Phenomena Arising from High Dimensionality

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1 Introduction

In this article I will attempt to describe the main principles and phenomena which govern the asymptotic behavior of high dimensional convex bodies and normed spaces. I presume that the same principles are applicable to any high parametric system in, say, asymptotic combinatorics or complexity. A few examples for such an analogy are already known, and I will mention them. Of course, this account is still in a very preliminary form based on our accumulated knowledge and will be corrected and adjusted in the future. But, I feel that some order should already be made of the mosaic of facts and results.

A number of surveys on this subject have recently been published (see [GM01], [GM03], [Mi00]), and I will try to avoid repetition by just referring to them. However, in some cases, I will not be able to avoid illustrating some principles with at least one or two examples, but I will be very brief. A number of books were written at the end of the 80s; [MiS86], [P89], [T89]. They present very well the state of the theory at that time, and I recommend them for in-depth reading.

Just as an appetizer, let us show a couple of very recent results where some surprises await us pointing to an interesting subject we should expect.

2 Symmetrizations

For some classical symmetrizations, let us check how many elementary steps are needed to approximate an ellipsoid starting with an arbitrary convex body $K \subset \mathbb{R}^n$. We will analyze two symmetrization procedures: Steiner symmetrization (see, e.g. [KlM03] for a definition) and Minkowski symmetrization, which is also called Blaschke symmetrization. In both cases,

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each elementary step consists of selecting a hyperplane h in $\mathbb{R}^n$. We will now define the lesser known Minkowski symmetrization. Consider the reflection $r_h K$ of the body K with respect to h, then the elementary step of this symmetrization is $S_h K := \frac{K + r_h K}{2}$, where by "+" we mean the Minkowski sum of sets. The elementary step of the Steiner symmetrization, associated with a hyperplane h, is denoted by $St_h K$.

Fix some $c > 1$. What is the smallest N s.t. for $\forall K \subset \mathbb{R}^n$ we may find $\{h_i\}_1^N$ such that there is an ellipsoid $\mathcal{E}$ and

$$\mathcal{E} \subset \prod_{h=1}^{N} St_{h_i} K := K_N \subset c \cdot \mathcal{E}\,? \tag{1}$$

Here is a brief history of this question:

Hadwiger (~ 55) estimated N from above by $\sim (c_1 \cdot n)^{n/2}$.

Bourgain–Lindenstrauss–Milman ($\sim$1988) made a dramatic improvement to $N \lesssim c_1 n \log n$ (for some $c \sim 3$). However, one does not even need a "log" factor and the answer is $N \leq \frac{3}{2} n$ (Klartag–Milman [KlM03]).

Actually, for every $\varepsilon > 0$ there is a constant $c(\varepsilon)$, depending only on ε, instead of c in (1) s.t.

$$N \leq (1+\varepsilon) n$$

(and, for some K, at least $n - c_1 \log n$ is necessary). If $\mathcal{D}$ is the standard euclidean ball then additional $n-1$ symmetrizations turns $\mathcal{E}$ into $r.\mathcal{D}$ and altogether $N \leq (2+\varepsilon)n$ steps are enough to approximate the euclidean ball. (Note, the answer has an isomorphic flavor. We did not approximate the euclidean ball "almost isometrically", but up to some constant, independent of dimension and the body K. Such isomorphic results in geometry actually have meaning as *asymptotic* results which are interesting for large dimensions.)

In the case of Minkowski symmetrizations the best known result on the same question, until 1987, was by [BLM88]: for any $\varepsilon > 0$ there is a $c(\varepsilon)$ such that one may find $\{h_i\}_1^N$ for $N \leq \frac{1}{2} n \log n + c(\varepsilon) n$ and (for some $r > 0$)

$$r\mathcal{D} \subset \prod_{i=1}^{N} S_{h_i} K \subset (1+\varepsilon) r \mathcal{D} \tag{2}$$

(and a random selection of h_i leads to (2) with high probability). Klartag later showed [Kl00] that, for some K and random selection of hyperplanes, this answer is precise (up to a constant factor), and gave an "isomorphic formula" for N for any body K.

