Interpolation and approximation of piecewise smooth functions

Francesc Arandiga, Albert Cohen, Rosa Donat and Nira Dyn

Abstract

This paper provides approximation orders for a class of nonlinear interpolation procedures for uniformly sampled univariate data. The interpolation is based on Essentially Non-Oscillatory (ENO) and Subcell Resolution (SR) reconstruction techniques. These nonlinear techniques aim at reducing significantly the approximation error for functions with isolated singularities, and are therefore attractive for applications such as shock computations or image compression. We prove that in the presence of isolated singularities, the approximation order provided by the interpolation procedure is improved by a factor of h relative to the linear methods, where h is the sampling rate. Moreover for h below a critical value, we recover the optimal approximation order as for uniformly smooth functions.

1. Introduction

This paper is concerned with the analysis of a class of univariate high order interpolation and approximation techniques for piecewise smooth functions, introduced by Ami Harten [**?**], namely *Essentially Non-Oscillatory* (ENO) and *Subcell Resolution* (SR) reconstructions. These methods automatically adapt near the singularities of the approximated function, and they are by essence data dependent and nonlinear.

While their initial motivation was in the context of finite volume methods for shock computations, ENO-SR methods have found natural applications in data compression algorithms, in particular through the development of multiscale decompositions, similar to wavelet expansions, which incorporate nonlinear reconstructions [**?**, **?**, **?**]. In such decompositions, the wavelet coefficients are interpreted as the errors between the sampled data and its reconstruction from a sampling at a twice coarser scale. When dealing with data sampled from a piecewise smooth function, the adaptive treatment of singularities results in more accurate reconstructions and therefore in sparser decompositions than when using standard wavelet basis. In recent years, ENO-SR techniques have been extended to 2D image data, either by tensor product [?, ?] or by intrinsically 2D reconstructions [?, ?]. Other related nonlinear multiscale representations have been introduced in [?, ?] in the context of the lifting scheme.

From a theoretical point of view, the adaptive treatment of singularities allows us to expect strictly better approximation rates than with linear methods in the case of piecewise smooth functions. A rigorous analysis of this improvement in the 1D case is the main objective of this paper. Based on this analysis, our future perspective is to study the approximation properties of edge-adapted techniques for 2D data such as in [?, ?].

Consider at first the following situation : from a set of uniformly sampled data $(f(kh))_{k\in\mathbb{Z}}$, we are interested in building an interpolant $\mathcal{I}_h f$ i.e. a function such that $\mathcal{I}_h(kh) = f(kh)$ for all $k \in \mathbb{Z}$. There are many ways to build an interpolant $\mathcal{I}_h f$ of a prescribed order $m > 0$, i.e. such that if $f \in C^m$ one has

$$|\mathcal{I}_h f - f| \leq Ch^m \sup |f^{(m)}|. \tag{1}$$

Basically, one can do it with a linear operator $\mathcal{I}_h$ which is (i) local, (ii) exact for polynomials of degree $m-1$ and (iii) stable. We are interested in the interpolation of continuous functions f which are smooth everywhere except at isolated points. For such functions, we can only expect an error bound of order $\mathcal{O}(h)$ with a linear method, independently of its order.

In order to explain in a nutshell the principles of the ENO and SR techniques, first consider a standard piecewise polynomial interpolation of the data $(f(kh))_{k\in\mathbb{Z}}$: to each interval

$$I_k := [kh, (k+1)h], \quad k \in \mathbb{Z}, \tag{2}$$

we attach the stencil S_k of size m around I_k i.e.

$$S_k := \{(k-m_1)h, \cdots, (k+m_2)h\}, \tag{3}$$

where $m_1 \geq 0$ and $m_2 > 0$ are fixed integers such that $m_1 + m_2 = m-1$. We define a unique polynomial $p_k \in \Pi_{m-1}$ which agrees with f on S_k. A linear interpolation operator is then defined by

$$\mathcal{I}_h f(x) = p_k(x), \quad x \in I_k. \tag{4}$$

This interpolant has accuracy of order $m:$ if f is C^m on $[(k-m_1)h,(k+m_2)h]$, we have the estimate

$$\|f-\mathcal{I}_h f\|_{L^\infty(I_k)} \leq Ch^m \|f^{(m)}\|_{L^\infty([(k-m_1)h,(k+m_2)h])}. \tag{5}$$

Clearly, for a smooth function f with an isolated singularity of f' situated in the interval I_k, the order of accuracy is reduced to $\mathcal{O}(h)$ on all the intervals I_{k+l} for $l=-m_2+1,\cdots,m_1$, due to the systematic use of a fixed stencil.

