LINEAR STRUCTURE OF BANACH SPACES; ASYMPTOTIC VIEW

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One of the main directions in Geometric Functional Analysis is the study of the linear structure of an infinite dimensional Banach space. But what do we understand by the words "structure" or "linear structure" of a normed space? There is a "classical" understanding which has developed since the time of Banach: search for some special, nice building blocks of a space; as extreme, say ℓ_p-subspaces or quotient spaces of a given space. Many beautiful results in this direction were proved in the 50's and 60's (see monographs of Lindenstrauss-Tzafriri [LT77] and [LT79], or survey [M70] and M71]).

However, such understanding of a structure for a generic Banach space failed. We have today counter-examples to almost every conjecture on a structure of an abstract Banach space in such classical direction (starting from Tsirelson space [Ts74], and most recently many examples of Gowers-Maurey and Gowers [GM93], [G1]).

In two lectures below, I will describe briefly two different possibilities, two opposite and, in a sense, complemented understanding of "linear structure" of infinite dimensional normed spaces.

The first of these approaches started and was developed much before the recent examples appeared. We changed our thinking and considered a *family* of finite dimensional subspaces (say, *all* finite dimensional subspaces of a given infinite dimensional Banach space) emphasizing an asymptotic behavior when dimension increases to infinity. And then such asymptotic view reveals regularities behind increasing (with increase of dimension) diversity of spaces. We call this Local Theory, or Asymptotic Finite Dimensional Theory. A short discourse on this theory is given in the second lecture.

This is the text of two lectures given at Wuhan International Conference on Banach Spaces (China); April 1994.

But in the first lecture I would like to describe another, very recent approach to *visualizing* some linear structure of infinite dimensional space, some geometric purely infinite dimensional phenomena; and our interest in this lecture is in phenomena which have *no* finite dimensional analogues.

1. Distortion and Asymptotic Infinite Dimensional Theory

The start of this purely infinite dimensional development stems from some facts of Local Theory. The following old observation may be seen today as just the "right" interpretation of the celebrated Dvoretzki theorem.

Let $(X, \|\cdot\|)$, $\dim X = \infty$, be a normed space and $r(x)$ an equivalent norm on X. We look for values $a \in \mathbf{R}$ such that for every $\varepsilon > 0$ and any integer n there exists a subspace E, $\dim E = n$ and for every $x \in E$, $x \neq 0$,

$$a(1-\varepsilon) \leq r(x)/\|x\| \leq a(1+\varepsilon) \ .$$

So, r is an almost isometry on E which is given by product on "a". We denote the collection of such numbers a by $\gamma(r)$ and call it the finite dimensional spectrum of r ("spectrum of isometry").

Fact (see [M69]). The finite dimensional spectrum of any equivalent norm on an infinite dimensional space X is never empty:

$$\gamma(r) \neq \emptyset \ .$$

Then, naturally, the following question arises:

Does a number $a \in \mathbf{R}$ exist such that for any $\varepsilon > 0$ there is an infinite dimensional subspace E such that for every $x \in E \setminus 0$

$$a(1-\varepsilon) \leq r(x)/\|x\| \leq (1+\varepsilon)a \ ?$$

The collection of all such numbers "a" we denote $\gamma_\infty(r)$ and call the infinite-dimensional spectrum of $r(x)$. So, the question is:

(i) Does $\forall\, X$, $\dim X = \infty$, $\forall\, r(x)$, $\gamma_\infty(r) \neq \emptyset$?

or

(ii) Does X exist and an equivalent norm $r(x)$ on X such that $\gamma_\infty(r) = \emptyset$?

If the case (ii) is realized then it is a pure infinite dimensional phenomenon. We see in this example one of the main roles of positive facts of Local Theory in extracting potential purely infinite dimensional phenomena as counter-examples to a natural extension of facts of Local Theory to an infinite dimensional setting.

To continue, we need to develop some terminology:

For a subspace E, consider an oscillation interval $I_r(E) = [a, b]$ where

$$a = \inf \left\{ r(x)/\|x\| \;\middle|\; x \in E \setminus 0 \right\} , \quad b = \sup \left\{ r(x)/\|x\| \;\middle|\; x \in E \setminus 0 \right\} .$$

Obviously, either these exists a sequence of embedded subspaces $\{E_1 \supset \cdots \supset E_n \supset E_{n+1} \supset \cdots\}$ such that $I_r(E_n) = [a_n, b_n]$, $b_n/a_n \to 1$, i.e. $[a_n, b_n] \to \{\alpha\}$, or there exists an interval $I = [\alpha, \beta]$, $\beta > \alpha$, and

(i) $\forall \varepsilon > 0$, $\exists$ a subspace E_ε, $I(E_\varepsilon) \subset [\alpha - \varepsilon, \beta + \varepsilon]$

(ii) for every infinite dimensional subspace $E' \subset E_\varepsilon$, $I(E') \supset (\alpha, \beta)$.

