Analysis of Hermite-interpolatory subdivision schemes

Nira Dyn and David Levin

School of Mathematical Sciences

Tel-Aviv University, Ramat-Aviv

Tel Aviv 69978, Israel

Abstract

The theory of matrix subdivision schemes provides tools for the analysis of general uniform stationary matrix schemes. The special case of Hermite-interpolatory subdivision schemes deals with refinement algorithms for the function and the derivatives' values, with matrix masks depending upon the refinement level, i.e., non-stationary matrix masks. Here we first show that a Hermite-interpolatory subdivision scheme can be transformed into a stationary process. Then, using special schemes for generating some Hermite-type divided differences, we give the theory and the tools for analyzing the convergence and smoothness of Hermite-interpolatory schemes.

1. Stationary Hermite-interpolatory schemes

Hermite-type subdivision schemes of order 2 were already considered in [14] and in [12]. In the present paper we are discussing the basic properties and the proper analysis tools for higher order Hermite-type subdivision schemes. The analysis presented here is an adaptation of the methods in [2], [4], [10], [15] and especially [12], exploiting the structure and the special significance of Hermite-type data. Examples and numerical implementation of the analysis tools are presented in [3].

The Hermite-interpolatory scheme of order m is of the form

$$f_n^{k+1} = \sum_{j\in \mathbb{Z}} A^{(k)}_{n-2j} f_j^k \ , \ n \in \mathbb{Z} \ , \tag{1.1}$$

where $\{f_n^k\}$ are vectors in $\mathbb{R}^m$, f_n^k is the vector attached to the diadic point $t = 2^{-k}n$ at level k of the subdivision. The $m \times m$ matrices $\{A_n^{(k)} \ , \ n \in \mathbb{Z}\}$, which form the 'mask' of the scheme at level k, are non-zero only for the finite set of indices I. In particular

$$A^{(k)}_{2n} = I_{m\times m}\delta_{n,0} \ ,$$

due to the interpolatory nature of the scheme. The scheme converges to a C^s limit function with $s \geq m-1$ if there exists $f \in C^s$ such that

$$f_n^k = (f^{(0)}(2^{-k}n), f^{(1)}(2^{-k}n), \frac{f^{(2)}(2^{-k}n)}{2!}, \ldots, \frac{f^{(m-1)}(2^{-k}n)}{(m-1)!})^t \ , \ \ n \in \mathbb{Z} \ , \ \ k \in \mathbb{Z}_+ \ .$$

The normalization with the factorials is for later convenience.

In the following we consider schemes which reproduce the space Π_ℓ of polynomials of degree ℓ or less, for $\ell \geq m-1$, namely schemes with the property that if the vectors at level k are of the form

$$f_i^k = (p(2^{-k}i), p'(2^{-k}i), \ldots, \frac{p^{(m-1)}(2^{-k}i)}{(m-1)!})^t \ , \quad i \in \mathbb{Z} \ ,$$

for $p \in \Pi_\ell$, then application of (1.1) results in

$$f_i^{k+1} = (p(2^{-k-1}i), p'(2^{-k-1}i), \ldots, \frac{p^{(m-1)}(2^{-k-1}i)}{(m-1)!})^t \ , \quad i \in \mathbb{Z} \ .$$

Later we show that this property is necessary for such a scheme to converge to C^ℓ limit functions. In fact we consider a subset of such schemes which we term stationary. Let us assume that the degree ℓ of the polynomial space is chosen so that a mask $\{A_n^{(0)} \ , \ n \in I\}$ exists with the property of reproduction of polynomials in Π_ℓ, namely

$$P(2^{-1}n) = \sum_{j \in \mathbb{Z}} A_{n-2j}^{(0)} P(j) \ , \ n \in \mathbb{Z} \ , \tag{1.2}$$

where $P(x)$ is a matrix of order $m \times \ell$ with polynomial elements of the form

$$(P(x))_{i,j} = \frac{1}{i!} \frac{d^i}{dx^i} x^j \ , \quad i = 0, \ldots, m-1 \ , \quad j = 0, \ldots, \ell - 1 \ . \tag{1.3}$$

We would like to establish the connection between the mask at level k to the mask at level zero. The system of equations that the mask at level $k > 0$ must satisfy can be written as

$$P(2^{-k-1}n) = \sum_{j \in \mathbb{Z}} A_{n-2j}^{(k)} P(2^{-k}j) \ , \ n \in \mathbb{Z} \ . \tag{1.4}$$