The principle of ENO (Essentially Non-Oscillatory) interpolation is to allow for data dependent stencils in order to reduce the influence of the singularity on the approximation. For this purpose, one typically introduces a measure of the oscillation of f on the stencil S_k. Since we are interested in detecting jump discontinuities in the first derivative, this measure is typically based on the evaluation of the second order differences,

$$\Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h), \tag{6}$$

for $x=(k-m_1)h,\cdots,(k+m_2-2)h$. For each k, we select among all the stencils $\{S_{k-m_2+1},\cdots,S_{k+m_1}\}$ which contain I_k the stencil $\tilde{S}_k$ which minimizes a chosen measure. The ENO interpolant is then given by

$$\mathcal{I}_h f(x) = \tilde{p}_k(x), \quad x \in I_k, \tag{7}$$

where $\tilde{p}_k$ is the polynomial which agrees with f on the stencil $\tilde{S}_k$. In comparison with the linear interpolation based on a fixed stencil, ENO interpolation has the same order of accuracy m and it reduces the effect of an isolated singularity, since the selected stencil will tend to avoid it. We therefore expect that the precision only deteriorates on the interval which contains the singularity.

The goal of the SR (Subcell Resolution) technique is to improve the approximation properties of the interpolant even on this interval. It is based on a detection mechanism which labels as B (bad) an interval I_k which is suspected to contain a singularity, in the sense that the selected stencils for its immediate neighbors tend to avoid it. Thus I_k is B if $\tilde{S}_{k-1}=S_{k-m_2}$ and $\tilde{S}_{k+1}=S_{k+m_1+1}$. Other intervals are labeled as G (good). On a G interval I_k, we use the above described ENO interpolation to define $\mathcal{I}_h f$. On a B interval I_k, we use the polynomials $\tilde{p}_{k-1}$ and $\tilde{p}_{k+1}$ to predict the location of the singularity : if these polynomials intersect at a single point a_k of I_k, we define for $x \in I_k$ the interpolant by

$$\mathcal{I}_h f(x) = \tilde{p}_{k-1}(x) \ \text{ if } \ x \le a_k, \ \ \tilde{p}_{k+1}(x) \ \text{ if } \ x \ge a_k. \tag{8}$$

In the case where these polynomials do not intersect at a single point of I_k, the interval is relabeled as G and the ENO interpolation is used.

An intuitive statement is that ENO-SR interpolation has accuracy of order $\mathcal{O}(h^m)$ for piecewise smooth functions. Our goal here is to investigate this statement in a rigorous way. For simplicity we consider functions which are smooth except at one unknown point a, but are globally continuous. We also assume that $f \in C^m(\mathbb{R} \setminus \{\mathrm{a}\})$ in the sense that its derivatives up to order m are uniformly bounded on $\mathbb{R} \setminus \{\mathrm{a}\}$. Thus the derivatives of f have jumps $([f'], [f''], \cdots)$ at the point a. Ideally we could hope for an estimate of the form

$$\|f - \mathcal{I}_h f\|_{L^\infty} \le C h^m \sup_{\mathbb{R} \setminus \{\mathrm{a}\}} |f^{(m)}|, \tag{9}$$

for all $h > 0$. Unfortunately, we shall see with a simple example that we cannot hope for such a result for $m > 2$. In fact (**??**) holds for h smaller than a fixed fraction of a critical scale h_c depending itself on the function f according to

$$h_c := \frac{[f']}{4 \sup_{x \in \mathbb{R} \setminus \{\mathrm{a}\}} |f''(x)|}. \tag{10}$$

This critical scale h_c corresponds to the minimal level of resolution which ensures the detection of the singularity. Therefore we can only achieve

$$\|f - \mathcal{I}_h f\|_{L^\infty} \le C h^m \sup_{\mathbb{R} \setminus \{\mathrm{a}\}} |f^{(m)}|, \ \ h \le K h_c(f), \tag{11}$$

Yet we shall yet prove that we have for all $h > 0$, an estimate of the form

$$\|f - \mathcal{I}_h f\|_{L^\infty} \le C h^2 \sup_{\mathbb{R} \setminus \{\mathrm{a}\}} |f''|, \tag{12}$$

i.e. at least a gain of one order of accuracy relative to any linear method. Note that since the interpolation process is local, our analysis applies also to the case of several isolated singularities which are sufficiently separated relative to the sampling scale (typically by mh).