However, the best selection of hyperplanes happened to be different and Klartag showed [Kl02] that the smallest N is always $N \leq 5n$ (!); this is true for $\varepsilon \sim c\sqrt{\frac{\log \log n}{\log n}}$. So, in this case, we observe an isomorphic answer which turns out to be "asymptotically isometric" for large dimension n.

3 Unified behavior, phase transitions, threshold, and other signs of high-dimensionality

We continue to accumulate examples.

3a Phase transition type behavior: "local" example.

Consider a subset $K \subset \mathbb{R}^n$. Let $|\cdot|$ be a euclidean norm in $\mathbb{R}^n$. D denotes the unit euclidean ball, $D(E) = D \cap E$, for a subspace $E \hookrightarrow \mathbb{R}^n$. Also, let $d(K)$ be the diameter of K, $P_E K$ the orthoprojection of K onto subspace E. Let $\mathcal{D}_\ell(K) := \mathbb{E}(d(P_E K) \mid \dim E = \ell)$. (So $\mathcal{D}_\ell(K)$ is the average diameter of orthoprojections with respect to Haar measure on the Grassmannian $G_{n,\ell}$.) We will also need the mean width of K,

$$\mathcal{D}_1(K) := w(K) = \int_{S^{n-1}} w(K,u) d\sigma(u)\,,$$

where $w(K,u)$ is the width in the direction $u \in S^{n-1}$,

$$w(K,u) = \sup\left\{(u,x) \mid x \in K\right\} - \inf\left\{(u,x) \mid x \in K\right\}.$$

Let

$$k^* = n\left(\frac{w(K)}{d(K)}\right)^2 .$$

Then *there is a* $c > 0$ *and* C *s.t. for any* n *and any set* $K \subset \mathbb{R}^n$ *for* $k^* \le \ell \le n$,

$$c\sqrt{\frac{\ell}{n}}\; d(K) \le \mathcal{D}_\ell(K) \le C\sqrt{\frac{\ell}{n}}\; d(K)$$

and

$$cw(K) \le \mathcal{D}_\ell(K) \le Cw(K)$$

for $1 \le \ell \le k^*$. We observe the stabilization of the function $\mathcal{D}_\ell(K)$ at the critical value k^*, and a unified form of behavior for the whole family of sets in high dimensional spaces (see [Mi00] for references).

Additional information:

The *only* reason for stabilization for small ℓ is that a "random" projection $P_E K$ for $\dim E \lesssim \varepsilon^2 k^*$ is an ε-net of a euclidean ball (of radius $w(K)/2$). If K is convex, $K = -K$, and $\|x\|_K$ is the norm with unit ball K, then, for a random subspace E of dimension $\lesssim \varepsilon^2 k^*$, the set $P_E K$ itself is an almost euclidean ball,

$$(1-\varepsilon)\frac{w(K)}{2}\mathcal{D}(E) \subseteq P_E K \subseteq (1+\varepsilon)\frac{w(K)}{2}\mathcal{D}(E)\,.$$

In this case $w(K,u) = 2(\|u\|_K)^* := 2\|u\|_{K^\circ}$, where K° is the polar of K and $w(K) := 2M^* = 2\int_{S^{n-1}} \|x\|_{K^\circ} d\sigma(x)$, where $\sigma(x)$ is the probability rotation invariant measure in S^{n-1}.

3b Corresponding global problems.

We would now like to approximate the euclidean ball $r\mathcal{D}$ by averaging rotations of K: $K_t = \frac{1}{t}\sum_{i=1}^{t} u_i K$, $u_i \in O(n)$. We will achieve this by studying decay of the diameter.

We present the results in the language of Functional Analysis.

Let $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$, $\quad b := \|Id : \ell_2^n \to X\|$, $\quad$ and $\quad M = \int_{S^{n-1}} \|x\| d\sigma(x)$. Consider a new averaging norm

$$|||x|||_t = \frac{1}{t}\sum_{1}^{t} \|u_i x\|, \quad u_i \in O(n)$$

and the space $X_t = (\mathbb{R}^n, |||\cdot|||_t, |\cdot|)$. Below, we use the notation $a \sim b$ when two universal constants exist, say $c > 0$ and $C < 0$, such that $cb \leq a \leq C.b$. Then, by [MiS97], with a very high probability (by selection u_i)

$$\|Id : \ell_2^n \longrightarrow X_t\| \sim \frac{1}{\sqrt{t}} \|Id : \ell_2^n \to X\|$$

for $t < (b/M)^2 = (\sup\limits_{|x|=1} \|x\| / \mathbb{E}_{x \in S^{n-1}} \|x\|)^2 := t_0$ and $|||x|||_t \sim M \cdot |x|$ for $t \gtrsim t_0$.

