ (or, which is the same for our purpose, $A = \{\pm 1\}^n$ – the set of vertices of the cube $[-1,1]^n$). The standard question of Statistical Physics is the following: Let $E \in G_{n,k}$ be a k-dimensional subspace of $\mathbb{R}^n$ and $k = \lambda n$. Let P_E be the orthogonal projection on E. We introduce the following function:

$$\varphi(\lambda; n) = \mathbb{E}\frac{d(P_E A)}{\sqrt{n}} \tag{$*$}$$

(where expectation $\mathbb{E}$ is over the Grassmannian $G_{n,k}$). The problem is to decide if $\lim_{n\to\infty} \varphi(\lambda; n) := \varphi(\lambda)$ exists (supposedly "Yes") and to show the existence of phase transition(s) for the function $\varphi(\lambda)$. This means that we expect function $\varphi(\lambda)$ to be continuous and even with continuous derivative beside one (or two) value of non-continuity for the derivative $\varphi'(\lambda)$ where we have a "phase transition".

Comparing with what we stated in 2.3.1, we see that although the problems which are investigated in both theories are exactly the same Isomorphic Geometry provides the answers up to universal factors and this is not enough for studying limits. (However, it provides the answers for any sets.) I would like to emphasize that it is not a sign that the methods and results of Asymptotic Geometric Analysis are not strong enough to prove existence of limits. Just isomorphic type results were the right ones in a theory which interested in understanding the asymptotic behavior of any set in $\mathbb{R}^n$ and was not in another theory which concentrates its attention on the study of special sets with many distinguished symmetries.

I believe that the "isomorphic" point of view may be adapted also in Statistical Physics, but here I will show what kind of general statements on existence of limits follow from Asymptotic Geometric Analysis.

2.3.5 "Outside" of the isomorphic phase transition. The existence of a limit, as (*), may be stated for an arbitrary set A (as in 2.3.1) but unfortunately only outside an interval around the "isomorphic" phase transition point $k^* = n(w(A)/d(A))^2$. In the language we adapted in 2.3.4, outside $\lambda_0 = k^*/n = (w(A)/d(A))^2$.

Let us put the corresponding result precisely. Let $d = d(A)$ be the diameter of the set A, $w := w(A)$ be the meanwidth of A, $k = \operatorname{rk} P_E$ be the rank of the orthoprojection onto a subspace E (i.e. $k = \dim E$). Then *there are universal constant $C > 0$ and $c > 0$ such that for any $\epsilon > 0$ and for $k > Ck^*/\epsilon^2$ (k^*, $d(P_E A)$ were defined in* 2.3.1)

$$\frac{1}{1+\epsilon}\sqrt{\frac{k}{n}} \le \frac{1}{d}\mathbb{E}_{G_{n,k}}(d(P_E A)) \le (1+\epsilon)\sqrt{\frac{k}{n}}$$

(*where $\mathbb{E}_{G_{n,k}}$ is the averaging over grassmanian $G_{n,k}$*) *and for $k \le c\epsilon^2 k^*$*

$$\frac{1}{1+\epsilon}w \le \mathbb{E}(d(P_E A)) \le (1+\epsilon)w\,.$$

So, outside an interval of length $\sim k^*/\epsilon^2$ around k^* we have "almost", up to ϵ, limit *uniformly* on *all* sets A.

Again, one may ironize that Statistical Physics is interested exactly in the interval where we don't have much information. But the information we have is exactly what Geometric Analysis was interested to know.

2.4 Approximation; what we expected from our old intuition and reality of the new one. The example in section 2.3.3 needs more attention. How it happens that just λn intervals ($\lambda > 1$) enough for the approximation of the Euclidean ball by Minkowski sum up to a constant $C(\lambda)$

independent of dimensionof the (Isomorphic representation of a ball). And why the sum of just around $\sim n/\epsilon^2$ intervals can approximate a Euclidean ball up to $1+\epsilon$ (almost isometrically)?

