Refining Oscillatory Signals by Non-Stationary Subdivision Schemes

N. Dyn, D. Levin, A. Luzzatto

Abstract

The paper presents a method for refining real highly oscillatory signals. The method is based upon interpolation by a finite set of trigonometric basis functions. The set of trigonometric functions is chosen (identified) by minimizing a natural error norm in the Fourier domain. Both the identification and the refining processes are computed by linear operations. Unlike the Yule-Walker approach, and related algorithms, the identification of the approximating trigonometric space is not repeated for every new input signal. It is rather computed off-line for a family of signals with the same support of their Fourier transform, while the refinement calculations are done in real-time. Statistical estimates of the point-wise errors are derived, and numerical examples are presented.

Introduction

Given a series of equidistant samples of a band limited real signal we would like to evaluate samples of the signal at intermediate points. I.e., we look for an approximation to a series of samples with a double sampling rate. We approach this problem as an interpolation problem, and the basic issue is to find an appropriate space of basis functions for interpolation. A naive local polynomial interpolation is not appropriate here since the signals may be highly oscillatory. It seems rather natural to try to interpolate by using trigonometric basis functions, and the main issue is to identify the optimal set of trigonometric functions which are suitable for local interpolation of the given signal at the given sampling rate. Based upon a previous work by the authors [3], we look for an 'orthogonal scheme', which is a scheme with the property of annihilating any signal taken from a related space of exponentials. Finding such an orthogonal scheme, following [3], is of course enough for identifying the space of exponentials. Moreover, by solving a linear system, this scheme also provides us with the rule for local interpolation by functions in the identified space.

Identifying the optimal orthogonal scheme for a given signal at a certain sampling rate is the same as identifying the optimal linear predictor. However, unlike the Yule-Walker approach to finding a linear predictor, which involves the solution of a linear system which varies with each portion of a signal, we are looking for an orthogonal scheme which will suit a class of signals with the same support Ω of their Fourier transforms. The computation of such a scheme may thus be done off-line, once and for all the signals in the prescribed class.

Both the orthogonal scheme and the related interpolating scheme are represented by associated Laurent polynomials. Assuming that the signals share the same support of their Fourier transforms, the actual algorithm for finding the optimal orthogonal scheme involves minimizing a certain weighted L_2- norm of the associated Laurent polynomial. It is shown that there exists always a minimizing Laurent polynomial with a full set of simple roots. This implies that the related space of exponentials is non-degenerate, and defines a space of trigonometric functions with frequencies in Ω.

In the rest of the paper we present estimates for the error resulting in refining signals by the new interpolating scheme, and we conclude with numerical experiments.

1 Preliminaries

In this paper we are interested in approximating signals by functions in finite dimensional spaces of exponential polynomials of the form

$$E_\ell^c = \{f : \mathbb{R} \to \mathbb{C},\ f \in C^\ell(\mathbb{R}),\ \sum_{n=0}^{\ell} c_n D^n f = 0\}\ , \tag{1.1}$$

where $c = (c_0, \ldots, c_\ell) \in \mathbb{R}^{\ell+1}$, and where D^n denotes differentiation of order n.

We can write explicitly the space E_ℓ^c in terms of the distinct roots of the polynomial $c(z) = \sum_{n=0}^{\ell} c_n z^n$ with their multiplicities $\{\gamma_n, \mu_n\}_{n=1}^{\nu}$, such that $\sum_{n=1}^{\nu} \mu_n = \ell$, as

$$E_\ell^c = \text{span}\{x^r \exp(\gamma_n x),\ r = 0, 1, \ldots, \mu_n - 1,\ n = 1, \ldots, \nu\}\ . \tag{1.2}$$

In the following we use the notation $\gamma_n = \alpha_n + i\beta_n,\ n = 1, \ldots \nu$. For oscillatory signals the relevant spaces are defined in terms of the parameters

$$\ell = 2m,\ \nu = \ell,\ \gamma_j = i\beta_j,\ \gamma_{j+m} = \bar{\gamma}_j = -i\beta_j,\ j = 1, \ldots, m\ . \tag{1.3}$$