By the form (1.3) of P it follows that

$$P(hx) = M_m(h)^{-1} P(x) M_\ell(h) \ , \tag{1.5}$$

where $M_j(h) = diag\{1, h, \ldots, h^{-j+1}\}$. Multiplying (1.2) by $M_m(2^{-k})^{-1}$ from the left and by $M_\ell(2^{-k})$ from the right we get,

$$M_m(2^{-k})^{-1} P(2^{-1}n) M_\ell(2^{-k}) =$$

$$\sum_{j \in \mathbb{Z}} M_m(2^{-k})^{-1} A_{n-2j}^{(0)} M_m(2^{-k}) M_m(2^{-k})^{-1} P(j) M_\ell(2^{-k}) \ .$$

Now, using relation (1.5) with $h = 2^{-k}$, it follows that (1.4) is satisfied for $k > 0$ with

$$A_n^{(k)} = M_m(2^{-k})^{-1} A_n^{(0)} M_m(2^{-k}) \ , \quad n \in I \ . \tag{1.6}$$

Thus given the mask at level $k = 0$ we can choose the mask at level $k > 0$ according to (1.6). We define such schemes as "stationary Hermite schemes". Although the scheme as given is not stationary since the mask depends on the level k, yet by a simple scaling the scheme becomes stationary. Defining the scaled vectors by $M_m(2^{-k})f_n^k$ these vectors are generated by a stationary scheme with the level independent mask

$$A_n = M_m(2^{-1})A_n^{(0)} \ , \quad n \in I \ . \tag{1.7}$$

2. Divided-difference operators for Hermite-interpolatory schemes

Our analysis of the convergence of stationary Hermite-interpolatory schemes which reproduce Π_ℓ , $\ell \geq m-1$ to C^s limit functions with $s \geq m-1$, is based on the existence of stationary subdivision schemes for the divided differences of the data vectors f_n^k , $n \in \mathbb{Z}$, of order j for $m-1 \leq j \leq \ell+1$. These divided differences are defined regarding every diadic point $t = n2^{-k}$ as having multiplicity m and interpreting the data there as

$$f_n^k = (f^{(0)}(t), f^{(1)}(t), \frac{f^{(2)}(t)}{2!}, \ldots, \frac{f^{(m-1)}(t)}{(m-1)!})^t \ , \quad n \in \mathbb{Z} \ , \quad k \in \mathbb{Z}_+ \ ,$$

for some function $f \in C^{m-1}$. Related to the data vectors f_n^k , $n \in \mathbb{Z}$, we consider the $(m-1)$th order divided differences defined as

$$u_k^{[m-1]}(mn+j) = [\tau_{j+1}, \tau_{j+2}, \cdots, \tau_{j+m}]f \ , \quad j = 0, \cdots, m-1 \ , \tag{2.1}$$

where

$$\tau_0 = \tau_1 = \cdots = \tau_{m-1} = (n-1)2^{-k} \ ,$$

and

$$\tau_m = \tau_{m+1} = \cdots = \tau_{2m-1} = n2^{-k} \ .$$

Thus, if a scalar stationary subdivision of the form

$$u_{k+1}^{[m-1]} = S_B u_k^{[m-1]} \ , \ k \in \mathbb{Z},$$

exists and converges uniformly to C^0 limit functions in the sense of convergence of scalar subdivision schemes, then the original Hermite-interpolatory scheme converges to C^{m-1} limit functions.

The vector $u_k^{[m-1]}$ can be obtained recursively from the original data vectors in the following way. Let

$$v_k^{[0]}(n) = f_n^k \ , \quad n \in \mathbb{Z} \ , \quad k \in \mathbb{Z}_+ \ , \tag{2.2}$$

and let

$$(v_k^{[r]}(n))_0 = 2^k((v_k^{[r-1]}(n))_0 - (v_k^{[r-1]}(n-1))_{r-1}) \ , \tag{2.3}$$

$$(v_k^{[r]}(n))_j = 2^k((v_k^{[r-1]}(n))_j - (v_k^{[r-1]}(n))_{j-1}) \ , \quad j = 1, \ldots, r-1 \ , \tag{2.4}$$