When dealing with functions f which are piecewise smooth with an isolated jump discontinuity in the function itself, there is no more hope that a nonlinear reconstruction of f from its samples $f(kh)$ brings any improvement on the interval that contains the jump point, since the location of this point cannot be resolved at a finer resolution from these samples. Moreover these samples are a-priori not well defined if f is not a continuous function. We should therefore replace the point value sampling by local averaging, in the sense that we are now given the cell averages $f_k^h := \frac{1}{h}\int_{kh}^{(k+1)h} f(t)dt$ for $k \in \mathbb{Z}$. We can build ENO-SR reconstruction procedures from such data in a similar way as for point value data. In fact, reconstruction from cell averages can be derived by differentiating the point value interpolant obtained from the discrete primitive values $\sum_{l=0}^{k-1} f_l^h$. In turn, the results that we establish for piecewise smooth continuous functions in the point value setting can be used to establish similar results for piecewise smooth discontinuous functions in the cell average setting.

Note that most nonlinear approximation methods that deal with local singularities are based on either adaptive mesh refinement or wavelet thresholding (see e.g. [**?**, **?**] for surveys on such nonlinear approximation). A specific feature of the present approach is that it does not rely on any local refinement of the sampled data : the function is accurately reconstructed from a given uniform sampling, by a locally defined data dependent operator. A similar approach, yet based on different tools (in particular Fourier analysis), has been developed in [**?**].

Our paper is organized as follows. We first show by an example in §2 that one cannot hope for more than second order accuracy when a singularity occurs (still better than first order with linear methods). We introduce in §3 a specific singularity detection mechanism together with an ENO-SR interpolation process, which slightly differs from the original ENO-SR, yet with the same basic principles, and we discuss the organization of the intervals which are detected by this mechanism. We prove in §4 that detection always occurs for $h < h_c$ and that the position of the singularity is accurately estimated. We then use these results in §5 to prove that our version of the ENO-SR interpolation technique has accuracy of order $\mathcal{O}(h^m)$ for h smaller than Kh_c where $0 < K < 1$ is a fixed constant, and that it is second order accurate for all $h > 0$, which is the best that we can hope for according to the example of §2. These findings are confirmed in §6 by numerical examples. Finally in §7, we derive similar approximation results for piecewise smooth discontinuous functions in the cell average setting, measuring the error in the L^p norm.

2. An important example

Consider the functions f_+ and f_- which depend on $h_0 > 0$:

$$f_+(x) = 0 \text{ if } x < 0, \quad f_+(x) = x(x - h_0) \text{ if } x \geq 0, \tag{13}$$

and

$$f_-(x) = 0 \text{ if } x < h_0, \quad f_-(x) = x(x - h_0) \text{ if } x \geq h_0. \tag{14}$$

We notice that both functions agree on $h_0\mathbb{Z}$ so that if $\mathcal{I}_h$ is *any* interpolation operator on the grid $\mathbb{Z}$h, we have when $h = h_0$:

$$\mathcal{I}_h f_+ = \mathcal{I}_h f_-. \tag{15}$$

Since $\|f_+ - f_-\|_{L^\infty} = h^2/4$, by the triangle inequality we either have

$$\|f_+ - \mathcal{I}_h f_+\|_{L^\infty} \geq h^2/8 \geq \frac{h^2}{16} \sup_{x \in \mathbb{R} \setminus \{0\}} |f_+''|, \tag{16}$$

or

$$\|f_- - \mathcal{I}_h f_-\|_{L^\infty} \geq h^2/8 \geq \frac{h^2}{16} \sup_{x \in \mathbb{R} \setminus \{\mathrm{h}\}} |f_-''|. \tag{17}$$

Since we also have

$$\sup_{x \in \mathbb{R} \setminus \{\mathrm{h}\}} |f_-^{(m)}| = \sup_{x \in \mathbb{R} \setminus \{0\}} |f_+^{(m)}| = 0, \quad m > 2, \tag{18}$$

this simple example shows us that (**??**) cannot be achieved with $m > 2$. Here h_0 plays the role of a *critical scale* above which singularities cannot be precisely detected. For $h << h_0$, our non-linear interpolation method gives an exact reconstruction of f_+ and f_-. However, we certainly cannot ensure more than second-order accuracy over all piecewise smooth functions and all $h > 0$.