(To connect geometric language of sum of sets and analytic language of sum of norms, observe that $\|x\|^*_{K_1+K_2} = \|x\|^*_{K_1} + \|x\|^*_{K_2}$ where $\|x\|_T$ denotes the norm with the unit ball T, i.e., the Minkowski functional of the convex set T.)

Again, we see phase transition at t_0, stabilization after this critical value, and the reason for stabilization: the averaging norm around the value t_0, with high probability, becomes euclidean.

First asymptotic phenomena:

These examples point to the following phenomena accompanying high dimensional processes:

(a) *"isomorphic" phase transition.* I emphasize that it is an isomorphic analogue of what is well known in Statistical Physics in the exact "isometric" form; our statements are true up to some universal factors, but they are applicable to any (convex) sets, without any special symmetries or structure. Also, until now, we observed, in some sense, "elementary" transitions: only one point of phase transition with constant behavior on one side.

(b) *The constant behavior* (at the "end" of a process) *corresponds to stabilization for one reason (only): "maximal" symmetry is achieved.*

(c) In addition, *in the "global" process*, i.e. when we study changes in the whole space, not in subspaces or projections, *"the best" possibility*, we are looking for, *coincides approximately with a "random" selection.* (However, there is an amount of freedom in identifying the right notion of randomness which may change from problem to problem.)

3c More complicated phase transitions.

The next example of phase transition has more complicated behavior, with "constant" behavior of a different nature (from Litvak–Milman–Schechtman [LiMS98]).

Again, let $X = (\mathbb{R}^n, \|\cdot\|, |\cdot|)$. Let, for $q \geq 1$,

$$M_q = \left(\int_{S^{n-1}} \|x\|^q d\sigma(x) \right)^{1/q} ,$$

and consider the L_q-average of rotated norms

$$|||x|||_{q;t} = \left(\frac{1}{t} \sum_1^t \|u_i x\|^q \right)^{1/q} , \quad u_i \in O(n) \, .$$

(our notation ignores dependence on operators $\{u_i\}$). Then

$$\begin{cases} M_q \sim M_1 \quad \text{for} \quad 1 \leq q \leq k(X) = n\left(\frac{M_1}{b}\right)^2 \\ M_q \sim b\sqrt{\frac{q}{n}} \quad \text{for} \quad k(X) \leq q \leq n \\ M_q \sim b \quad \text{for} \quad q > n \, . \end{cases} \tag{M_q}$$

So, we see two (isomorphic) phase transitions: $q = k(X)$ and $= n$.

Let $t_q := t_q(X)$ be the *smallest* t s.t. $\exists u_i \in O(n)$ and $\forall x \in \mathbb{R}^n$

$$\frac{1}{2} M_q |x| \leq \left(\frac{1}{t} \sum_1^t \|u_i x\|^q \right)^{1/q} \leq 2M_q|x| \, .$$

Then:

(i) t_q is $\sim$ the same for random choice u_i

and (ii) $t_q \sim t_1$ for $1 \leq q \leq 2$

$t_q^{2/q} \sim t_1 \left(\frac{M_1}{M_q}\right)^2$ for $q \geq 2$

It looks as though we have one phase transition, $q = 2$. But put (M_q) inside the formula and we have another phase transition for $q = k(X)$. So altogether, our process has two phase transitions for t_q on the most interesting interval $1 < q < n$.

We may now summarize **additional phenomena**:

a$'$) *More complicated processes with two (or more) phase transitions are superpositions of "elementary" ones.*

b$'$) *The constant behaviour (but at the "start" of a process) corresponds to* ***inertia****; in the behaviour of M_q, it is a* ***concentration phenomenon*** *and in the case of t_q (at $q = 2$) the reason behind it is a* ***convexification****:* Khinchine's inequality transforms L_p-averages to $L_{p/2}$-averages which, for $1 \le p \le 2$, behave in the same way as L_1 because of the lack of convexity (i.e. the same as for the case $p = 2$).