Such spaces are termed here *strictly oscillatory.*

For a given space E_ℓ^c, there exists a reproducing subdivision scheme of minimal rank [3]. Such a scheme is an interpolatory subdivision scheme, which consists of reproducing refinement rules of minimal rank $\{R_k\}_{k\geq k_0}$. The rule R_k maps samples of any signal f from the space E_ℓ^c at refinement level k, $f^k = \{f_j^k = f(2^{-k}j) \ : \ j \in \mathbf{Z}\}$, to its samples at the next refinement level, namely at the points $2^{-(k+1)}\mathbf{Z}$ in the following way,

$$f_{2j}^{k+1} = f_j^k, \quad f_{2j+1}^{k+1} = \sum_{n=-M}^{N-1} a_{2n+1}^{[k]} f_{j-n}^k, \ j \in \mathbf{Z}, \tag{1.4}$$

where $M = [\frac{\ell}{2}]$, $N = [\frac{\ell+1}{2}]$. The operation in equation (**??**) is written formally as $f^{k+1} = R_k f^k$. The symbol of the rule R_k is defined as $a^{[k]}(z) = \sum_{n=-M}^{N-1} a_{2n+1}^{[k]} z^{2n+1} + 1$. The following theorem characterizes the symbol of the rule R_k [2]. It is used later for the derivation of a linear system defining this rule.

Theorem 1.1. *The symbol $a^{[k]}(z)$ is the symbol of a reproducing refinement rule of minimal rank at level k for a space E_ℓ^c of the form (**??**), if and only if*

$$D^r a^{[k]}(z_{\gamma_n,k+1}) = 2\delta_{r,0}, \ \ D^r a^{[k]}(-z_{\gamma_n,k+1}) = 0, \ \ r = 0,\ldots,\mu_n-1, \ n = 1,\ldots,\nu, \tag{1.5}$$

with $z_{\gamma_n,k} = \exp(-2^{-k}\gamma_n)$, $n = 1, \ldots \nu$.

There might be values of k for which a reproducing refinement rule of minimal rank (of the form (**??**)) does not exist. Yet, such rules exist if k is large enough [3]. There is a sufficient condition for the existence and uniqueness of the reproducing refinement rule of minimal rank at level k.

Theorem 1.2. *For a space E_ℓ^c of the form (**??**), if for any pair $m, n \in \{1, \ldots, \nu\}$ such that $m \neq n$ and $\alpha_n = \alpha_m$, we have*

$$\beta_n - \beta_m \neq 2^{k+1} s\pi, \quad s \in \mathbf{Z} \tag{1.6}$$

then a unique reproducing refinement rule of minimal rank at level k exists.

Remark

If

$$\beta_n < 2^{k_0}\pi, \ n = 1, \ldots, \nu, \tag{1.7}$$

then by Theorem 1.2 there exists at each level $k \geq k_0$ a unique reproducing refinement rule of minimal rank for the space E_ℓ^c, and therefore there exists a unique minimal rank reproducing scheme for $k \geq k_0$.

In the case of a strictly oscillatory space E^c_{2m}, conditions (**??**) on $\{\beta_n \ : \ n = 1\dots,2m\}$ guarantee that the sampling rate of 2^k samples per unit at level k is above the Nyquist rate [5].

The minimal rank reproducing scheme of a space E^c_ℓ, can refine exactly samples of a signal from that space. Yet, for a given signal in a space E^c_ℓ , the space or its minimal rank reproducing scheme have first to be identified. The tool for a suboptimal identification of an E^c_ℓ space, given a signal from this space with or without added small noise, is the notion of orthogonal schemes [3, 4]. A minimal rank orthogonal scheme for E^c_ℓ consists of orthogonal refinement rules of minimal rank $\{O_k \ : \ k \geq 0\}$ such that O_k refines and maps to zero the samples of any signal from E^c_ℓ at level k, in the following way,

$$f^{k+1}_{2j+1} = 0, \quad f^{k+1}_{2j} = \sum_{n=-N}^{M} p^{[k]}_{2n} f^k_{j-n} = 0, \tag{1.8}$$

where $N = [\frac{\ell}{2}]$, $M = [\frac{\ell+1}{2}]$ as before. Thus, the symbol of an orthogonal refinement rule of minimal rank at level k has the form

$$p^{[k]}(z) = \sum_{n=-N}^{M} p^{[k]}_{2n} z^{2n}. \tag{1.9}$$

For an orthogonal refinement rule of minimal rank there exists a characterization of its symbol [3].