3. A modified ENO-SR detection and interpolation mechanism

For a given approximation order m, our detection mechanism defines a set of intervals labeled as B, which potentially contain the singularity, according to the following rules :

1. If

$$|\Delta_h^2 f((k-1)h)| > |\Delta_h^2 f((k-1 \pm n)h)|, \quad n = 1, \cdots, m. \tag{19}$$

both I_{k-1} and I_k are labeled as B. Notice that (**??**) indicates that the point kh lies at the center of the largest second divided difference (among those being compared). Hence either I_{k-1} or I_k could potentially contain the singularity.

2. If

$$|\Delta_h^2 f(kh)| > |\Delta_h^2 f((k+n)h)|, \quad n = 1, \cdots, m-1, \tag{20}$$

and

$$|\Delta_h^2 f((k-1)h)| > |\Delta_h^2 f((k-1-n)h)|, \quad n = 1, \cdots, m-1. \tag{21}$$

then I_k is labeled as B. In this case the two largest divided differences involved in the comparison process include I_k, which is then a candidate to contain the singularity.

All other intervals are labeled as G.

This detection mechanism is designed in such a way that for h sufficiently small, the interval I_k containing the singularity a is labeled as B, while all intervals labeled as G are in smooth regions of f. On the other hand it is also possible that an interval I_k might be labeled as B in a smooth region at an arbitrarily small scale. In case of such false alarms, it is crucial that the polynomials which are used to construct the interpolation are built from stencils which only contain G intervals, i.e. from smooth regions. This is ensured by the following lemma which describes the organization of the B and G intervals.

Lemma 1. *The groups of adjacent B intervals are at most of size* 2. *They* *are separated by groups of adjacent G intervals which are at least of size $m-1$.*

Proof : Assume that I_0 and I_k are B with $1 < k < m$. We have three cases :

1. I_0 and I_k have been labeled B by the second rule. Then it follows that both $|\Delta_h^2 f(0)| > |\Delta_h^2 f((k-1)h)|$ and $|\Delta_h^2 f(0)| < |\Delta_h^2 f((k-1)h)|$ which is a contradiction.

2. I_0 has been labeled B by the second rule and I_k has been labeled B by the first rule. Then either I_{k-1} or I_{k+1} is also a B interval. Hence we obtain that both $|\Delta_h^2 f(0)| > |\Delta_h^2 f(qh)|$ and $|\Delta_h^2 f(0)| < |\Delta_h^2 f(qh)|$ for some $q \in \{k-1, k\}$, which is a contradiction. The case where I_0 has been labeled B by the first rule and I_k has been labeled B by the second rule is treated in a similar way.

3. I_0 and I_k have been labeled B by the first rule, hence each one is a member of a B-pair (two adjacent B intervals). Hence we obtain that both $|\Delta_h^2 f(ph)| > |\Delta_h^2 f(qh)|$ and $|\Delta_h^2 f(ph)| < |\Delta_h^2 f(qh)|$ for some $p \in \{-1, 0\}$ and $q \in \{k-1, k\}$, which is a contradiction.

We therefore obtain that no two B intervals can have difference of indices strictly between 1 and m which concludes the proof. □

Based on the above described detection mechanism, we propose the following interpolation procedure:

1. If I_k is a G interval, define $\mathcal{I}_h f$ on I_k as a polynomial p_k of degree $m-1$ obtained by interpolation of f on a stencil $\{ph, \cdots, (p+m-1)h\}$ such that $p \le k < k+1 \le p+m-1$ and such that this stencil only contains G intervals. Such a stencil always exists according to Lemma 1, yet is not unique. In practice, we may choose the stencil which is the most centered around the interval I_k or we may use the standard ENO procedure.

2. If I_k is an isolated B interval, we obtain polynomials p_k^- and p_k^+ of degree $m-1$ by interpolation of f on the stencils $\{(k-m+1)h, \cdots, kh\}$ and $\{(k+1)h, \cdots, (k+m)h\}$ and use them to predict the location of the

singularity : if these polynomials intersect at a single point y of I_k, we define for $x \in I_k$ the interpolant by

$$\mathcal{I}_h f(x) = p_k^-(x) \ \text{if} \ x \le y, \ \ p_k^+(x) \ \text{if} \ x \ge y. \tag{22}$$

In the case where these polynomials do not intersect at a single point of I_k, the interval is relabeled as G and we return to the previous case.