In fact, we observed this type of phase transition form behavior long ago (without recognizing it). The "simplest" one known to us (since 1976 [FLM77]) is the Dvoretzky type theorem for ℓ_p^n spaces

$$k(\ell_p^n) \sim \begin{cases} n & 0 < p \le 2 \\ pn^{2/p} & 2 < p \le \log n \end{cases}$$

(and, of course $\ell_{\log n}^n \simeq \ell_\infty^n$). Here $k(\ell_p^n)$ denotes the largest dimension k such that ℓ_p^n contains a 2-isomorphic copy of ℓ_2^k.

3d Threshold (in Problems of Approximation).

Let $x_i \in S^{n-1}$ and $I_i = [-x_i, x_i]$ be an interval. We approximate a euclidean ball $r\mathcal{D}$ (of some radius $r \sim \frac{1}{\sqrt{n}}$) by $K_N = \frac{1}{N}\sum_1^N I_i$. We will measure the difference between two centrally symmetric convex bodies, K and T, by geometric (multiplicative) distance $d_g(K,T) = \min\{a \cdot b \mid K \subset bT,\, T \subset aK\}$. Later, we also use the Banach–Mazur distance $d(K,T) = \inf\{d_g(uK,T) \mid u \in GL_n\}$. Then (Kashin [K77])

$$\inf_{\{x_i\}} d_g(K_N, \mathcal{D}) = \begin{cases} \infty & N < n \\ \sqrt{n} & N = n \\ C(\lambda) & N = \lambda n, \lambda > 1 \end{cases}$$

Also (let $\lambda < 2$), by Gluskin [Gl03]

$$C(\lambda) \sim \min\left\{\sqrt{n}, \sqrt{\left(\log \frac{1}{\lambda - 1}\right) \Big/ (\lambda - 1)}\right\}.$$

We see a sharp *threshold* at "n".

A similar picture may be observed for different convex bodies K in the problem of approximating $\mathcal{D}$ by averaging $u_i K$, $u_i \in O(n)$. And, changing the parameter of study from "decay of diameter of a generic average K_N" to "distance to the euclidean ball" of this average, often changes the behavior we observe from "phase transition" type to "threshold" type behavior. It would be interesting to demonstrate similar changes from threshold type behavior to phase transition type in problems of Asymptotic Combinatorics, which usually deals with thresholds.

4 Complexity Connection

The Theory of Complexity is, of course, another subject where asymptotic behavior of high parametric systems is in the spotlight at present. I would like to show here how the ideology of complexity brings very correct conjectures to High Dimensional Convexity and Asymptotic Geometric Analysis. Let us give one example of such influence.

The scheme works as follows: Fix convex K (say, cross-polytope $K = \operatorname{conv}\{\pm e_i\}_1^n$). Let $u \in GL_n$ be some (unknown) operator. Let T be convex body (unknown) s.t. $d(T, D) \le 10$. Finally, let

$$K_0 = \begin{cases} \text{either} & uK \\ \text{or} & T \end{cases}$$

Select (say, randomly) $\{x_i \in K_0\}_1^N$. Let N be polynomial in dimension n. What test may be performed on $\{x_i\}_1^N$ to distinguish the case of "uK" from the case of "T"? Can we distinguish K (presented in the form uK) from slightly distorted ellipsoid T?

The intuition coming from the Complexity circle of ideas suggests that it should not be possible in the scale above logarithmic in dimension. For example, one may present a test of complexity, linear in dimension, which will show that a cube, i.e. the unit ball of ℓ_∞^n, has distance at least $\sqrt{\log n}$ from the euclidean space. However, the real distance is $\sqrt{n}$.

Actually, it is not a "negative" conjecture but, on the contrary, a very "positive" one. Indeed, it means the following:

Conjecture. *Any test of polynomial (in dimension) computational complexity performed on any convex body K will give, up to some logarithmic factor, the same answer as for some ellipsoid.*

(The level of impreciseness in the formulation of this conjecture is also influenced by the Complexity Theory connection.)