Theorem 1.3. *The symbol $p^{[k]}(z)$ is the symbol of an orthogonal refinement rule of minimal rank at level k for the space E^c_ℓ of the form (**??**), if and only if*

$$D^r p^{[k]}(\pm z_{\gamma_n,k+1}) = 0, \quad r = 0,\dots,\mu_n - 1, \ n = 1,\dots,\nu, \tag{1.10}$$

with $z_{\gamma_n,k} = \exp(-2^{-k}\gamma_n), \ n = 1,\dots \nu$.

Note that in contrast to Theorem 1.1, from which the explicit form of the symbol of the reproducing refinement rule of minimal rank cannot be deduced, this theorem implies that up to a multiplicative constant (normalization), the symbol of O_k has the form,

$$z^{-2N}\Pi^{\nu}_{n=1}(1 - exp(-2^{-k}\gamma_n)z^2)^{\mu_n}. \tag{1.11}$$

When the multiplicative constant is real, we see from (**??**) that the coefficients of the symbol of O_k are real.

Hereafter we term a refinement rule with real coefficients in its symbol as a real refinement rule.

The existence of an orthogonal refinement rule of minimal rank at any level k is a consequence of Theorem 1.3. The following Theorem establishes its uniqueness and the existence, up to normalization, under conditions of the type (**??**) [3].

Theorem 1.4. *An orthogonal refinement rule of minimal rank at level k (of the form (**??**)) exists and is unique up to normalization, if the conditions in Theorem 1.2 hold for $k-1$.*

Remark

Note that by the last theorem, the conditions of Theorem 1.2 on the structure of an E_ℓ^c space, guarantee the existence and uniqueness of a reproducing refinement rule of minimal rank at level k, and of a unique orthogonal refinement rule of minimal rank at level $k+1$, up to normalization factor. We assume hereafter that the free coefficient of the symbol of the orthogonal refinement rule is not zero, and thus it may be normalized so that $p_0^{[k]} = 1$.

For a given space E_ℓ^c, there is a simple relation between the symbols of the reproducing refinement rule of minimal rank at level k and the orthogonal refinement rule of minimal rank at level $k+1$ with free coefficient 1, as can be deduced from Theorems 1.1, 1.3 [3, 4].

Theorem 1.5. *Let $a^{[k]}$ be the symbol of a reproducing refinement rule of minimal rank at level k for a space E_ℓ^c, and let $p^{[k+1]}$ with $p_0^{[k+1]} = 1$ be the symbol of an orthogonal refinement rule of minimal rank at level $k+1$ for the same E_ℓ^c space. Then*

$$a^{[k]}(z^2) - 2 = p^{[k+1]}(z) q^{[k]}(z), \tag{1.12}$$

where $q^{[k]}(z) = \sum_{n=-(N-1)}^{M-1} q_{2n}^{[k]} z^{2n}$. The coefficients of $q^{[k]}$ are determined uniquely as the solution of the linear system

$$\sum_{n=-(N-1)}^{M-1} p_{2(2j-n)}^{i[k+1]} q_{2n}^{[k]} = -\delta_{j,0}, \quad j = -(N-1), \dots, M-1. \tag{1.13}$$

It follows from the last theorem that once the symbol of the orthogonal refinement rule of minimal rank with free coefficient 1 at level $k+1$ is known for an E_ℓ^c space, then the reproducing refinement rule of minimal rank at level k

can be obtained by solving a non-singular linear system, and that this scheme is real, since the coefficients of the given symbol are real.

In the next section we discuss how to choose the orthogonal refinement rule of minimal rank at level k, based on some knowledge about the Fourier transform of the signal.

2 Choosing the Approximating Exponential Space

We now turn to one of the main issues of this paper, namely the identification of an approximating space E_ℓ^c for a given band-limited signal.

To be able to make a valuable choice, we assume prior knowledge about the Fourier transform of the signal f, namely, knowledge of $|\widehat{f}|$, or at least the support of $\widehat{f}$. Both cases may appear in real applications, and we shall start with the first, namely, we assume that $|\widehat{f}|$ is known.