$$(v_k^{[r]}(n))_j = (v_k^{[r-1]}(n))_j \ , \quad j = r, \ldots, m-1 \ , \ , \tag{2.5}$$

for $r = 1, 2, \ldots, m-1$. We note that

$$v_k^{[r]}(n) = ([\tau_0, \ldots, \tau_{r-1}, \tau_m]f, [\tau_0, \ldots, \tau_{r-2}, \tau_m, \tau_{m+1}]f, \ldots,$$

$$[\tau_0, \tau_m, \ldots, \tau_{m+r-1}]f, f^{(r)}(n), \ldots, f^{(m-1)}(n))^t \ ,$$

and $u_k^{[m-1]}$ as defined in (2.1) is related to $v_k^{[m-1]}$ by

$$u_k^{[m-1]}(mn+j) = (v_k^{[m-1]}(n))_j \ , \quad j = 0, \ldots, m-1 \ , \quad n \in \mathbb{Z} \ .$$

The vectors $v_k^{[r]}(n)$, $n \in \mathbb{Z}$, can be obtained recursively by using the formalism of vector generating functions and of matrix Laurent polynomials. Let us introduce the vector generating functions

$$F_k^{[r]}(z) = \sum_{n \in \mathbb{Z}} v_k^{[r]}(n) z^n \ , \quad r = 0, \ldots, m-1 \ , \quad k \in \mathbb{Z}_+ \ ,$$

and the matrix Laurent polynomials

$$\Gamma_k^{m,r}(z) = (2^k \Gamma^r(z)) \oplus I_{(m-r-1)\times(m-r-1)} \ ,$$

where $\Gamma^r(z)$ is a matrix of order $(r+1) \times (r+1)$, which for $r = 1, \cdots, m-1$ is given by its non-zero elements:

$$\Gamma^r_{1,1} = 1 \ , \ \Gamma^r_{1,r+1} = -z \ , \ \ \Gamma^r_{i,i} = 1 \ , \ \Gamma^r_{i,i-1} = -1 \ , \quad i = 2, \cdots, r+1 \ ,$$

while for $r = 0$ its only element is $1 - z$. For this formalism we also define the symbol of a mask $\{A_n^{(k)} \ n \in I\}$ as the matrix Laurent polynomial

$$D_k^{[0]}(z) \equiv A_k(z) = \sum_{n \in I} A_n^{(k)} z^n \ .$$

With these notations we can rewrite relations (2.2)-(2.5) as relations between the generating functions,

$$F_k^{[r+1]}(z) = \Gamma_k^{m,r}(z) F_k^{[r]}(z) \ , \quad r = 0, \ldots, m-2 \ . \tag{2.6}$$

Formally, let us assume that the symbol of the subdivision scheme for the vectors $v_k^{[r]}$ is $D_k^{[r]}(z)$. That is we have

$$F_{k+1}^{[r]}(z) = D_k^{[r]}(z) F_k^{[r]}(z^2) \ .$$

Then, using (2.6), the symbol of the subdivision scheme for $v_k^{[r+1]}$ should be given by

$$D_k^{[r+1]}(z) = \Gamma_{k+1}^{m,r}(z) D_k^{[r]}(z) (\Gamma_k^{m,r}(z^2))^{-1} \ . \tag{2.7}$$

We are now ready to state and prove the existence of subdivision schemes for the vectors $v_k^{[r]}$, $\ r = 0, \ldots, m-1$.

Theorem 1. *Consider a Hermite-interpolatory subdivision scheme of order m, of the form (1.1). If the scheme reproduces Π_ℓ, $\ell \leq m-2$, then there exist subdivision schemes, with finitely supported masks, defined by their symbol $D_k^{[r]}(z)$ for the refinement of the vectors $v_k^{[r]}$, for $r = 1, \dots, \ell+1$.*

Proof: The proof is by recursion on r. First we observe that since the original scheme reproduces the constant polynomials, the data of constant vector $f_n^k = (1,0,\dots,0)^t$, $n \in \mathbb{Z}$, is an eigenvector of the subdivision scheme, namely if $f_n^0 = (1,0,\dots,0)^t$, $n \in \mathbb{Z}$, then $f_n^k = (1,0,\dots,0)^t$, $n \in \mathbb{Z}$, for all $k \in \mathbb{Z}_+$. This is equivalent to the following property of the symbols of the scheme,

$$A_k(1)(1,0,\dots,0)^t = (2,0,\dots,0)^t \ , \quad A_k(-1)(1,0,\dots,0)^t = (0,0,\dots,0)^t \ . \tag{2.8}$$