To show that such thinking may be very useful let us formulate one recent theorem which shows that some test which was suggested to distinguish some spaces from a euclidean one, does not in fact distinguish between them.

Theorem (E. Gluskin, V. Milman [GlM02]). *$\exists\, c > 0$ s.t. $\forall\, n$ and $\forall\, X = (\mathbb{R}^n, \|\cdot\|_K)$ random iid uniform in K variables $\{x_i \in K\}_{i=1}^n$ satisfy with probability exponentially close to 1,*

$\forall\, \lambda_i \in \mathbb{R},\ i = 1, \dots, n,$

$$\operatorname*{Ave}_{\pm} \Big\| \sum_{1}^{n} \pm \lambda_i x_i \Big\| \geq c \sqrt{\sum_{i}^{n} |\lambda_i|^2}.$$

i.e. every finite dimensional normed space has "random" cotype 2.

5 Concentration Phenomenon

The "Concept of Concentration" is a concept of behavior of large systems which (roughly) states that any reasonably good (in the sense of "smoothness") function of too many variables actually degenerates to a constant. Or, more precisely, we are unable to check that it is not an almost constant. This may be viewed as a parallel observation to the so called "self-averaging" principle in Statistical Physics.

In fact it looks as though a more general outcome should occur. In many examples we observed a "self-shaping" phenomenon, say, the example in 3b may be seen in that way: very few rotations "shape" an approximately euclidean ball from any starting "shape" K. Actually, much lighter "intervention" turns any convex K into an almost ellipsoid. (For a new example of this, see [MiP03].)

This phenomenon led to a complete reversal of our intuition on high dimensional results. Instead of a chaotic diversity with an increase in dimension, which previous intuition suggested, we observe well organized and simple patterns of behaviour.

It should be mainly viewed as a (technical) tool. But, on an ideological level, it connects Analysis with Geometry; Geometry with Probability and Combinatorics. It extends deviation-type inequalities to the non-linear setting instead of the previously considered linear ones, and leads, in many instances, to removing the traditional probabilistic conditions (such as "independence", "martingale", and so on). We understand now that the importance of these conditions was, mainly, in ensuring high-dimensionality. However, high dimension is enough to achieve the same type of results.

In Combinatorics, for specific graphs, the concentration property is equivalent to the notion of "expanders".

There are many surveys and books on this subject, and I have discussed it enough in the past. So, I will just give a few references [Mi88], [L01], [M00], [Gr99]

6 Ramsey Type Results, or Concept of Spectrum/Distortion

The understanding behind this concept is that reasonably good functions are almost constants on large substructures. (Similarly, for maps from very high dimensional structures to much smaller dimension.) In the discrete (combinatorial) setting we may often achieve a constant value exactly (and derive Ramsey type results), but not in Analysis, where "almost" becomes essential and plays against the meaning of "good maps". The first papers where this was realized are [Mi69], [Mi71a], [Mi71b]; the main example on which this concept was illustrated is the famous Dvoretzky theorem. See [Mi88] for some historical connections. Later Gromov [Gr83] analyzed these connections; for a very recent turn of an infinite dimensional flavor see [P00], [P02]. There is also another line of such results, on the one hand discrete but on the other isomorphic, and in the spirit of the Concept of Spectrum, see [BFM86], [BLMN].

However, the size of these substructures, where our map is almost constant, in general is not large enough, even very small (say, logarithmically in the original dimension) if functions are not good Lipschitz functions, and this blocks many potential applications in Analysis. (To my surprise, isomorphic discrete examples may be very different [BLMN].)

At that point another powerful principle picks up the challenge.

7 Polarity

(Quotient of Subspace or "QS"–principle)

As the notion of "polarity" involves some convexity type structures, this principle is well developed only under some convexity assumptions, however, in fact, very weak convexity type conditions are already enough.

The following example should help to show what I mean. Returning to the problem of approximating euclidean norm:

$$|x| \sim \frac{1}{t}\sum_{i=1}^{t} \|u_i x\|_K, \quad u_i \in O(n) \ .$$

Even if we use the larger family, $u_i \in GL_n$, for some K minimal t may be as large as $n/\log n$.

So, we cannot hope to use only one u, or, say, two: u_1, u_2. However, let us also allow use of "dual" operations.