We shall identify the space E_ℓ^c through identifying the orthogonal scheme $p^{[k]}$, annihilating k-level samples of functions in E_ℓ^c. With the special normalization $p_0^{[k]} = 1$, the quantity

$$e_j = \sum_{n=-N}^{M} p_{2n}^{[k]} f_{j-n}^k \ , \tag{2.1}$$

may be viewed as the approximation error in predicting the value f_j^k by interpolation based on a function in E_ℓ^c, interpolating the values $\{f_{j-n}^k \ : \ 0 < |n| \le \ell\}$, at the points $\{2^{-k}(j-n) \ : \ 0 < |n| \le \ell\}$. Thus, we search for the orthogonal scheme $p^{[k]}$ that minimizes $\|\{e_j\}\|_\infty$.

Using the equality

$$e_j = \sum_{n=-N}^{M} p_{2n}^{[k]} f_{j-n}^k \ = \frac{1}{2\pi}\int_{-\infty}^{\infty} p^{[k]}(e^{-i2^{-k-1}\omega})\widehat{f}(\omega)e^{i\omega 2^{-k}j}d\omega \ , \tag{2.2}$$

and denoting by Ω the support of $\widehat{f}$, we have

$$\|\{e_j\}\|_\infty \le \frac{1}{2\pi}\int_\Omega |p^{[k]}(e^{-i2^{-k-1}\omega})||\widehat{f}(\omega)|d\omega \ . \tag{2.3}$$

Note that Ω is symmetric since f is real.

Let us identify the r.h.s. of (**??**) as a weighted L_1 norm of the polynomial $p^{[k]}$, and denote it by $\|p^{[k]}\|_{1,\widehat{f}}$. We also have

$$\|p^{[k]}\|_{1,\widehat{f}} \leq |\Omega|^{\frac{1}{2}} \|p^{[k]}\|_{2,\widehat{f}} \equiv \frac{|\Omega|^{\frac{1}{2}}}{2\pi} \{ \int_{\Omega} |p^{[k]}(e^{-i2^{-k-1}\omega})|^2 |\widehat{f}(\omega)|^2 d\omega \}^{\frac{1}{2}} \ , \qquad (2.4)$$

where $|\Omega|$ is the measure of Ω.

The relation (**??**) implies that, in the case of prior knowledge about $\widehat{f}$, one would like to define the orthogonal scheme $p^{[k]}$ by minimizing $\|p^{[k]}\|_{1,\widehat{f}}$. Yet, since it is much simpler to minimize $\|p^{[k]}\|_{2,\widehat{f}}$, we define $p^{[k]}$ to be the minimizer of the $|\widehat{f}(\omega)|^2$ weighted L_2 norm. Let us further assume that $\ell = 2N$, that Ω contains at least ℓ distinct points and that $\Omega \subset 2^{k-1}(-\pi, \pi)$.

The following results are central in the characterization of the zeros of $p^{[k]}$ which minimizes $\|p^{[k]}\|_{2,\widehat{f}}$.

Lemma 2.1. *The polynomial $p^{[k]}$, of the form (**??**) with $p_0^{[k]} = 1$, minimizing $\|p^{[k]}\|_{2,\widehat{f}}$, is a symmetric polynomial. Namely, $p_{2n}^{[k]} = p_{-2n}^{[k]}$, $n = 1, .., N$.*

Proof. Setting

$$c_n = p_{2n}^{[k]} + p_{-2n}^{[k]} \ , \quad d_n = p_{2n}^{[k]} - p_{-2n}^{[k]} \ , \qquad (2.5)$$

we get

$$p^{[k]}(e^{-i2^{-k-1}\omega}) = \eta(\omega) - i\phi(\omega) \ , \qquad (2.6)$$

where

$$\eta(\omega) = 1 + \sum_{n=-N}^{N} c_n cos(2^{-k}\omega n) \ , \quad \phi(\omega) = \sum_{n=-N}^{N} d_n sin(2^{-k}\omega n) \ . \qquad (2.7)$$

Consequently we have

$$\|p^{[k]}\|_{2,\widehat{f}}^2 = \frac{1}{2\pi} \int_{\Omega} (|\eta(\omega)|^2 + |\phi(\omega)|^2) |\widehat{f}(\omega)|^2 d\omega \ . \qquad (2.8)$$

It thus follows that for the minimizing polynomial $\phi(\omega) \equiv 0$, which is equivalent to the claim of the Lemma. □

Remark

Under the conditions of Lemma 2.1, it is possible to show by its method of proof that if $\ell = 2N + 1$ then $p_{2N+2}^{[k]} = 0$.