As shown below, property (2.8) guarantees that

$$D_k^{[1]}(z) = \Gamma_{k+1}^{m,1}(z) A_k(z) (\Gamma_k^{m,1}(z^2))^{-1} \ , \tag{2.9}$$

is a finite matrix Laurent polynomial. Hence, by (2.6), there is a finite mask consisting of the matrix coefficients of $D_k^{[1]}(z)$ which maps $v_k^{[1]}$ on $v_{k+1}^{[1]}$.

To see that indeed $D_k^{[1]}(z)$ in (2.9) is indeed a finite matrix Laurent polynomial, we recall that $\Gamma_k^{m,1}(z) = diag\{2^k(1-z), 1, \dots, 1\}$. By condition (2.8) it follows that all the elements in the first column of $A_k(-1)$ vanish and that all but the first element of $A_k(1)$ vanish. Therefore, all the elements of the first column of $\Gamma_{k+1}^{m,1}(z)A_k(z)$ are divisible by $1-z^2$, and hence $D_k^{[1]}(z)$ is a Laurent polynomial.

To do the recursive step we assume that for a given r, $0 < r < m-1$, there exists a matrix Laurent polynomial $D_k^{[r]}(z)$ defining the subdivision scheme generating the vectors $v_k^{[r]}$. We observe that the first $r+1$ components of $v_k^{[r]}(n)$ are the divided difference of order r of the original data f^k. Hence if f_n^0 is taken from $f(x) = x^r$ then by the polynomial reproduction property of the original scheme the corresponding vectors $v_k^{[r]}$ are all of the same form, namely

$$v_k^{[r]}(n) = e^{(r)} \ , \ n \in \mathbb{Z} \ , \ k \in \mathbb{Z}_+ \ ,$$

where $e^{(r)} = (1,1,\dots,1,0,0,\dots,0)^t$, with 1 repeated $r+1$ times and 0 repeated $m-r-1$ times. This is equivalent to

$$D_k^{[r]}(1)e^{(r)} = 2e^{(r)} \ , \quad D_k^{[r]}(-1)e^{(r)} = 0 \ . \tag{2.10}$$

We further observe that

$$\Gamma_k^{m,r}(z)^{-1} = \left(\frac{1}{1-z} 2^{-k} E^{(r)}(z)\right) \oplus I_{(m-r-1)\times(m-r-1)} \ ,$$

where $E^{(r)}(z)$ is a matrix of order $(r+1)\times(r+1)$ with $E^{(r)}(z)_{i,j} = 1$ for $i \geq j$ and $E^{(r)}(z)_{i,j} = z$ for $i < j$. It is enough to show that the first $r+1$ columns of

$$\Gamma_{k+1}^{m,r}(z) D_k^{[r]}(z) (E^{(r)}(z^2)) \oplus I_{(m-r-1)\times(m-r-1)}$$

vanish at $z = \pm 1$. This follows by observing that the first $r+1$ columns of $E^{(r+1)}(z^2)$ equal $e^{(r)}$. Hence, by (2.10), the first $r+1$ columns of $D_k^{[r]}(z)(E^{(r+1)}(z^2)) \oplus I_{(m-r-1)\times(m-r-1)}$ are zero at $z = -1$ and equal $e^{(r)}$ at $z = 1$. The result now follows by noting that left operation with $\Gamma_{k+1}^{m,r}(1)$ gives differences of pairs of the first $r+1$ rows.

3. Conditions for smoothness of Hermite-interpolatory schemes

In this section we first show that a C^s continuous stationary Hermite-interpolatory subdivision scheme must reproduce Π_s. Then we give necessary and sufficient conditions for C^s continuity, which can be put in as an algorithm for checking smoothness of Hermite-interpolatory schemes.