We denote the family of such intervals I by $\widetilde{\gamma}(r)$ and call it the tilda-spectrum of $r(x)$. It is convenient to consider γ_∞ as being part of $\widetilde{\gamma}(r)$ (i.e. an interval-point $I = \{\alpha\} \in \widetilde{\gamma}(r)$).

Then the asymptotic multiplicative oscillation $d(I = [\alpha, \beta] \mid I \in \widetilde{\gamma}(r)) = \beta/\alpha$ and

$$d(r) = \sup \left\{ d(I) \;\middle|\; I \in \widetilde{\gamma}(r) \right\} .$$

So, we have an obvious alternative:

Either $d(r) = 1$ (i.e. Spectrum $\gamma(r) \neq \emptyset$)

or $d(r) > 1$, and we call this case *a distortion.*

We call space X distortable if there is a norm $r(x)$ on X, s.t. $d(r) > 1$, i.e., if

$$d(X) = \sup \left\{ d(r) \;\middle|\; r \text{ is an equivalent norm on } X \right\} > 1 .$$

At the end of the 60s, the following theorem was proved (see [M71]):

Theorem 1. *If $d(x) = 1$ then for any $\varepsilon > 0$, X contains either $(1+\varepsilon)$-isometric copy of ℓ_p (for some $1 \le p < \infty$) or $(1+\varepsilon)$-isometric copy of c_o.*

[In fact, it is not important to consider a family of equivalent norms for such a conclusion, but a family of functions of two variables. $\varphi_\lambda(x, y) = \|x + \lambda y\|$ $(\|x\| = 1 = \|y\|)$, which correspond to "curvature-type" behavior around point x on the unit sphere; and for some spaces, such as L_p, such functions do not create "distortion".]

The next step was the construction of a new type of normed spaces by Tsirelson [Ts74]. The huge, not decreasing influence of this construction on the infinite dimensional Banach space theory we experience till now. Tsirelson built a normed space T which contains neither an isomorphic copy of ℓ_p (for any $1 \le p < \infty$) nor c_0. We define this norm first on c_{00} (i.e. on all finite support vectors of $\mathbf{R}^\infty$). We write $E_i = [n_i, m_i] \subset \mathbf{N}$, $E_i < E_j$ if $m_i < n_j$ and $n < E_1$ meaning $n < n_1$. Also, let $\{e_k\}_1^\infty$ be the natural basis of $\mathbf{R}^\infty$ and Ex denote the projection of x onto "interval $E = [n, m]$", i.e. $E\left(\sum a_i e_i\right) = \sum_n^m a_i e_i$. Then the "$T$-norm" is the solution of the following equation: $\forall x \in c_{00}$

$$\|x\|_T = \max\left\{\|x\|_{c_0}, \tfrac{1}{2}\sup\left(\sum_1^n \|E_i x\|_T \;\middle|\; n \in \mathbf{N}\ ,\ E_i \text{ are any intervals s.t. } n < E_1 < \cdots < E_n\right\}\right. . \tag{1.1}$$

Then T is a Banach space receiving by the completion of $(c_{00}, \|\cdot\|_T)$.

I consider this to be the first "non-classical" normed space where the norm is defined by "an equation", and not by a "formula". This construction was immediately investigated by [FJ74] and many consequent papers (see [CS89]). However, at this stage, the following consequence from Theorem 1 was not investigated: T is distortable, $d(T) > 1$ (!) Note that no "size" of distortion follows from Theorem 1. Only after 1988, answering a question of H. Rosenthal, T. Odell first checked that $d(T) \ge 2$ (and what the real value of $d(T)$ is still an open question). Then Odell (unpublished) substituted $\frac{1}{2}$ by $\frac{1}{\lambda}$ in the definition (1.1) of the space T (let us call such spaces T_λ) and he showed that $d(T_\lambda) \ge \lambda$.

So, we have learnt that there are spaces with (large) distortion.