Thus the minimizer of $\|p^{[k]}\|_{2,\widehat{f}}$ has the form

$$p^{[k]}(z) = 1 + \sum_{n=1}^{N} p_{2n}^{[k]}(z^{2n} + z^{-2n}) \ , \quad N = [\frac{\ell}{2}] \ .$$

With the substitution $z = \exp(-i2^{-k-1}\omega)$, the above polynomial becomes

$$p^{[k]}(\exp(-i2^{-k-1}\omega) = \tilde{p}^{[k]}(\omega) = 1 + \sum_{n=1}^{N} p_{2n}^{[k]} cos(2^{-k}n\omega) \ . \qquad (2.9)$$

Lemma 2.2. *The system $\{cos(j\cdot)\}_{j=1}^{N}$ is a Haar system on $[0, \frac{\pi}{2})$.*

By the substitution $x = \cos t$, this lemma is equivalent to the result,

Lemma 2.3. *The system of Chebyshev polynomials $\{T_j\}_{j=1}^{N}$ is a Haar system on $(0, 1]$.*

Proof. It is sufficient to show that any linear combination of the elements of the system $\{T_j\}_{j=1}^{N}$ can have at most $N-1$ zeros in $(0, 1]$. To prove this we will show that any algebraic polynomial of degree at most N with N zeros in $(0, 1]$, has the representation $\sum_{j=0}^{N} t_j T_j(x)$, with $t_0 \neq 0$.

Let p be a polynomial of degree N with N zeros in $(0, 1]$. Then $p(x) = C\pi_{j=1}^{N}(x - \xi_j)$, with $\xi_j \in (0, 1]$, $j = 1, \ldots, N$, and C a constant. It is easy to conclude from this form of p that the sign of $p(x)$, $x \in (-1, 0)$ is fixed and that $|p(x)| > |p(-x)|$, $x \in (-1, 0)$. Thus $|\int_{-1}^{0} p(x)(1-x^2)^{-\frac{1}{2}}dx| > |\int_{0}^{1} p(x)(1-x^2)^{-\frac{1}{2}}dx|$, and

$$t_0 = \int_{-1}^{1} p(x)(1-x^2)^{-\frac{1}{2}}dx \neq 0.$$

□

With Lemma 2.2 proved, we can use ideas from the theory of orthogonal algebraic polynomials, in order to conclude that,

Theorem 2.4. *There exists a unique polynomial $p^{[k]}$ of the form (**??**) with $p_0^{[k]} = 1$, minimizing $\|p^{[k]}\|_{2,\widehat{f}}$. The roots of $p^{[k]}$ are simple and are of the form $\{\pm \exp(\pm i2^{-k-1}\beta_q)\}_{q=1}^{N}$ with $\{\beta_q\}_{q=1}^{N} \subset \Omega$.*

proof. First we observe that the minimizing polynomial should be symmetric. Next, we define the following inner product on the space of symmetric trigonometric polynomials.

$$< p, q >_{\widehat{f}} = \frac{1}{2\pi} \int_{\Omega} p(\omega)q(\omega)|\widehat{f}(\omega)|^2 d\omega \ . \qquad (2.10)$$

If $p^{[k]}$ with $p_0^{[k]} = 1$, minimizes $\|p^{[k]}\|_{2,\widehat{f}}$, then $\tilde{p}^{[k]}$ must satisfy the orthogonality conditions

$$< \tilde{p}^{[k]}, cos(2^{-k} j\cdot) >_{\widehat{f}} = 0 \ , \quad j = 1, ..., N \ . \tag{2.11}$$

Using Lemma 2.2, it follows from (**??**) that $\tilde{p}^{[k]}(\omega)$ must have exactly n simple roots in Ω. Otherwise, since $\Omega \subset 2^k[0, \frac{\pi}{2})$, we can find a linear combination of the functions $\{cos(2^{-k} j\omega)\}_{j=1}^N$ which has the same sign structure as $\tilde{p}^{[k]}(\omega)$ for $\omega \in \Omega$. This would imply a contradiction to the orthogonality conditions (**??**).