To analyze the smoothness of the scheme we have to introduce higher order divided differences of the vector $u_k^{[m-1]}$, denoted by $u_k^{[r]}$, representing the divided differences of f of order $m \leq r \leq s+1$, taking into account that each point $2^{-k}n$, $n \in \mathbb{Z}$, has multiplicity m. $u_k^{[r]}$ are defined recursively by

$$u_k^{[r+1]}(n) = 2^k(\Lambda^{m,r})^{-1}(u_k^{[r]}(n) - u_k^{[r]}(n-1)) \ , \quad n \in \mathbb{Z} \ , \tag{3.1}$$

where $\Lambda^{m,r}$ is a diagonal matrix of order $m \times m$ with elements

$$\Lambda_{j,j}^{m,r} = [\frac{m+r-j+1}{m}] \ , \quad j = 1, 2, \cdots, m \ . \tag{3.2}$$

Theorem 2. *Consider a stationary Hermite-interpolatory subdivision scheme of order m, of the form (1.1), with $A_i^{(k)}$ satisfying (1.6). If the scheme is C^s then it reproduces polynomials of degree s, i.e., is satisfies (1.4) with $\ell = s+1$. Furthermore, there exist subdivision schemes, with finitely supported masks, for the refinement of the divided differences of orders $\leq s+1$.*

Proof: The proof is iterative. First we observe that the original scheme must reproduce the constant polynomial, i.e., $(1, 0, \ldots, 0)^t$ is an eigenvector of the subdivision scheme. To see this we transform the process (1.1), using (1.6), into the stationary process

$$M(2^{-k})f_i^{k+1} = \sum_{j\in\mathbb{Z}} M(2^{-k})A_{i-2j}^{(k)}M(2^{-k})^{-1}M(2^{-k})f_j^k = \sum_{j\in\mathbb{Z}} A_{i-2j}^{(0)}M(2^{-k})f_j^k \ , \ i \in \mathbb{Z} \ . \tag{3.3}$$

Since $M(2^{-k}) = diag\{1, 2^{-k}, 2^{-2k}, \ldots, 2^{-(m-1)k}\}$ and the vectors $\{f_j^k\}$ are values on the continuous limit function $(f, f', \ldots, f^{(m-1)})^t$, the values $g_j^k = M(2^{-k})f_j^k$ are on the continuous function $(f, 0, \ldots, 0)^t$. Therefore, as in the scalar case, it follows that the constant vector data $(1, 0, \ldots, 0)^t$ is a fixed point of the process. It thus follows that the scheme reproduces constant polynomials. The original scheme generates the vector values $f_n^k = (f_k^{(0)}(n), f_k^{(1)}(n), \ldots, f_k^{(m-1)}(n)/(m-1)!)^t$ forming the limit vector function $f(t) = (f^{(0)}(t), f^{(1)}(t), \ldots, f^{(m-1)}(t)/(m-1)!)^t$. The vectors $v_k^{[1]}(n) = (2^k(f_k^{(0)}(n) - f_k^{(0)}(n-1)), f_k^{(1)}(n), \ldots, f_k^{(m-1)}(n)/(m-1)!)^t$ form the limit vector function $v^{[1]}(t) =$

$(f^{(1)}(t), f^{(1)}(t), \ldots, f^{(m-1)}(t)/(m-1)!)^t$. By Theorem 1, since Π_0 is reproduced, the vectors $\{v_k^{[1]}(n)\}$ are generated by a subdivision process defined by the finite Laurent polynomial matrix

$$D_k^{[1]}(z) = \Gamma_{k+1}^{m,1}(z) A_k(z) (\Gamma_k^{m,1}(z^2))^{-1} \ .$$

Using this form and the form of $\Gamma_k^{m,1}(z)$ it can be shown here that the scheme $D_k^{[1]}$ is also stationary, but here the normalization matrix is

$$M^{(1)}(2^{-k}) = diag\{1, 1, 2^{-k}, 2^{-2k}, \ldots, 2^{-(m-2)k}\} \ ,$$

i.e., $D_k^{[1]} = M^{(1)}(2^{-k})^{-1} D_0^{[1]} M^{(1)}(2^{-k})$. Normalizing the process generating $\{v_k^{[1]}(n)\}$, and using the assumption that the limit vector function $v^{[1]}(t)$ is continuous, we now obtain that the constant vector data $(1, 1, 0, \ldots, 0)^t$ is reproduced by the subdivision defined by $D_k^{[1]}$. Since this vector data is the data $\{v_k^{[1]}(n)\}$ corresponding to the linear data $\{f_n^k = n2^{-k}\}$, itt thus follows that the original scheme reproduces Π_1. We have thus completed the first step of the iterative proof and we are ready to prove the general step:

For $1 < r < m$ we assume that we have a finite polynomial $D_k^{[r]}(z)$ defining the subdivision scheme generating the data $\{v_k^{[r]}(n)\}$. Normalizing the process $\{D_k^{[r]}\}$ with

$$M^{(r)}(2^{-k}) = diag\{1, \ldots, 1, 2^{-k}, 2^{-2k}, \ldots, 2^{-(m-r-1)k}\} \ ,$$

with the 1 repeated $r+1$, and using the continuity of $f^{(r)}$, it follows that

$$v_k^{[r]} \equiv (1, 1, \ldots, 1, 0, 0, \ldots, 0)^t \ ,$$

with the 1 repeated $r+1$ times, is a fixed point of the process. Therefore, it can be concluded that the original process reproduces Π_r, and Theorem 1 can be used again to establish the existence of $D_k^{[r+1]}(z)$.

The symbols of the subdivision schemes generating divided differences $u_k^{[r]}$ of order $m \le r \le s+1$, are independent of the level k, and are recursively given by

$$D^{[r+1]}(z) = 2(\Lambda^{m,r+1})^{-1} \Gamma^m(z) D^{[r]}(z) (\Gamma^m(z^2))^{-1} \Lambda^{m,r} \ . \tag{3.4}$$

These schemes exist and are of finite support by the same arguments used in Theorem 1, and the above inductive process can thus be continued. Finally, the continuity of $f^{(s)}$ implies the existence of $D_k^{[s+1]}$. ∎

Theorem 3. *Consider a stationary Hermite-interpolatory subdivision scheme of order* m*, of the form (1.1), with* $A_i^{(k)}$ *satisfying (1.7). The scheme is* C^s *if and only if the subdivision scheme for the refinement of the divided differences of orders* s *is converging, or equivalently, the scheme for the differences of the divided differences of orders* s *is contractive.*

To prove the above Theorem we need the following lemma on divided differences:

Lemma 4. *Consider the set of distinct points $X = \{x_i\}_{i\geq 0} \subset I\!R$, and another set of points $T = \{t_j\}$ such that*

$$t_j = x_i \ , \quad m_i \leq j < m_{i+1} \ ,$$

where $m_0 < m_1 < ... < m_i < m_{i+1} < ....$ Then the divided difference operators of order k on the set X can be expressed as a finite combination of the divided difference operators of order k on the set T. In particular,

$$[x_0, x_1, \ldots, x_k]f = \sum_j w_j \ [t_j, t_{j+1}, \ldots, t_{j+k}]f \ , \quad \sum_j w_j = 1 \ . \tag{3.5}$$

We note that in the case of equidistant points X and uniform multiplicity the combination is even convex, $w_j \geq 0$.

Proof: The proof is by induction on k. For $k = 1$ the result is obvious. Let us assume that (3.5) holds, together with

$$[x_1, x_2, \ldots, x_{k+1}]f = \sum_j v_j \ [t_j, t_{j+1}, \ldots, t_{j+k}]f \ , \quad \sum_j v_j = 1 \ . \tag{3.6}$$

Hence, it follows that

$$[x_1, x_2, \ldots, x_{k+1}]f - [x_0, x_1, \ldots, x_k]f =$$

$$\sum_j c_j([t_{j+1}, t_{j+2}, \ldots, t_{j+k+1}]f - [t_j, t_{j+1}, \ldots, t_{j+k}]f) \ , \tag{3.7}$$

with $c_j = \sum_{i=i_0}^{j}(v_i - w_i)$. Since both $\{w_j\}$ and $\{v_j\}$ sum up to 1 and are of finite support, then $\{c_j\}$ are also of finite support. The terms in the right hand side of (2.22) with $t_{j+k+1} = t_j$ are zero and can be removed from the sum. Now each term in the right hand side can be divided by $t_{j+k+1} - t_j$ and the left hand side by $x_{k+1} - x_0$ to give a representation relating the finite differences of order $(k + 1)$

$$[x_0, x_1, \ldots, x_{k+1}]f = \sum_j d_j \ [t_j, t_{j+1}, \ldots, t_{j+k+1}]f \ . \tag{3.8}$$