To see why it looks impossible (even though this is a proven fact), one should realize that any set of Cn points $\{x_i\}$ on the Euclidean unit sphere belongs to some narrow strip between two parallel hyperplanes at a distance $\sim \sqrt{\log n/n}$. So, we expect a terrible dissymmetry in the sum of such intervals. From this point of view we would expect that we need an exponential (by dimension n) number of intervals to be able to approximate a Euclidean ball. However, the true answer is a logarithm of what our intuition expects. The same happened with many other problems of approximation. We saw in 2.3.2 that, starting any convex set K (or even any set A, but the understanding of the result should be adjusted), not more than Cn random rotations of K will average into $\sim$ a Euclidean ball. In Bourgain-Lindenstrauss-Milman [BoLM1,2], many examples of this nature were considered (the number of steps of Minkowski or Steiner symmetrizations needed to approximate Euclidean ball is of special interest). And in all examples we studied the results are of the order of logarithm (!) of previously expected estimates. As we already discussed in 2.1, the reason behind such unexpected behavior is Concentration Phenomenon which we will discuss next.

2.5 General references. I have just briefly outlined only a very few directions of a very rich theory, with many subjects of study and many unexpected turns. I recommend the following books and survey articles for different directions and points of view on this theory: [MiS1], [To], [Pi1], [Pi2], [LT], [LiM], [GiM1].

3 Isomorphic Form of Isoperimetric problems; Concentration Phenomenon

3.1 The standard form. Let (X, ρ, μ) be a metric (defined by ρ) probability (with respect to a measure μ) space. Let A be a measurable subset of X and $\mu(A \subset X) \geq 1/2$. Define $A_\epsilon = \{x \in X, \rho(x, A) \leq \epsilon\}$. We introduce the function $\alpha(X;\epsilon) = 1 - \inf_A \mu(A_\epsilon)$ which we call the *concentration function* of X.

Important observation: in many examples $\alpha(X,\epsilon)$ is small. But what does this mean? "Small" relative to which quantity or parameter? To continue our discussion let us consider a few typical examples:

The euclidean unit sphere $S^n \subset \mathbb{R}^{n+1}$, or the Stieffel manifold $W_{n,2}$, or the special orthogonal group $SO(n)$, all equipped with the natural Riemannian $SO(n)$-invariant metrics and the rotation invariant (Haar) probability measures;
the n-dimensional space $F_2^n = \{\pm 1\}^n$ over $\mathbb{Z}_2$, or symmetric (=permutation) group Π_n, equipped with the Hamming metric normalized so that the diameters of these sets are equal to 1 and with the counting probability measures;
$SL_2(\mathbb{Z}_p)$ with the counting probability measure and the word distance (i.e. Cayley-graph distance) defined through a special set of generators (see [AM] for precise description and discussion, or the book [AS]).

So we, in fact, consider "families" $\{X_n\}$ of spaces and not individual spaces. Taking this into account, we introduce the notion of a *Levi family* defined by the condition:

$$\alpha(X_n;\epsilon)@ >> n \to \infty > 0 \qquad \text{(for any but fixed } \epsilon > 0\text{)}\,. \tag{3.1}$$

Therefore, we add another parameter, n, a "natural" numeration of a "natural" family $\mathcal{X} = \{X_n\}$ (but what does "natural" mean?) Moreover, in the *natural* examples we often have the following estimates:

$$\alpha(X_n;\epsilon) \le c_1 \exp(-c_2\epsilon^2 n) \tag{3.2}$$

or

$$\le c_1 \exp(-c_2\epsilon\sqrt{n})\,. \tag{3.3}$$

Don't expect from me to explain what I mean by "natural". The theory of what we call Lèvy families, normal Lèvy families (satisfying estimate (3.2) above) is still, for me, the theory of examples. By the way, two types of estimates we observe correspond to families which one may like to identity as "elliptic type" families ($S^n; W_{n,2}; SO(n); F_2^n, \Pi_n$) and which are *normal Lèvy families* (i.e. satisfy the estimate (3.2)) and another one satisfying estimate (3.3), which clearly should be identified as "hyperbolic type" families ($SL_2(\mathbb{Z}_p)$, but also $SL_k(\mathbb{Z}_p)$ for a fixed $k > 2$ and "p" playing the role of "n"). I would like to see a justification for such (obvious) terminology and to find a reason behind the feeling.