Then the following statement is correct.

For any n and any norm $\|\cdot\|$ in $\mathbb{R}^n$ there are two linear maps u_1, u_2 which are enough for approximation: for every $x \in \mathbb{R}^n$ let

$$|||x||| := \|x\| + \|u_1 x\|$$

and, dualizing, consider the new norm $|||x|||^* + |||u_2 x|||^*$ which is already C-equivalent to a euclidean norm (and C is a universal constant).

A similar picture is seen for any pair of dual operations.

The first fact of this kind was the following "Quotient of a Subspace Theorem" ([Mi85]):

Let $1/2 \le \lambda < 1$ and X be an n-dimensional normed space. Then $\exists$ subspaces $F \hookrightarrow E \hookrightarrow X$ with

$$k = \dim E/F \ge \lambda n\,,\ \operatorname{dist}(E/F, \ell_2^k) \le c\frac{|\log(1-\lambda)|}{1-\lambda}\,.$$

There is a very interesting recent development which demonstrates a similar phenomenon in the case of discrete finite metric space [MN]. At the same time, there are remarkable differences between the linear and discrete cases. M. Mendel and A. Naor [MN] observe strange phase transitions in the discrete case which we don't see in the linear one, and it is not yet clear if there is any corresponding analogue in the case of normed spaces.

Let me provide another simply formulated example to demonstrate the influence of polarity on the results of "Spectrum type" in the linear setting. Let $f \in C(S^{n-1})$, $f > 0$ and 1-Lip. Then $\forall \varepsilon > 0$ there is a subspace $E \hookrightarrow \mathbb{R}^n$, $k = \dim E$, $k \ge c\varepsilon^2 n/\log 1/\varepsilon$, c – universal number, s.t.

$$Osc\{f \mid x \in S^{n-1} \cap E\} < \varepsilon$$

(such subspaces form a set in $G_{n,k}$ of exponentially close to 1 measure).

However, important families $\{f_n \in C(S^{n-1})\}$ of functions do NOT have uniformly bounded Lip. constants which often, on the contrary, increase very fast with an increase of dimension.

Then another scheme follows:

Extend $f > 0$ by homogeneity $\hat{f}(x) = |x| f\left(\frac{x}{|x|}\right)$, $(x \ne 0)$. For $x \in E$, a subspace of $\mathbb{R}^n$, define

$$(q_E f)(x) = \inf\{\hat{f}(z) \mid z = x + y,\ y \in E^\perp\}\,.$$

Now, under very mild convexity type conditions on $\hat{f}$ (say, $\hat{f}$ being convex is enough, or $\hat{f}$ being quasi-convex with a quasi-convexity constant C), we may state:

There is an ellipsoid $\mathcal{E}$ (and euclidean structure connected with it) s.t.

$$\forall\lambda\,,\ 0 < \lambda < 1\,,\ \text{and}\ \forall\mu\,,\ 0 < \mu < \lambda$$

[say, $\lambda = 1/2$, $\mu = 1/4$] for "random" (in $\mathcal{E}$-structure) subspaces E, $\dim E = [\lambda n]$ and $F \hookrightarrow E$ $\dim F = [\mu n]$

$$Osc\{(q_E f) \mid \mathcal{E} \cap F\} < C(\lambda; \mu)$$

a constant depending only on $\lambda < 1$ and $\mu < \lambda$ (but not on dimension n or a function f from the selected class).

Also now for a "random" subspace F_ε, $\dim F_\varepsilon \sim (\varepsilon^2 \mu) n / \log 1/\varepsilon$

$$Osc\{(q_E f) \mid \mathcal{E} \cap F_\varepsilon\} < \varepsilon\,.$$

Note, belonging to such classes means some global type "smoothness" conditions, not a local type: the star body $K = \{x \mid \hat{f}(x) \le 1\}$ should not have many needles, it should not look like a hedgehog.

8 Conclusion

Let me conclude with one general remark about my intuition on this subject.

I deal with high-dimensional systems, as some random media behind the scenes "govern" them (and I feel it this way). Different "realizations" of concrete objects in such media may look very different (as, say, two realizations of Brownian motion look different). But they reveal the same, almost identical, statistical pattern. And our general theory describes these patterns.

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