That $\sum_j d_j = 1$ follows from the application of (3.8) to $f(x) = x^{k+1}$. ∎

Proof of Theorem 3:

The proof here follows the same lines of the proof of a similar result for the scalar case in [1], [8] and [11]. The only difference is in the step of proving that f is C^s if the differences of the divided differences of orders s tend uniformly to zero. Here the divided differences involve also derivatives' values, being based on a set of points of multiplicity m. Lemma 4 provides us the missing link since it says that any simple divided difference can be expressed as a finite linear combination of divided differences of the same order, based on points with multiplicity. The same holds for differences of divided differences, and thus we can conclude that if the scheme for the differences of the divided differences of orders

s is contractive then the original scheme is C^s. The existence of that scheme follows by Theorem 2, and its matrix Laurent polynomial is given by

$$\widehat{D}^{[s]}(z) = \Gamma^m(z) D^{[s]}(z) (\Gamma^m(z^2))^{-1} \ . \tag{3.9}$$

■

Theorem 3, together with the recursive relations (2.7) and (3.4) and the definition (3.9) of $\widehat{D}^{[s]}$, form the basis for a smoothness checking algorithm for stationary Hermite-interpolator schemes.

References

[1] Cavaretta, A.S., W. Dahmen and C.A. Micchelli, *Stationary Subdivision*, Memoirs of AMS, No. 453, 1991.

[2] Cohen, A., Daubechies I. and Plonka G., On the regularity of refinable vectors, submitted to Journal of Fourier Analysis and Applications, 1995.

[3] Donoho, D. L., Dyn, N., Levin, D. Yu T., Smooth Multiwavelet Duals of Alpert Bases By Moment-Interpolating Refinement and Applications in Recursive Partitioning, Preprint, 1996.

[4] Cohen, A., Dyn, N. and Levin, D., Stability and inter-dependence of matrix subdivision schemes, To appear in "Advanced Topics in Multivariate Approximation", F. Fontanella, K. Jetter, P.J. Laurent (eds.), 1996.

[5] Daubechies, I. and C. Lagarias, Two scale difference equations II. Local regularity, infinite products of matrices and fractals, SIAM J. Math. Anal. **23** (1992), 1031-1079.

[6] Deslauriers, G. and S. Dubuc, Interpolation dyadique, in *Fractals, dimensions non entieres et applications*, G. Cherbit (ed.), Masson, Paris, 1989, 44-55.

[7] Dubuc, S., Interpolation through an iterative scheme, J. Math. Anal. and Appl. **114** (1986), 185-204.

[8] Dyn, N., Subdivision schemes in computer-aided geometric design, in *Advances in Numerical Analysis II, Wavelets, Subdivision algorithms, and Radial Basis Functions*, W.A. Light (ed.), Clarendon Press, Oxford, 1992, 36-104.

[9] Dyn, N., J.A. Gregory and D. Levin, A four-point interpolatory subdivision scheme for curve design, CAGD **4** (1987) , 257-268.

[10] Dyn, N., J.A. Gregory and D. Levin, Analysis of linear binary subdivision schemes for curve design, Constr. Approx. **7** (1991), 127-147.

[11] Dyn, N. and D. Levin, Interpolating subdivision schemes for the generation of curves and surfaces, in *Multivariate Interpolation and Approximation*, W. Haussmann and K.Jetter (eds.), Birkhauser Verlag, Basel, 1990, 91-106.

[12] Dyn, N. and D. Levin, Analysis of Hermite-type subdivision schemes, in *Approximation Theory* **VIII**, Charles K. Chui and Larry L. Schumaker (eds.), World Scientific Publishing, 1995.

[13] Goodman, T. N. T. and Lee, S. L., Wavelets of multiplicity r, Trans. Amer. Math. Soc. **342** (1994) 307-324.

[14] Merrien, J.L., A family of Hermite interpolants by bisection algorithms, Numerical Algorithms **2** (1992) 187-200.

[15] Plonka, G., Approximation order provided by refinable function vectors, preprint, Universität Rostock, 1995.

[16] Shen, Z., Regularity and stability of refinable vectors, preprint Singapore University, 1995.

[17] Strang, G. and Strela, V., Orthogonal Multiwavelets with vanishing moments, SPIE Proceedings, Orlando 1994, Optical Engineering.