In fact, the concept of a Lévy family (and especially a normal Lévy family) generalizes the concept behind the law of large numbers in two directions: a) the measures are not necessarily the product of measures (that is, we have no condition of "independence") and b) Lipschitz functions on the space are considered instead of linear functionals only.

Returning to our understanding of the concentration function $\alpha(X_n,\epsilon)$

being small, we see that it is connected with a family of spaces. For example, it means that for (any) $A_n \subset S^{n+1}$, $\mu(A_n) = 1/2$, and $\epsilon > 0$ but fixed, $\mu(A_n)_\epsilon$ tends to 1 when $n \to \infty$ (and exponentially quickly). This is not a geometric point of view. Classical geometry fixes a space (meaning "n") and considers dependence on $\epsilon > 0$ and then it does not observe the phenomenon! However, we don't consider one set "A" but a family of sets $\{A_n \subset S^{n+1}, \mu(A_n) \geq 1/2\}$ and the statement, which is an *isomorphic inequality*, gives a uniform bound for ϵ-extensions of sets of this family.

At the same time in many (most) interesting cases, we know how to estimate $\alpha(X;\epsilon)$ without having any idea of the structure of the extremal sets. The exact structure of extremal sets or exact value of $\alpha(X;\epsilon)$ are known only for two or three cases and are not known even for such "simple" cases as $X = \mathbb{T}^n = \prod_i^n S_i^1$ or the cube.

However, we don't need the exact values of $\alpha(X;\epsilon)$. An isomorphic form of inequalities is enough for all applications [and also the only one typically available]. So the isomorphic (=asymptotic) view on isoperimetric problems has freed us from the necessity of solving (exactly) the isoperimetric problems.

The above estimate (3.2) and (3.3) happen to be typical and imply the so-called "*concentration phenomenon*".

To explain the reason for such terminology and also outline why a bound of the form (3.1) is so crucial, let us consider a 1-Lip function $f(x)$ defined on (X, ρ, μ), i.e.

$$\left|f(x) - f(y)\right| \leq \rho(x,y)\,.$$

Denote by L_f the median of $f(x)$, which is defined by

$$\mu\left\{x \in X \;\middle|\; f(x) \geq L_f\right\} \leq \tfrac{1}{2} \quad \text{and} \quad \mu\left\{x \in Z \;\middle|\; f(x) \leq L_f\right\} \geq \tfrac{1}{2}\,.$$

Then

$$\mu\left\{x \in X \;\middle|\; |f(x) - L_f| < \epsilon\right\} \geq 1 - 2\alpha(X,\epsilon)\,. \tag{3.4}$$

So, if the value of $\alpha(X,\epsilon)$ is very small, then the values of Lipschitz function "concentrate" in the measure around one value, meaning it is almost constant with high probability. This is the case when $X = S^n$ and dimension n is large, as well as for large n for $\mathbb{T}^n$ or SO_n or the other examples we mentioned above. It is, in fact, a general property of high dimensional metric probability spaces which is called "*concentration phenomenon*".

Such a "concentration" of measure (this type of estimates) balances the exponentially high entropy of n-dimensional spaces (or other n-parametric families) and leads to a "regularity" in high dimension, keeping "diversity" under control. The absolute constants involved in the examples we saw are needed to balance rates of exponential decay (coming from Concentration) and exponential expansion (coming from covering/entropy). Surprisingly, both exponents have "roughly" the same order of decay via expansion by dimension and only a factor is needed to compensate them.