3. If (I_k, I_{k+1}) is a B-pair, we treat $I_k \cup I_{k+1}$ as I_k in the previous case, i.e. we obtain polynomials p_k^- and p_{k+1}^+ of degree $m-1$ by interpolation of f at stencils $\{(k-m+1)h, \cdots, kh\}$ and $\{(k+2)h, \cdots, (k+m+1)h\}$ and use them to predict the location of the singularity : if these polynomials intersect at a single point y of $I_k \cup I_{k+1}$, we define for $x \in I_k \cup I_{k+1}$ the interpolant by

$$\mathcal{I}_h f(x) = p_k^-(x) \ \text{if} \ x \le y, \ \ p_{k+1}^+(x) \ \text{if} \ x \ge y. \tag{23}$$

In the case where these polynomials do not intersect at a single point of $I_k \cup I_{k+1}$, both intervals are relabeled as G and we return to the first case.

Note that the interpolation operator $\mathcal{I}_h f$ described above does not make use of the data at mid-points of B-pairs. Hence $\mathcal{I}_h f$ does not interpolate f at these points. This is a specific feature of our modified ENO-SR interpolation which greatly facilitates the proof of our main approximation result in section §5.

4. Properties of the detection mechanism

The goal of this section is to establish some properties of the detection mechanism which shall be used in §5 for proving the improved approximation order of $\mathcal{I}_h f$ announced in the introduction.

The properties are expressed by two lemmas. The first one ensures that the singularity is always detected under some critical scale.

Lemma 2. *Let f be a globally continuous function with a bounded second* *derivative on* $\mathbb{R}\setminus\{\mathrm{a}\}$ *and a discontinuity in the first derivative at a point* a*. Define the critical scale*

$$h_c := \frac{|[f']|}{4\sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}}|f''(x)|}, \tag{24}$$

where $[f']$ *is the jump of the first derivative* f' *at the point* a*. Then for* $h < h_c$*, the interval* I_k *which contains* a *is labeled as* B*. Moreover, if* a *is close to one edge of the interval* I_k *by at most quarter of its size, then the interval adjacent to this edge is also labeled as* B*.*

Proof : Without loss of generality, we can assume that a is located on the first half of the interval I_0, i.e. $0 \le a \le h/2$. For $k > 0$ and $k < -1$, we find that

$$|\Delta_h^2 f(kh)| \le h^2 \sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}} |f''(x)|. \tag{25}$$

For $k = -1$ and $k = 0$, the second order finite differences can be estimated by decomposing f into

$$f(x) = f_1(x) + f_2(x), \tag{26}$$

with $f_1(x) = [f'](x-a)_+$ and $f_2(x)$ is a C^1 function with a bounded second derivative on $\mathbb{R}\setminus\{\mathrm{a}\}$, such that

$$\sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}} |f_2''(x)| = \sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}} |f''(x)|. \tag{27}$$

We therefore have for all $k \in \mathbb{Z}$

$$|\Delta_h^2 f_2(kh)| \le h^2 \sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}} |f''(x)|. \tag{28}$$

On the other hand, we have

$$|\Delta_h^2 f_1(-h)| = |(h-a)[f']|. \tag{29}$$

It follows that

$$|\Delta_h^2 f(-h)| \ge |(h-a)[f']| - h^2 \sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}} |f''(x)|, \tag{30}$$

and therefore

$$|\Delta_h^2 f(-h)| \geq \frac{h}{2}|[f']| - h^2 \sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}} |f''(x)|. \tag{31}$$

So if $h < h_c$, we get by (**??**)

$$|\Delta_h^2 f(-h)| > h^2 \sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}} |f''(x)| \tag{32}$$

Combining this with (**??**), we find that if $h < h_c$

$$|\Delta_h^2 f(-h)| > |\Delta_h^2 f(kh)|, \tag{33}$$

for $k < -1$ and $k > 0$. In the case where $|\Delta_h^2 f(-h)| > |\Delta_h^2 f(0)|$, we find that I_{-1} and I_0 are a B-pair according to the first detection rule. Otherwise, if $|\Delta_h^2 f(-h)| \leq |\Delta_h^2 f(0)|$, we find that I_0 must be labeled as B according to the second detection rule.