To continue, let us give a geometric interpretation of the phenomenon of distortion. Let $d(r) > 1$ and $I = [\alpha, \beta] \in \widetilde{\gamma}(r)$, $\alpha < \beta$. Let Y be a subspace such that for every infinite dimensional subspace $E \hookrightarrow Y$, $I_r(E) \supset (\alpha, \beta)$. Fix $0 < \varepsilon < \frac{\beta - \alpha}{2}$. Define two subsets of the sphere of Y:

$$A = \left\{x, \|x\| = 1 \;\middle|\; r(x) \le \alpha + \varepsilon\right\}, \quad B = \left\{x \in S(Y) \;\middle|\; r(x) \ge \beta - \varepsilon\right\}.$$

Then $\forall\, E \hookrightarrow Y$, $\dim E = \infty$, $A \cap E \neq \emptyset$ and $B \cap E \neq \emptyset$. We call such sets *asymptotic* sets on $S(Y)$.

So the geometric interpretation of distortion is the existence of two asymptotic sets A and B on the sphere of some Banach space such that the distance $d(A,B) > 0$. Note, that a similar phenomenon cannot hold for any (large) integer N: if, for any Banach space X, A and B are such sets on $S(X)$ that for any N-dimensional subspace E of X, $A \cap E \neq \emptyset$ and $B \cap E \neq \emptyset$, then the distance $d(A,B) = 0$.

Let us return to the study of distortion. Schlumprecht [S1] put $\log n$ instead of λ in (1.1) (it is enough then to consider $1 < E_1 < \cdots < E_n$) and constructed the space S such that $d(S) = \infty$. Moreover, this space has a sequence of asymptotic sets $\{A_i\}_{i=1}^{\infty}$ on its sphere such that distance $d(A_i, A_j) \geq 1$ for any $i \neq j$.

[The existence of such a sequence of asymptotic sets was the starting point of all the following constructions of spaces by Gowers and Maurey.]

Till now, we have discussed a construction of special spaces which are distortable. But what about classical spaces? It is easy to see (James [J64]) that $d(\ell_1) = 1$ as well as $d(c_0) = 1$, i.e. ℓ_1 and c_0 are not distortable. However, the similar question on ℓ_p spaces for $1 < p < \infty$ was open from 1969 till the recent and very novel work of Odell and Schlumprecht.

Theorem 2 ([OSch93]). *For any p, $1 < p < \infty$, $d(\ell_p) = \infty$. Moreover, there are asymptotic sets $A_i \subset S(\ell_2)$ such that, for $i \neq j$ and $\forall\, x \in A_i$, $\forall\, y \in A_j$*

$$\big|(x,y)\big| < \frac{1}{\min(i,j)} \ ,$$

where (x,y) is the inner product in ℓ_2. Of course, the similar statement is true for every ℓ_p, $1 < p < \infty$. [We call this a "biorthogonal distortion"].

This is a new and purely infinite-dimensional geometry of Hilbert spaces!

Note that B. Maurey [Ma1], using Theorem 2, constructed a distorted norm $r(x)$ on ℓ_2 which is a symmetric norm on ℓ_2.

Combining Theorems 1 and 2, we see

Corollary. *Every Banach space X which does not contain isomorphic copies of ℓ_1 and c_0 has a distortion norm: $d(X) > 1$ (so, $d(X) = 1$ implies X contains either an isomorphic copy of ℓ_1 or an isomorphic copy of c_0).*

However, how large can this distortion be? Can spaces X exist such that $1 < d(X) < \infty$? We studied these questions with N. Tomczak-Jaegermann [MTJ93]. To formulate the corresponding result, we have to introduce a new type of spaces and a different view: classes of Banach spaces are defined by their asymptotic properties. We call space X with a basis $\{e_i\}_1^\infty$ a (stabilized) asymptotic ℓ_p-space if there is a constant C such that for any integer n, any n consecutive blocks $\{x_i\}_1^n$ such that $\|x_i\| = 1$ and $n < \operatorname{supp} x_1 < \operatorname{supp} x_2 < \cdots < \operatorname{supp} x_n$ are C-equivalent to the natural basis of ℓ_p^n:

$$\frac{1}{\sqrt{C}}\Big(\sum_1^n |a_i|^p\Big)^{1/p} \le \Big\|\sum_1^n a_i x_i\Big\| \le \sqrt{C}\Big(\sum_1^n |a_i|^p\Big)^{1/p} .$$

The above Tsirelson space is an example of a stabilized asymptotic ℓ_1-space which is not isomorphic to ℓ_1 and moreover it does not contain an isomorphic copy of ℓ_1. Similar Tsirelson spaces may be constructed for every $1 \le p < \infty$ and c_0: stabilized asymptotic ℓ_p-spaces which do not contain any isomorphic copies of ℓ_p.