On a slightly more technical level, let, for example, X_n be S^{n+1} (the euclidean unit sphere in $\mathbb{R}^{n+2}$). Then $\alpha(X_n,\varepsilon) \le c_1 \exp(-\varepsilon^2 n/2)$ and (3.4) implies that for any set $\mathcal{N} = \{x_i\}_1^N \subset S^{n+1}$, with $N < c_1 \exp(\varepsilon^2 n/2)$ one may find a rotation $u \in O(n)$ such that

$$f(ux_i) \underset{\varepsilon}{\sim} L_f\,. \tag{3.5}$$

So, our function $f(x)$ is almost (up to ε) constant on the configuration $u\mathcal{N}$. Choosing various functions and configurations we obtain various geometric consequences. (Naturally, similar applications for other Lèvy families are possible and used.)

[This is, of course, only a scheme. The whole theory was developed to use in a delicate and correct way the concept of "concentration", see [MiS1], [Pi2], [GiM1]. To study the concentration phenomenon itself, consult [Mi2], [T1], [T2], [Gr1]; one may also consult Gromov's article in this collection [Gr2].]

Today, the concentration phenomenon has many applications outside this Asymptotic Theory (Geometric Analysis) where it was actually born. It is used broadly in Probability Theory, Complexity Theory (construction of fast algorithms and so on), Discrete Mathematics (especially in its branch I would call Asymptotic Combinatorics) and even in PDEs and others. Because of its widespread applicatons let me comment briefly on two ways of extending this concept.

(i) (X,ρ) – metric spaces. We often want to apply the same technique and ideas as we learned from the use of the concept of concentration in situations where no natural measure may be introduced (say, for infinite dimensional spaces). Also, we often have a measure and use it, receiving answers "with very high probability" when in fact, we are just searching for "existence" results. So, the measure is not natural in these problems.

(ii) (X,μ) – probability spaces. In many examples of clearly "concentration" type results the metric we use is not natural and not connected with concentration. Also, in some situations, it cannot even exist (say, in the framework of non-commutative geometry).

So, let us check how one may define such "concentration properties".

For this let us turn to two main schemes of applicatons.

3.2 Metric G-spaces (X, ρ). Let $X = (X, \rho, \mu)$ be (still) a metric probability space and let X also be a G-space (meaning that a group G of metric and measure preserving maps acts on X). Note, that if $A \subset X$ has not too small measure, say $\mu(A) \geq 1/10$, but $\alpha(X, \epsilon)$ is very small then it is easy to see that $\mu(A_\epsilon)$ will be close to 1. Let, say, $\alpha(X, \epsilon)$ be so small that $\mu(A_\epsilon) > 1 - \frac{1}{100}$. Then, for any $\{g_i\}_1^{100} \subset G$ the intersection $\bigcap_1^{100} g_i(A_\epsilon) \neq \emptyset$. Therefore, for *any* partition of $X = \bigcup_1^{10} A_i$ one may find A_{i_0} such that for any 100 "rotation" g_i's from G intersection $\bigcap_1^{100} g_i((A_i)_\epsilon) \neq \emptyset$.

This is one of the important schemes of application of the concentration property. But it essentially deals only with a metric property of X and the action on the group G on it.

Now let (X, ρ) be a metric G-space (so without any measure structure on it). To introduce a notion of concentration on X with respect to G, I will deal with one example (and future generalizations are obvious).

Let $X = S^\infty = S(H)$ be the unit sphere of a Hilbert space H. Let G be a subgroup of unitary (orthogonal, if our field is $\mathbb{R}$) operators U.

We call a set $A \subset S^\infty$ *essential* with respect to G iff $\forall \epsilon > 0$ $\forall n$ and $\forall g_1, \ldots, g_n \subset G$

$$\bigcap_1^n g_i(A_\epsilon) \neq \emptyset .$$

We say that (S^∞, G) has a "*concentration property*" iff $\forall$ finite partition $S^\infty = \bigcup_1^N A_i$, $A_i \subset S^\infty$, there is A_{i_0} which is essential.

EXAMPLES. 1) Fix an orthonormal basis $\{e_i\}_1^\infty$ and let $U(n)$ be a subgroup of U which is the unitary group on span $\{e_i\}_1^n$ and the identity action on $\{e_i\}_{i=n+1}^\infty$. Let $G = \bigcup_1^\infty U(n)$. Then (S^∞, G) has the concentration property.