We finally notice that

$$|\Delta_h^2 f_1(0)| = |a[f']|, \tag{34}$$

so that

$$|\Delta_h^2 f(0)| \leq |a[f']| + h^2 \sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}} |f''(x)|, \tag{35}$$

Therefore combining (**??**) and (**??**) we are always in the case of I_{-1} and I_0 constituting a B-pair whenever

$$2h^2 \sup_{x\in\mathbb{R}\setminus\{\mathrm{a}\}} |f''(x)| < (h-2a)|[f']|, \tag{36}$$

which holds whenever $h < h_c$ and $a < h/4$. □

The next lemma expresses the fact that the location of the singularity is accurately estimated when h is less than a fixed fraction of the critical scale.

Lemma 3. *There exist constants $C > 0$ and $0 < K < 1$ such that for all continuous f with uniformly bounded m-th derivative on $\mathbb{R} \setminus \{\mathrm{a}\}$ and for $h < Kh_c$ with h_c defined by (**??**), the following holds :*

1. *The singularity a is contained in an isolated B interval I_k (case 1) or in a B-pair (I_k, I_{k+1}) (case 2).*

2. *The two polynomials (p_k^-, p_k^+) (case 1) or (p_k^-, p_{k+1}^+) (case 2) which are used in the definition of $\mathcal{I}_h f$ have only one intersection point y inside I_k (case 1) or inside $I_k \cup I_{k+1}$ (case 2).*

3. *The distance between a and y is bounded by*

$$|a - y| \leq C \frac{h^m \sup_{\mathbb{R}\setminus\{a\}} |f^{(m)}|}{|[f']|} \tag{37}$$

Proof : Since $K < 1$, the first statement has already been proved in Lemma 2. Without loss of generality, we assume that $0 \leq a \leq h/2$. In this case we know by Lemma 2 that I_0 is B for $h < h_c$. For the sake of notational simplicity we denote by $I = [b, c]$ the interval where we do the subcell resolution process, which is either I_0 (case 1) or $I_{-1} \cup I_0$ (case 2) or $I_0 \cup I_1$ (case 2). By Lemma 2, we are ensured that $I = I_{-1} \cup I_0$ when $a < h/4$, and therefore

$$\min\{|a - b|, |a - c|\} \geq h/4. \tag{38}$$

We also denote by (p_-, p_+) the polynomials which are used in the subcell resolution of I. Finally we note that for any $2 \leq k \leq m$ we can write

$$f = f_- \chi_{]-\infty,a]} + f_+ \chi_{[a,+\infty[}, \tag{39}$$

where f^- and f^+ are functions which are globally C^k over $\mathbb{R}$ and such that

$$\sup_{x\in\mathbb{R}} |f_\pm^{(k)}(x)| \leq \sup_{x\in\mathbb{R}\setminus\{a\}} |f^{(k)}(x)|. \tag{40}$$

For example, we can define these functions by extension of f using its left or right Taylor expansion of order k at the point a. In order to prove the second statement of the theorem, we choose $k = 2$. We note that p_- and p_+ can also be viewed as Lagrange interpolation of f_- and f_+. It then follows from classical results on Lagrange interpolation that there exists a constant D independent of f such that for all $t \in I$,

$$|f_\pm(t) - p_\pm(t)| \leq Dh^2 \sup_{x\in\mathbb{R}} |f''_\pm(x)| = Dh^2 \sup_{x\in\mathbb{R}\setminus\{a\}} |f''(x)|, \tag{41}$$

and

$$|f'_\pm(t) - p'_\pm(t)| \le Dh \sup_{x\in\mathbb{R}} |f''_\pm(x)| = Dh \sup_{x\in\mathbb{R}\setminus\{a\}} |f''(x)|. \tag{42}$$

Since $|t-a| \le 2h$ when $t \in I$, we also have

$$|f'_\pm(t) - f'_\pm(a)| \le 2h \sup_{x\in\mathbb{R}\setminus\{a\}} |f''(x)|, \tag{43}$$

therefore we get from (**??**)

$$|f'_\pm(a) - p'_\pm(t)| \le (D+2)h \sup_{x\in\mathbb{R}\setminus\{a\}} |f''(x)|, \quad t \in I \tag{44}$$