Now, since $\tilde{p}^{[k]}$ is a symmetric trigonometric polynomial its roots are of the form

$$\pm\beta_q \ , \ \beta_q \in \Omega \ , \ q = 1, \ldots, N \ .$$

Hence by (**??**) the roots of $p^{[k]}(z)i$, which is an even polynomial, are of the form $\{\pm \exp(\pm i 2^{-k-1}\beta_q)\}_{q=1}^N$ with $\{\beta_q\}_{q=1}^N \subset \Omega$.
□

A direct consequence of Theorems 2.4 and 1.3 is:

Corollary 2.5. *The space E_ℓ^c determined by the $2N$ roots of the polynomial $p^{[k]}$ minimizing $\|p^{[k]}\|_{2,\widehat{f}}$ is strictly oscillatory.*

The space in the last Corollary is denoted hereafter by $E_\ell^{c^*}$ with $c^* = c^*(|\widehat{f}|, k, \ell)$.

In the case that only the support Ω of the spectrum is known, we are restating the problem as follows:

Consider the set F_Ω of all signals f having a Fourier transform with support contained in Ω, and such that $|\widehat{f}| \leq 1$. We would like to find ian optimal E_ℓ^c space, corresponding to a minimal rank orthogonal rule at level k, which is as closely orthogonal as possible to the k-level samples of all signals in F_Ω. Using (**??**) we observe that for any given polynomial $p^{[k]}$ we can find a function in F_Ω such that

$$\|\{e_j\}\|_\infty = \frac{1}{2\pi} \int_\Omega |p^{[k]}(e^{-i2^{-k-1}\omega})| d\omega \ . \tag{2.12}$$

This implies that

$$\inf_{p^{[k]}} \{ \sup_{f \in F_\Omega} \|\{e_j\}\|_\infty \} = \inf_{p^{[k]}} \{ \frac{1}{2\pi} \int_\Omega |p^{[k]}(e^{-i2^{-k-1}\omega})| d\omega \} = \inf_{p^{[k]}} \{ \|p^{[k]}\|_{1,\chi_\Omega} \} \ , \tag{2.13}$$

where χ_Ω is the characteristic function of Ω. Therefore, the optimal solution to the above stated problem is to find the polynomial $p^{[k]}$ minimizing $\|p^{[k]}\|_{1,\chi_\Omega}$. Here again, for practical reasons, we shall be content with a sub-optimal solution defined by minimizing $\|p^{[k]}\|_{2,\chi_\Omega}$.

3 Refining Oscillatory Signals

The problem we solve with the tools presented in the previous sections is:

Given in real-time consecutive samples at level k of a zero-mean real signal $f \in L^2(\mathbb{R})$

$$f^k = \{f_j^k = f(2^{-k}j) \ : \ j \in J\}, \tag{3.14}$$

and a compact set $\Omega \in \mathbb{R}$, which is the support of $\hat{f}$,

compute in real-time values

$$\tilde{f}^{k+1} = \{\tilde{f}_j^{k+1} \ : \ \ j \in I\}$$

approximating

$$f_j^{k+1} = f(2^{k+1}j) \ : \ \ j \in I\},$$

with I a finite set of consecutive indices in $\mathbf{Z}$, which is almost the set $2J$. This problem is termed hereafter *the spectral refinement problem.*

In view of the results in the previous sections the procedure we suggest for solving the spectral refinement problem is:

1. Choose $\ell = 2N$ (the dimension of the approximating exponentials space. The bigger ℓ is the more accurate are the results).
2. Compute $p^{[k+1]}$ by minimizing $\|p^{[k+1]}\|_{2,\chi_\Omega}$.
3. Compute the symbol of R_k, $a^{[k]}$, from $p^{[k+1]}$, by solving a linear system.
4. Compute in real-time $\tilde{f}^{k+1} = R_k f^k$.

The procedure above is termed *spectral refinement* or *spectral interpolation.* Next, we analyze the error in our procedure, under a reasonable statistical assumption, and bound the variance of the point-wise error.