2) Let u be any unitary operator on H and $G = \{u^n\}_{-\infty}^\infty$. Then (S^∞, G) has the concentration property.

3) Let $\mathfrak{A}$ be a family of pairwise commuting unitary operators in H. Then $(S^\infty, \mathfrak{A})$ has the concentration property.

References. These notions appeared in our discussions with M. Gromov at the end of the 70s and beginning of the 80s. See [Mi2].

Later V. Pestov ([P1]) connected this property with the *amenability* of G-action on X (i.e. with the existence of invariant mean; i.e. with existence of some substitution for measure !). I will cite only one of Pestov's results in this direction (and recommend the interested reader to consult the original papers for deeper connections):

A locally compact group G is amenable iff the uniform action $(S_{L_2(G)}, G)$ has the concentration property.

3.3 Probability spaces (X, μ). Again, let (X, ρ, μ) be a metric probability space with small concentration function $\alpha(X, \epsilon)$. Then (3.4) implies that $\mathbb{E}|f|$ is close to $L_{|f|}$. Similarly $(\mathbb{E}|f|^p)^{1/p} :=$ L_p-norm of a Lip function f is around the median $(L_{|f|^p})^{1/p} = L_{|f|}$. So, different L_p-norms are, in fact, equivalent (up to a factor depending on $p > 0$ and Lip – constant of f) which means that Hölder inequality may be reversed:

$$\|f\|_{L_p(X,\mu)} \leq c(p)\|f\|_{L_1(X,\mu)} \tag{3.6}$$

and the order of the constant $c(p)$ as $p \to \infty$ reflects the degree of concentration.

Such inverse Hölder inequalities appear often in the context of probability spaces. For example, linear functionals on a convex body K with volume 1 satisfy the inequality $\|f\|_{L_p(K;dx)} \leq cp\|f\|_{L_1(K;dx)}$ where $c > 0$ is an absolute constant. More generally, Bourgain has shown that if $f : K \to \mathbb{R}$ is a polynomial of degree m, then $\|f\|_p \leq c(p,m)\|f\|_2$ for every $p > 2$, where $c(p,m)$ depends only on p and on the degree m of f. Talagrand showed that an analogous statement holds true for the class of convex functions on E_2^n. In a joint work with Giannopoulos [GiM2] we started a regular study of the level of concentration with respect to a given *class of functions*. Our goal was to understand concentrations of "random" sets in $\mathbb{R}^n$ with respect to linear functions on $\mathbb{R}^n$. Of course, there are many different meanings of randomness in $\mathbb{R}^n$ and also, depending on the cardinality of our subsets, we may achieve different levels of concentration. In a typical example, we have a log-concave probability measure on $\mathbb{R}^n$, a "random", with respect to this measure, set S of points and then we have very exact connection between the level of concentration of linear functions on S (measured by constants $c(p)$ in (3.6)) and the cardinality of S.

The subject is very fresh, possible applications and connections with other parts of the theory are very promising. But it is too early to discuss it here and I am referring to the original paper [GiM2].

3.4 Functional point of view. Let us return to (3.5). There is a way of vastly extending this consequence of concentration which we, again, will demonstrate on one example of Lèvy family. Consider a family of probability measures $\{\mu_i\}_1^N$ on S^{n+1} and let $f(x)$ be a 1-Lip. function on S^{n+1}.

Fact. *If $N \sim e^{\varepsilon^2 n/8}$ then there is a rotation $u \in O(n)$ such that $\left| \int_{S^{n+1}} f(ux) d\mu_i(x) - L_f \right| < \varepsilon$ for every $i = 1, \dots, N$.*

Of course, when μ_i are δ-measures, we return to (3.5). However, the opposite case of μ_i having good densities may be more interesting in many cases.

This also opens the way to understanding how the purely geometric idea of concentration may be extended to a non-commutative setting.