It follows that for all $t \in I$,

$$|p'_+(t) - p'_-(t)| \ge |[f']| - 2(D+2)h \sup_{x\in\mathbb{R}\setminus\{a\}} |f''(x)|. \tag{45}$$

Thus, for $h < \frac{2}{D+2}h_c$ the function $p_+ - p_-$ is strictly monotone on I and has at most one root. Therefore, we are ensured that p_+ and p_- intersect at most at a single point inside I. In order to prove that this point y exists, we need to show that $p_+ - p_-$ has a sign change inside I. Without loss of generality, assume here that $[f'] > 0$. By second order Taylor expansion at the point a, we find that

$$(f_+ - f_-)(b) \le -(a-b)[f'] + (a-b)^2 \sup_{x\in\mathbb{R}\setminus\{a\}} |f''(x)|, \tag{46}$$

and

$$(f_+ - f_-)(c) \ge (c-a)[f'] - (c-a)^2 \sup_{x\in\mathbb{R}\setminus\{a\}} |f''(x)|. \tag{47}$$

Combining with (**??**), we thus obtain

$$(p_+ - p_-)(b) \le -(a-b)[f'] + ((a-b)^2 + 2Dh^2) \sup_{x\in\mathbb{R}\setminus\{a\}} |f''(x)|, \tag{48}$$

and

$$(p_+ - p_-)(c) \ge (c-a)[f'] - ((c-a)^2 + 2Dh^2) \sup_{x\in\mathbb{R}\setminus\{a\}} |f''(x)|. \tag{49}$$

Using (**??**), we therefore obtain,

$$(p_+ - p_-)(b) \le -\frac{h}{4}[f'] + (2D+4)h^2 \sup_{x\in \mathbb{R}\setminus\{a\}} |f''(x)|, \tag{50}$$

and

$$(p_+ - p_-)(c) \ge \frac{h}{4}[f'] - (2D+4)h^2 \sup_{x\in \mathbb{R}\setminus\{a\}} |f''(x)|. \tag{51}$$

It follows that for $h < \frac{1}{4D+8}h$, we have

$$(p_+ - p_-)(b) \le -\frac{h}{8}[f'] < 0, \tag{52}$$

and

$$(p_+ - p_-)(c) \ge \frac{h}{8}[f'] > 0, \tag{53}$$

so that there exists a single intersection point $y \in I$. So defining $K := \frac{1}{4D+8}$, we have proved the two first statements of the theorem.

In order to prove the third statement (**??**), we now choose $k = m$ in the definition of the extensions f_+ and f_-. It again follows from classical results on Lagrange interpolation that there exists a constant $\tilde{D}$ such that for all $t \in I$,

$$|f_\pm(t) - p_\pm(t)| \le \tilde{D}h^m \sup_{x\in \mathbb{R}\setminus\{a\}} |f^{(m)}(x)|, \tag{54}$$

and therefore, if we define $g = f_+ - f_-$ and $q = p_+ - p_-$, we obtain for all $t \in I$,

$$|g(t) - q(t)| \le 2\tilde{D}h^m \sup_{x\in \mathbb{R}\setminus\{a\}} |f^{(m)}(x)|, \tag{55}$$

so that for $t = a$,

$$|q(a)| \le 2\tilde{D}h^m \sup_{x\in \mathbb{R}\setminus\{a\}} |f^{(m)}(x)|. \tag{56}$$

Now note that for $t \in I$ and $h < Kh_c$, by (**??**)

$$|q'(t)| = \frac{|[f']|}{2}, \tag{57}$$

and therefore

$$|q(a)| = |q(y) - q(a)| \ge |y-a||[f']|/2. \tag{58}$$

Combining this with (**??**), we therefore obtain (**??**) with $C = 4\tilde{D}$. □

5. Approximation properties of $\mathcal{I}_h$

We are now ready to derive our main approximation result.