Theorem 3 [MTJ93]**.** *IF space X has a bounded distortion, $d(X) \le d$, then there is p, $1 \le p \le \infty$, and a subspace $E \hookrightarrow X$ which is a stabilized asymptotic ℓ_p-space for constant $C \sim d^2$.*

B. Maurey [Ma2] used this fact to prove that if X has an unconditional basis and type $p > 1$ then $d(X) = \infty$.

Asymptotic infinite-dimensional structure. To continue the line of [MTJ93] and to classify Banach spaces by their asymptotic properties we introduce in [MMT] the notion of *asymptotic structure* of space X.

The main idea behind it is a stabilization at infinity of finite dimensional subspaces which appear everywhere far away. This further leads to an infinite-dimensional construction resulting in a notion of an *asymptotic version* of X.

Let us describe the intuition of an asymptotic structure of an infinite dimensional Banach space X. Such a structure is defined by a family $\mathcal{B}(X)$ of infinite dimensional subspaces of X satisfying a filtration condition which says that for any two subspaces from $\mathcal{B}(X)$ there is a third subspace from $\mathcal{B}(X)$ contained in both of them; the main example is the family $\mathcal{B}^0(X)$ of all subspaces of finite codimension in X. Then, for every k, we define the family $\{X\}_k$ of asymptotic k-dimensional spaces associated to this asymptotic structure as follows (exact definitions are better given through the "game" approach introduced for similar purposes by Gowers [G2] – see [MMT]).

Fix k and $\varepsilon > 0$. Consider a "large enough" number N_1, a "far enough" subspace E_1 of $\operatorname{codim} E_1 = N_1$, and an arbitrary vector $x_1 \in S(E_1)$. Next consider a number $N_2 = N_2(x_1)$, depending on x_1 and again "large enough", a "far enough" subspace $E_2 \subset E_1$ of codimension $N_2(x_1)$ and an arbitrary vector $x_2 \in S(E_2)$. In the last kth step, we have already chosen normalized vectors $x_1, \dots, x_{k-1}$ and subspaces $E_{k-1} \subset \cdots \subset E_2 \subset E_1$; we then choose a "far enough" $E_k \subset E_{k-1}$ with $\operatorname{codim} E_k = N_k(x_1, \dots, x_{k-1})$ and an arbitrary vector $x_k \in S(E_k)$.

We call a space $E = \operatorname{span}[x_1, \dots, x_k]$ a permissible subspace (up to $\varepsilon > 0$) and $\{x_i\}_1^k$—a permissible k-tuple if for an *arbitrary* choice of N_i and E_i (with $\operatorname{codim} E_i = N_i$) we would be able to choose normalized vectors $\{y_i \in E_i\}$ so that a basic sequence $\{y_i\}_1^k$ is $(1+\varepsilon)$-equivalent to $\{x_i\}_1^k$.

Now we can also clarify the imprecise notion of "far enough" subspaces E_i: by this we mean that an *arbitrary* choice as above of $x_i \in E_i$ results in a permissible (up to $\varepsilon > 0$) k-tuple $\{x_i\}_1^k$ and a permissible (up to $\varepsilon > 0$) subspace $E = \operatorname{span}[x_i, \dots, x_k]$. The existence of such subspaces "far enough" and of associated N_i's, is proved by a compactness argument.

If $F(k;\varepsilon)$ is the set of all k-dimensional ε-permissible subspaces then we put $\{X\}_k = \bigcap_{\varepsilon>0} F(k;\varepsilon)$, and we call every space from $\{X\}_k$ a k-dimensional asymptotic space of X. Thus , permissible subspaces are $(1+\varepsilon)$-realizations of asymptotic spaces.

Finally, a Banach space Y is an asymptotic version of X, if Y has a monotone basis $\{y_i\}_1^\infty$ and for every n, $\{y_i\}_1^n$ is a basis in an asymptotic space of X i.e. $\operatorname{span}[y_i]_1^n \in \{X\}_n$.