4 Concluding Remarks

4.1 I considered in the paper behavior of sets (often convex sets) in $\mathbb{R}^n$ with the emphasis on asymptotics when dimension $n \to \infty$. But, in fact, I see them just as examples, better understood but typical, of features of the world of very high degree of freedom. One should realize that this world looks very different from what was expected when we entered it:

- The exponential estimates we expected in approximation procedures happened to be essentially linear (logarithmical of what "old intuition" prepared us to expect);
- In many problems, the expected diversity is essentially reduced to the orbit of GL_n. However, we have to take a new angle and adapt the isomorphic point of view in Geometry;
- Quite non-trivial functions depending on huge number of variables are, with high probability, almost constant ("concentration phenomenon").
- Also, in a geometric sense, a similar phenomenon may be described. I mean that complicated functions look almost constant on huge substructures (the concept of "spectrum" in my old terminology – see [Mi2], or "Ramsey type" theorem in Combinatorics – see [P2]).

We know of many more results which do not correspond to the intuition we inherit, and I believe even more surprising results will be discovered in the near future.

If one would ask what is the reason behind these phenomena (but not on the technical level, we discussed in the section on Concentration Phenomenon) I would refer to a perceived random nature of high dimension as being at the root of the reasons and I would add that the patterns such "high dimensional randomness" produces create the unusual phenomena we observe (see [Mi4] where this point of view persists).

4.2 Speculations. At what stage of its development is Asymptotic Geometric Analysis, as described in this article? What is the perspective of its development? To attempt to answer this almost philosophical question I will speculate even more than is required.

There are a few stages in the development of our knowledge at large which are repeated in all small (and even smaller) components, in the directions we discover, in problems and questions we ask and try to solve.

First, we are just *curious*. At this stage we ask natural questions ("bad" questions as Gromov described them in one of our discussions at the Conference). However, these "natural" questions force us to start to think on the subject, they slowly prepare us for the next stage.

Then we already *think* about the subject and our questions are already much less "natural", they are already deep and unexpected, and the answers we accumulate begin to piece together into a puzzle, but the picture is not yet complete.

At the next stage, comes *understanding*, and later on the stage of "*knowing*". Only at the end, do we finally *possess* our knowledge: we may use it, we may apply it, but, in fact, more that this, we *possess* it.

The subject I discussed in the paper has long ago passed the first stage of curiosity; we are deeply inside the "thinking" stage and even, sometimes, we have *hutzpa* to think we *understand*.

5 Some Open Problems of Asymptotic Geometric Analysis

(I advise also to consult [Mi1] for some list of open problems; only a few of them will be repeated below).

I General problem-directions.

1a. Randomness in the structure of high dimensional spaces. (See [Mi4] for a vague discussion of this direction.)

1b. What is the level of complexity (deterministic or "randomized") of different properties of n-dimensional normed spaces (or n-dimensional convex bodies)? (Of course, "complexity" is not a well defined notion, and, in fact, it is also interesting to develop different forms of understanding of the complexity involved.) An example of a very surprising fact (discovered and developed by Dyer, Frieze, Kannan, Lovasz and Simonovits, see, e.g. [Bol]) is that the volume of an n-dimensional body is computable (by a randomized algorithm) in polynomial time (say, $\sim n^5$ in dimension n). Although it cannot be computed in less than exponential time deterministically. We now know of more examples which show that a slight modification of the question, or the right point of view, changes problems with "exponential" complexity to similar ones with only "polynomial" complexity.

2. Different levels of "randomness". We often observe that "the best choice" (whatever that means in concrete problems) is equivalent to "random" choice. (But not in some other cases.) To give conditions when we should expect such equivalence.
3. Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $K = -K$. What may be said about the distribution of volume for a random projection of K onto a k-dim. space? One may expect it to approach a Gaussian distribution when $n \to \infty$ and k is small enough. I would think that k of order $O(k^*)$, k^* as defined in 2.3.1, should work. However, we don't know much even for $k = 1$. I would like to specially emphasise some interesting partial results obtained recently by K. Ball and his students (see, e.g. [AnBP]).
4. Fix $0 < \lambda < 1$. A generic $k = [\lambda n]$-dim. subspace of an arbitrary n-dim. space X has already some very special properties which do not hold for arbitrary k-dim. spaces. (This phenomenon was first observed in [BoM].) We know very little about these properties. Also, we know that "improvements" which accompany generic k-dim. subspaces and generic k-dim. quotients are different. But we don't know much about either of them.