Theorem 1. *For all continuous f with derivatives up to degree m uniformly bounded on $\mathbb{R} \setminus \{a\}$, the nonlinear interpolant $\mathcal{I}_h f$ satisfies*

$$\|f - \mathcal{I}_h f\|_{L^\infty} \le C h^2 \sup_{\mathbb{R}\setminus\{a\}} |f''|, \tag{59}$$

*for all $h > 0$, with $C > 0$ independent of f. Moreover there exists $0 < K < 1$ independent of f such that for $h < K h_c$ with h_c defined by (**??**), we have*

$$\|f - \mathcal{I}_h f\|_{L^\infty} \le C h^m \sup_{\mathbb{R}\setminus\{a\}} |f^{(m)}|. \tag{60}$$

Proof : We choose for K the constant in Lemma 3. Note first that for $h < K h_c$, according to Lemma 1 and Lemma 2, all the polynomials which are used in the construction of $\mathcal{I}_h$ are built from stencils over which the function is smooth. It follows from classical results on Lagrange interpolation that the estimate

$$|f(x) - \mathcal{I}_h f(x)| \le C h^m \sup_{\mathbb{R}\setminus\{a\}} |f^{(m)}|, \tag{61}$$

holds whenever x belongs to a G interval or to an isolated B interval or B-pair which does not contain a (i.e. false alarms do not deteriorate the convergence rate). Let us now assume that x belongs to the group of adjacent B intervals which contains a. Here, we shall assume again without loss of generality that $0 \le a \le h/2$ and use the notations $I = [b, c]$, p_+, p_-, f_+, f_- that were introduced in the proof of Lemma 3. We also assume that $a \le y$, the case $y \le a$ being treated in a similar way. For $x \in [b, a]$, we have the estimate

$$|f(x) - \mathcal{I}_h f(x)| = |f_-(x) - p_-(x)| \le C h^m \sup_{\mathbb{R}\setminus\{a\}} |f^{(m)}|, \tag{62}$$

and for $x \in [y, c]$, we have the estimate

$$|f(x) - \mathcal{I}_h f(x)| = |f_+(x) - p_+(x)| \le C h^m \sup_{\mathbb{R}\setminus\{a\}} |f^{(m)}|. \tag{63}$$

It remains to consider the case $a < x < y$. In this case, we have

$$|f(x) - \mathcal{I}_h f(x)| = |f_+(x) - p_-(x)| \leq |f_+(x) - f_-(x)| + |f_-(x) - p_-(x)|. \quad (64)$$

The second term is again bounded by $Ch^m \sup_{\mathbb{R}\setminus\{a\}} |f^{(m)}|$. For the first term, we use second order Taylor expansion to derive

$$\begin{aligned} |f_+(x) - f_-(x)| &\leq |[f']|(y-a) + (y-a)^2 \sup_{\mathbb{R}\setminus\{a\}} |f''| \\ &\leq (y-a)(|[f']| + h \sup_{\mathbb{R}\setminus\{a\}} |f''|). \end{aligned}$$

Since $h < h_c$, this gives

$$|f_+(x) - f_-(x)| \leq \frac{5}{4}|[f']|](y-a). \quad (65)$$

Combining this with the estimate (**??**) of Lemma 3, we also obtain the bound $Ch^m \sup_{\mathbb{R}\setminus\{a\}} |f^{(m)}|$ for $|f_+(x) - f_-(x)|$ which concludes the proof in the case $h < Kh_c$.

In the case $h \geq Kh_c$, we are ensured of the estimate

$$|f(x) - \mathcal{I}_h f(x)| \leq Ch^m \sup_{\mathbb{R}\setminus\{a\}} |f^{(m)}|, \quad (66)$$

only when x is at distance at least $(m+1)h$ from a. We also have the lower order estimate

$$|f(x) - \mathcal{I}_h f(x)| \leq Ch^2 \sup_{\mathbb{R}\setminus\{a\}} |f''|. \quad (67)$$

Let us now prove that this estimate remains valid if $|x - a| \leq (m+1)h$. For this purpose we consider the decomposition $f = f_1 + f_2$ used in the proof of Lemma 2. The errors of polynomial interpolation of f_1 and f_2 are respectively bounded by $Ch|[f'||$ and $Ch^2 \sup_{\mathbb{R}\setminus\{a\}} |f''|$. Since $h \geq Kh_c$ the second bound dominates the first one so that the above estimate is valid. The proof of the theorem is now complete. □

6. Numerical examples

We consider the following functions

$$f_\varepsilon(x) = \begin{cases} (x - \pi/6)(x - \pi/6 - \varepsilon) + \sin(\pi x/8)/8 & x < \pi/6 \\ \sin(\pi x/8)/8 & \text{otherwise} \end{cases} \quad (68)$$