Families of asymptotic spaces and asymptotic versions of a given Banach space have interesting properties and reveal a new structure of the original space. For example, we proved in [MMT] that for a fixed p, with $1 \le p < \infty$, if X is a Banach space such that there exists C such that for every n, every space $E \in \{X\}_n$ is C-isomorphic to ℓ_p^n, then every asymptotic version Y of X is isomorphic to ℓ_p and the natural basis of Y is equivalent to the natural basis of ℓ_p. It means that in such a space (called an *asymptotic l_p-space*) all permissible subspaces lie only along its natural l_p basis.

Some properties of families of asymptotic spaces $\{X\}_n$ can be demonstrated through the notion of envelopes. For any sequence with finite support $a \in c_{00}$ the *upper envelope* is a function $r(a) = \sup \|\sum_i a_i e_i\|$, where the supremum is taken over all natural bases $\{e_i\}$ of asymptotic spaces $E \in \{X\}_n$ and all n. Similarly, the *lower envelope* is a function $g(a) = \inf \|\sum_i a_i e_i\|$, where the infimum is taken over the same set. Remarkably, the functions r and g are always very close to some l_p- (and l_q)-norms. An interesting general property of asymptotic versions is that some of them are, in a sense, stable under iteration. Precisely, for an arbitrary space X there is a special asymptotic version Y, called *universal*, such that its asymptotic structure is the same as for X. In particular this implies that not every space X, even with an unconditional basis, can be a universal asymptotic version of any Banach space. We direct the reader to [MMT] to receive more precise information on this subject.

2. Local Theory/Asymptotic view of high dimensional spaces in Functional Analysis and Convex Geometry

In this lecture we discuss results in Local Theory which stand between Geometry and Functional Analysis. The theory was built during the last two decades. It considers geometric problems via a Functional Analysis point of view. Consequently, typical for geometry "isometric" problems and view are substituted by "isomorphic" ones. This became possible with the *asymptotic* approach (with respect to dimension increasing to infinity) to the study of high dimensional spaces. My goal in this lecture is to demonstrate, with some examples, a new intuition which corresponds to high dimensional spaces and present a few results to support our point of view. We recommend the following books: [MSch86], [P86], [P89], [TJ89] and surveys [M88a], [M92],[L92], [LM93]. Consider a finite dimensional normed space $X = (\mathbf{R}^n, \|\cdot\|)$. Such a space is defined by its unit ball $K_X = \{x \in \mathbf{R}^n, \|x\| \leq 1\}$. Inversely, if K is a convex centrally-symmetric body in $\mathbf{R}^n$, then $X_K = (\mathbf{R}^n, \|\cdot\|_K)$ is the normed space with the unit ball K. Let $|x|$ be the canonical euclidean norm in $\mathbf{R}^n$, (x, y) be the standard inner product and $\mathcal{D}$ denote the standard euclidean ball, i.e. $\mathcal{D} = K_{(\mathbf{R}^n, |\cdot|)}$.

A few examples.

We start with two observations:

a) [MP89] Fix $0 < \delta < \frac{1}{2}$. Define the floating body K_δ of K as the intersection of the half spaces $\{x \in K \mid (x, \theta) \le m_\delta(\theta)\}$, $\theta \in \mathbf{R}^n$, where $m_\delta(\theta)$ is defined by

$$\mathrm{Vol}\left\{x \in K \;\middle|\; (x, \theta) > m_\delta(\theta)\right\} = \delta \,\mathrm{Vol}\, K \ .$$

Then, *there is a number $C(\delta)$, independent of dimension n or $K \subset \mathbf{R}^n$, such that for any symmetric convex body K the floating body K_δ is uniformly, up to a factor $C(\delta)$, isomorphic to an ellipsoid; this means that there is an ellipsoid $\mathcal{E}$, s.t.*

$$\mathcal{E} \subset K_\delta \subset C(\delta)\mathcal{E} \ .$$

[Note that the initial body K could be very far from any ellipsoid; it could be, say, a crosspolytope (= the unit ball of ℓ_1^n) or a cube, but described above "regularization" by cutting a fixed portion of volume in any direction brings us to a $C(\delta)$-neighborhood of an ellipsoid.]

Moreover, this ellipsoid is homothetic to the Legendre ellipsoid of inertia of K.