II Some concrete central problems of the theory

5. Entropy duality problems. (Going back to Carl and Pietch) (see [BoPST]; or the more recent [MiSz] for discussion). On the geometric finite dimensional language the problem is equivalent to the following question:
   Do universal constants a and b exist such that the following is correct: If K and T are convex centrally symmetric compact bodies in $\mathbb{R}^n$; and K^0, T^0 are their polar bodies; then
   $$N(K, aT) \leq N(T^\circ, K^\circ)^b\,?$$
6. Isotropic constant – problem (going back to J. Bourgain).
   Let K be convex centrally symmetric body in $\mathbb{R}^n$, v.r. $K = 1$ (i.e. $|K| = |D|$) and K be in the isotropic position, meaning that for $\forall y, z \in \mathbb{R}^n$
   $$\int_K (x, y)(z, x) \frac{dx}{|K|} = L_K (z, y)\,.$$

Does a universal constant C exist such that $L_K \leq C$ (for any n and $K \subset \mathbb{R}^n$)? (The problem has many connections with different directions in classical convexity and Local Theory, see [MiP1]. Some experts would consider it to be the most attractive open question of Asymptotic Convexity Theory. The best known estimate by Bourgain is $L_k \lesssim n^{1/4}\sqrt{\log n}$). See [D] for a nice presentation of Bourgain's result and other references.

7. Does for every $\epsilon > 0$ a constant $C(\epsilon)$ exist such that for any n and n-dimensional normed space $X = (\mathbb{R}^n, \|\cdot\|)$ there is a quotient space qX of X such that
$$\dim qX \geq n/2 \qquad \text{and} \qquad C_{2+\epsilon}(qX) \leq C(\epsilon) \ ?$$
($C_q(Y)$ is a cotype q constant of the space Y).
See [Mi6] for many connections of this problem with other open problems of Local Theory.

III Problems in the direction of Dvoretzky type theorems

8. It is an old and well known fact (originated in Kashin [K]) that for any $\lambda < 1$ and $k = [\lambda n]$ the space ℓ_1^n contains an isomorphic copy (up to a constant $C(\lambda)$ depending on λ only) of space ℓ_2^k. Moreover, it is shown in [FLM] that, for $k \sim \epsilon^2 n$ such ℓ_2^k-copy may be found almost, up to $1+\epsilon$, isometrically ℓ_2^k and with a very high probability k-dim subspace will be such copy. However, we cannot construct ("present") such ℓ_2^k subspace even for k just larger than $n^{1/2}$.
So, how to construct "explicitly" ("present", whatever our intuition says it means) ℓ_2^k copies in ℓ_1^n for k being proportional to n?
9. What is the best order of estimate in ϵ-version of Dvoretzky theorem? Precisely: Let $N_3(\epsilon)$ be the smallest integer such that any $N_3(\epsilon)$-dim. normed space contains an $(1+\epsilon)$-isometric copy of ℓ_2^3 (3-dim. subspace which is, up to $1+\epsilon$, Euclidean). What is order of growth of this function? Is it polynomial by $1/\epsilon$? Naturally, the same question for any $k \geq 3$ (see [Mi7], for discussion on this problem; the best known estimate is worse then exponential by $1/\epsilon$).
10. In fact, I would think that the worst, or almost worst, embedding of ℓ_2^k copy in a normed space $X = (\mathbb{R}^n, \|\cdot\|)$ should be for X being ℓ_∞^n space. I think, this may be true for any n and $k < n$.
11. Algebraic form of Dvoretzky type theorems (see the problem in [Mi7, Section 1].

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