4 Statistical error analysis

In this section we present an error analysis based upon a statistical assumption on the signal, which is valid in many communication applications. The statistical assumption at level k on $f \in L^2(\mathbb{R})$ with zero mean and with Fourier transform of compact support Ω, is:

Any set of consecutive samples of f at level k, w.l.o.g.

$$f_j^k = f(2^{-k}j) \ , \quad j = -N, \ \dots \ , N + L \ ,$$

for k such that $2^{-k}\Omega \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$, and for large L, can be written as

$$f_j^k = g(2^{-k}j) + \nu_j^k \ , \quad j = -N, \ \dots \ , N + L \ , \tag{4.1}$$

with $g \in E_\ell^{c^*}$, $c^* = c^*(\chi_\Omega, k, \ell)$, $\ell = 2N$, and where $\{\nu_j^k : j = -N, ..., N+L\}$ are samples of an ergodic random process which are independent Gaussian random variables with zero mean and variance σ^2.

Under this statistical assumption we can bound σ.

Lemma 4.1. *Under the statistical assumptions at level $k+1$,*

$$\sigma \le \|p^{[k+1]}\|_{2,\chi_\Omega} \|f\|_2 + o(1) \ , \ as \ L \to \infty \ , \tag{4.2}$$

where $p^{[k+1]}(z)$ is the symbol of the orthogonal rule of minimal rank for $E_\ell^{c^*}$, $c^* = c^*(\chi_\Omega, k+1, \ell)$, with a free coefficient 1.

Proof. Denote $p^{[k+1]}(z) = 1 + \sum_{n=-N}^{N} p_{2n}^{[k+1]} z^{2n}$. Then for $j \in \{0, 1, ..., L\}$, by (**??**) and since $g \in E_\ell^{c^*}$,

$$\sum_{n=-N}^{N} p_{2n}^{[k+1]} f_{j-n}^{k+1} = \sum_{n=-N}^{N} p_{2n}^{[k+1]} \nu_{j-n}^{k+1} \ . \tag{4.3}$$

Defining the random variables $\xi_j = \sum_{n=-N}^{N} p_{2n}^{[k+1]} \nu_{j-n}^{k+1}$, $j \in \{0, 1, ..., L\}$, we observe that each ξ_j has zero mean, and

$$var(\xi_j) = \sigma^2 \sum_{n=-N}^{N} (p_{2n}^{[k+1]})^2 \ . \tag{4.4}$$

By the ergodic assumption, and since L is large,

$$var(\xi_j) = \frac{1}{L+1} \sum_{j=0}^{L} \xi_j^2 + o(1) \ , \tag{4.5}$$

while by (**??**) and (**??**)

$$\xi_j = \frac{1}{2\pi} \int_\Omega p^{[k+1]}(e^{-i2^{-k-2}\omega}) \widehat{f}(\omega) e^{i\omega 2^{-k-1}j} d\omega \ , \tag{4.6}$$

and therefore

$$|\xi_j|^2 \leq \|p^{[k+1]}\|^2_{2,\chi_\Omega} \|\widehat{f}\|^2_2 \ . \tag{4.7}$$

Then, by (**??**)

$$var(\xi_j) \leq \|p^{[k+1]}\|^2_{2,\chi_\Omega} \|\widehat{f}\|^2_2 + o(1) \ , \tag{4.8}$$

and since $\sum_{n=-N}^{N}(p_{2n}^{[k+1]})^2 \geq 1$, we conclude (**??**) from (**??**) and (**??**). □

Without loss of generality we bound the error incurred by step 4 of our procedure, namely the variance of the point-wise error

$$\epsilon_j^{k+1} = \tilde{f}_j^{k+1} - f_j^{k+1} \ , \quad j \in I = \{0, 1, \ldots, 2L\}.$$

The same bound applies everywhere. Observe that under the statistical assumption at level $k+1$

$$f_j^{k+1} = g(2^{-k-1}j) + \nu_j^{k+1} \ , \quad j = -N, \ldots, N+L \ , \tag{4.9}$$

with $g \in E_\ell^{c^*}$, $c^* = c^*(\chi_\Omega, \ell, k)$, and with $var(\nu_{2j+1}^{k+1}) = \sigma^2$, where σ is bounded by (**??**). Since the operation $\tilde{f}^{k+1} = R_k f^k$ is described by (**??**), step 4 generates samples at level $k+1$ of the form,

$$\tilde{f}_{2j}^{k+1} = f_j^k, \quad \tilde{f}_{2j+1}^{k+1} = \sum_{n=-N}^{N-1} a_{2n+1}^{[k]} f_{j-n}^k, \ 2j, 2j+1 \in I \ , \tag{4.10}$$

we have $\epsilon_{2j}^{k+1} = 0$, $2j \in I$, and by the statistical assumption at level $k+1$,

$$\epsilon_{2j+1}^{k+1} = \sum_{n=-N}^{N-1} a_{2n+1}^{[k]} f_{j-n}^k - f_{2j+1}^{k+1} = \sum_{n=-N}^{N-1} a_{2n+1}^{[k]} \nu_{2(j-n)}^{k+1} - \nu_{2j+1}^{k+1} \ .$$