Let us formally introduce a multiplicative geometric distance $d(\cdot, \cdot)$ between convex bodies K and T: $d(K, T) = \inf\{b/a \mid K \subset bT \text{ and } aT \subset K\}$. The Banach-Mazur distance between two normed spaces X_K and X_T is $d(X_K, X_T) = \inf\{d(K, uT) \mid u \in GL_n\}$.

b) *Lattice tiling.* Let the inner part $\overset{\circ}{K} \neq \emptyset$. Denote $K_i = K + x_i$ for $x_i \in \mathbf{R}^n$. We call $\{K_i\}$ a tiling of $\mathbf{R}^n$ (and say that K produces a tiling by shifts) if (i) $\bigcup_i K_i = \mathbf{R}^n$ and (ii) $\overset{\circ}{K}_i \cap \overset{\circ}{K}_i = \emptyset$ for $i \neq j$. A lattice tiling is a tiling such that the set $\{x_i\}$ is a lattice, i.e. $\{x_i\} = A\mathbf{Z}^n$, where A is an invertible linear map and $\mathbf{Z}^n \subset \mathbf{R}^n$ is the set of all integer vectors of $\mathbf{R}^n$.

Trivially, an affine image of the cube $[-1, 1]^n$ produces a lattice tiling and the euclidean ball does not. However, do uniformly isomorphic versions K_n of the euclidean balls $\mathcal{D}_n$

exist which produce a tiling? Surprisingly, the answer is "Yes". Exactly, *for any integer n there is a convex symmetric body $K_n \subset \mathbf{R}^n$, $\mathcal{D}_n \subset K_n \subset 3 \cdot \mathcal{D}_n$, such that K_n produces a lattice tiling of $\mathbf{R}^n$.* (This observation of Alon-Milman follows immediately from some known results on lattice covering-packing; see [M92] for details and references.)

The above two examples lead us to a notion of "isomorphic ellipsoid": a family of convex bodies $\{K_\alpha\}$ represents an "isomorphic ellipsoid" if there is a constant C and a family of ellipsoids $\{\mathcal{E}_\alpha\}$ such that $\mathcal{E}_\alpha \subset K_\alpha \subset C\mathcal{E}_\alpha$ for every K_α in the family, i.e. $d(K_\alpha, \mathcal{E}_\alpha) \le C$. So, in example a) we can state that for a fixed δ, $0 < \delta < \frac{1}{2}$, the family of δ-floating bodies $\{K_\delta \mid \forall n \; \forall K \subset \mathbf{R}^n\}$ is an isomorphic ellipsoid.

Our next example is already a non-trivial theorem originally proved in 1988 (see [M92]).

Theorem 1a. *There is a constant C such that for any integer n, any $K \subset \mathbf{R}^n$ (centrally symmetric, convex body) two operators $u_, u_2 \in GL_n$ exist such that for some ellipsoid $\mathcal{E}$, $d(\mathcal{E}, Q) \le C$ where $Q = \mathrm{conv}(P \cup u_2 P)$ and $P = K \cap u_1 K$. So, the corresponding family $\{Q\}$ is an isomorphic ellipsoid.*

In the language of Functional Analysis, the same fact can be reformulated in the following form:

Theorem 1b. *For every finite dimensional normed space $X = (\mathbf{R}^n, \|\cdot\|)$ there are three linear operators $T_1, T_2, T_2 \subset GL_n$, such that the following relation holds:*

Consider $p(x) = \|T_1 x\| + \|T_2 x\|$ and a convolution

$$q(x) = p(x) * p(T_3 x) \quad [\text{by definition } p(x) * p(ux) \equiv \inf_{y+z=x} \{p(y) + p(uz)\}] \; ;$$

then $q(x)$ is C-isomorphic to the standard euclidean norm in $\mathbf{R}^n$:

$$|x| \le q(x) \le C|x| \; .$$

Note, C does not depend on the dimension n or on the initial norm $\|\cdot\|$.

Geometric inequalities.

We are going to outline a few geometric ideas and concepts behind the technique of Asymptotic Theory. Because my only goal in this lecture is to give a feeling of a new intuition of high-dimensional normed spaces, I am leaving aside some crucial concepts and theorems, such as notions of 'type' and 'cotype', ideal operator norms, factorizations and distances between spaces.

Isomorphic isoperimetric inequalities/concentration phenomenon.

We start with the simplest example. Let $S^{n-1} \subset \mathbf{R}^n$ be the standard euclidean sphere (i.e. $S^{n-1} = \partial \mathcal{D}_n$), ρ – the geodesic distance on S^{n-1}, μ – rotation invariant probability measure on S^{n-1}. Consider a closed set $A \subset S^{n-1}$, $\mu(A) \geq \frac{1}{2}$, and denote $A_\varepsilon = \{x \in S^{n-1} \mid \rho(x, A) \leq \varepsilon\}$. Then the classical isoperimetric inequality on S^{n-1} (proved first for the application described here by P. Levy in 1919) implies: $\mu(A_\varepsilon) \geq \mu(\varepsilon$ – extension of a semisphere$) \geq 1 - \sqrt{\frac{\pi}{8}} e^{-\varepsilon^2 n/2}$ (see [MSch86] for details). So, for n large, whatever remains from the sphere after taking ε-neighborhood of (any) set of half-measure has an exponentially (by $n \nearrow$) small measure. We apply this to a study of $1 -$ Lip functions $f(x)$ (i.e. $|f(x) - f(y)| \leq \rho(x, y)$). Denote L_f being the median of $f(x)$, i.e. $\mu\{x \in S^{n-1} \mid f(x) \geq L_f\} \geq \frac{1}{2}$ and $\mu\{x \in S^{n-1} \mid f(x) \leq L_f\} \geq \frac{1}{2}$. Then

$$\mu\big\{x \in S^{n-1} \bigm| |f(x) - L_f| < \varepsilon\big\} \geq 1 - \sqrt{\frac{\pi}{2}} e^{-\varepsilon^2 n/2} \ .$$

So, if dimension n is large then values of Lipschitz function "concentrate" by a measure around one value. It is, in fact, a general property of high dimensional metric probability spaces which we call "concentration phenomenon". Many different technique were developed to treat different examples of metric probability spaces and to prove concentration phenomenon for them (see survey [M88a]). Let us describe just one more example considered by B. Maurey [Ma79].

Let Π_n be the group of permutations of $[1, \ldots, n]$ equipped with the counting probability measure $\mu(A) = \frac{\#A}{n!}$ (for any $A \subset \Pi_n$) and Hamming distance $\rho(s,t) = \frac{\#\{i \in [1,\ldots,n] \mid s(i) \neq t(i)\}}{n}$ for any two permutations $s, t \subset \Pi_n$. Then $\mu(A_\varepsilon) \geq 1 - c_1 e^{-c_2 \varepsilon^2 n}$ for absolute constants c_1 and c_2 where $\mu(A \subset \Pi_n) \geq \frac{1}{2}$ and $A_\varepsilon = \{t \in \Pi_n \mid \rho(t, A) \leq \varepsilon\}$. Again, we have an inequality of "isoperimetric type" but in a new, isomorphic vision, which is enough for what we call "concentration phenomenon".

This phenomenon is responsible for many "unexpected", "strange" properties of high dimensional spaces. Behind intuitively expected properties of high dimensional spaces stands the behavior of ε-entropy which increases exponentially with increase in dimension. However, this exponential increase is compensated by also exponential effect of concentration phenomenon. As a result, we often observe only linear behavior where a priori intuition expects an exponential. Some Examples:

Approximation by Minkowski sums.

Let $A+B = \{x+y \mid x \in A, y \in B\}$ be the Minkowski sum of two sets A and B in $\mathbf{R}^n$. Let $I_i = [-x_i, x_i] \subset \mathbf{R}^n$ be intervals of length, say, 1. Consider $T = \sum_{i=1}^{N} I_i$. We want to approximate a Euclidean ball by such sums, that is, for a given $\varepsilon > 0$ we would like to have $d(T, D) \leq 1+\varepsilon$. Obviously, if $N = n$ then $d(T, D) \geq \sqrt{n}$ and, by an entropy consideration, it looks as if we need at least an exponential by n of a number of intervals to achieve a good approximation to D. However, as was observed in [M86], an easy geometric interpretation of an old result from [FLM76] shows that there exist intervals $I_i \subset \mathbf{R}^n$, $i = 1, \ldots, N_0$, for $N_0 \leq c\frac{n}{\varepsilon^2} \log \frac{1}{\varepsilon}$ (c is a numerical constant) such that $d\big(\sum_1^{N_0} I_i, D\big) \leq 1 + \varepsilon$.

This direction was recently treated intensively in [BLM89] and [BLM88]. It is shown there that the above situation is essentially preserved when we substitute intervals by other convex bodies or if we consider approximation by sums of other convex bodies instead of D. For example

Theorem 2 [BLM88]. *Let a convex compact body $K \subset \mathbf{R}^n$ be given. There exist orthogonal operators $A_i \in SO_n$, $i = 1, \ldots, N_0$ for $N_0 < c\frac{n}{\varepsilon^2} \log \frac{1}{\varepsilon}$ such that $T = \frac{1}{N_0} \sum_1^{N_0} A_i K$ satisfies $d(T, D) \leq 1 + \varepsilon$ (as usual, c is a numerical constant).*

The family of positions. One of the best known and important geometric inequalities is the Brunn-Minkowski inequality which states that for any two measurable sets A and B in $\mathbf{R}^n$

$$\big[\,\mathrm{Vol}(A+B)\big]^{1/n} \geq (\mathrm{Vol}\, A)^{1/n} + (\mathrm{Vol}\, B)^{1/n}\ . \qquad (*)$$

The isoperimetric inequality in $\mathbf{R}^n$ is the direct consequence of $(*)$. We will discuss below only the case of convex bodies A and B. In what sense can we hope to satisfy the inverse to $(*)$? Does a constant C exist such that

$$\big(\mathrm{Vol}(A+B)\big)^{1/n} \le C\big[(\mathrm{Vol}\,A)^{1/n} + (\mathrm{Vol}\,B)^{1/n}\big] \ ?$$

Trivial examples show that such a constant C does not exist. However, for every $K \subset \mathbf{R}^n$ there exists a linear transform $u_K \in SL_n$ such that for $\hat{K} = u_K K$ the situation is different.

We call *a position* of K any affine image uK for $u \in GL_n$. Clearly, every position of K produces the unit ball of isometrically the same normed space as X_K. It is an interesting feature of the(asymptotic) high dimensional theory of convex sets that we are, in fact, forced to consider the family of all positions of a given K (that is, all affine images uK, $u \in GL_n$) even when we are aiming at some volume inequalities or other properties of an individual K.

Theorem 3 (see [M88b]). *There is a constant C such that for any integer n and any convex bodies K and T in $\mathbf{R}^n$ there are volume preserving positions $\hat{K} = uK$ and $\hat{T} = vT$ for $u, v \in SL_n$ and*

$$\mathrm{Vol}(\hat{K} + \hat{T})^{1/n} \le C\big[(\mathrm{Vol}\,K)^{1/n} + (\mathrm{Vol}\,T)^{1/n}\big] \ .$$

The main step in the proof of this isomorphic inequality is the existence of a special ellipsoid (M_K-ellipsoid) associated to a convex body K and such that from the point of view of *volume radius*, any K behaves as a suitable ellipsoid. Precisely:

For any convex compact body K there exists an ellipsoid $\mathcal{M}_K$ such that $\mathrm{vol}\,.K = \mathrm{vol}\,.\mathcal{M}_K$ *and for any other convex body T*

$$\frac{1}{C^n}\,\mathrm{vol}\,.(\mathcal{M}_K + T) \le \mathrm{vol}\,.(K+T) \le C^n\,\mathrm{vol}\,.(\mathcal{M}_K + T)$$

(C is a universal constant).

In the above fact the family of ellipsoids does not play a special role. We could take any fixed P and replace the ellipsoid $\mathcal{M}_K$ by some affine image of P.

Quotient of subspace theorem ([M85]). In the heart of the above global results stand methods of Functional Analysis which were developed to understand the structure of subspaces of a given space. Dvoretzky's theorem was initial result in this direction. But it had to develop a long way before this structure was understood so well that it could be used for the study of global properties of space. One of the crucial links between linear structure (the so called "local properties") and global structure is the following theorem:

Theorem 4. *Fix $0 < \lambda < 1$. Every finite dimensional normed space $X = (\mathbf{R}^n, \|\cdot\|)$ contains a subspace sX and a quotient $q(sX) = qsX$ of the subspace sX such that*

$$i)\ k = \dim(qsX) \geq \lambda n \text{ and } ii)\ d(qsX, \ell_2^k) \leq C(\lambda) \sim \frac{1}{1-\lambda}\log\frac{1}{1-\lambda}.$$

We finish this lecture with a few remarks on polarity. In the above theorem we use two "polar" operations: taking subspace and quotient. And this brought an improvement from the logarithmic estimate on euclidean section in Dvoretzky theorem to (any) proportion λn of original dimension. Similarly, in Theorem 1a (or 1b) the dual pair of operations (intersection and convex hull of the union; or sum of norms and convolution of norms) gave, by one step, regularization of the initial norm to a euclidean one. I still consider this role of duality to be mystical. I direct the reader to [M91] for more facts and observations in this direction.

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