Recalling the statistical properties of $\{\nu_j^{k+1}\}$ we get,

$$var(\epsilon_{2j+1}^{k+1}) = \sum_{n=-N}^{N-1} (a_{2n+1}^{[k]})^2 + 1)\sigma^2 \ , \quad 2j+1 \in I \ . \tag{4.11}$$

Finally, using Lemma **??**, we obtain the bound

$$(var(\epsilon_j^{k+1}))^{\frac{1}{2}} \leq (\sum_{n=-N}^{N-1} (a_{2n+1}^{[k]})^2 - 1)^{\frac{1}{2}} \|p^{[k+1]}\|_{2,\chi_\Omega} \|f\|_2 + o(1) \ , \quad j \in I. \tag{4.12}$$

with $o(1)$ small relative to L. Thus, in order to bound the variance of the point-wise error, we derive in the next section a bound on $\|p^{[k+1]}\|_{2,\chi_\Omega}$ in terms of Ω, ℓ, k.

5 Bounding $\|p^{[k]}\|_{2,\chi_\Omega}$

In this section we derive a bound on $\|p^{[k]}\|_{2,\chi_\Omega}$, in terms of Ω, ℓ, and k. This bound is then used in bounding the refinement error analyzed in the previous section.

Let us assume that $\Omega = [-b,-a] \cup [a,b]$, and that $\ell = 2N$. Since $p^{[k]}$ is a symmetric trigonometric polynomial, we may write

$$p^{[k]}(e^{-i2^{-k-1}\omega}) = 1 + 2\sum_{n=1}^{N} p_{2n}^{[k]} cos(2^{-k}n\omega) \ . \tag{5.1}$$

Substiuting $x = cos(2^{-k}\omega)$ we obtain an algebraic polynomial representation

$$p^{[k]}(e^{-i2^{-k-1}\omega}) = 1 + 2\sum_{n=1}^{N} p_{2n}^{[k]} T_n(x) \ , \ x = cos(2^{-k}\omega) \ , \tag{5.2}$$

where T_n is the nth degree Chebyshev polynomial of the first kind. As ω varies in Ω, $x \in [c_k, d_k] = [cos(2^{-k}b), cos(2^{-k}a)]$, and for large enough k, this is an interval of length $O(2^{-2k})$ near $x = 1$.

In order to bound $\|p^{[k]}\|_{2,\chi_\Omega}$, we recall its optimal property as the minimizer of $\|q\|_{2,\chi_\Omega}$, among all trigonometric polynomials q of the form (**??**), and bound $\|p^{[k]}\|_{2,\chi_\Omega}$ by $\|h^{[k]}\|_{2,\chi_\Omega}$, where $h^{[k]}$ is of the form (**??**). To obtain a trigonometric polynomial for which we can estimate its $\|\cdot\|_{2,\chi_\Omega}$-norm, we consider the Chebyshev polynomial of degree N transformed to the interval $[c_k, d_k]$,

$$g(x) = T_N\left(\frac{x - \frac{c_k+d_k}{2}}{\frac{d_k-c_k}{2}}\right) \ . \tag{5.3}$$

and normalize it to be a polynomial of the form (**??**), namely with a free coefficient 1 in its Chebyshev expansion,

$$\overline{g}(x) = g(x) / \int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} g(t)dt \equiv 1 + 2\sum_{n=1}^{N} g_{2n} T_n(x) \ . \tag{5.4}$$

Analysing the integral in (**??**), we have

$$\int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} g(t)dt \geq g(0)(1 + O(2^{-2k})) = \frac{2^{N-1}}{(\frac{d_k-c_k}{2})^N}(1 + O(2^{-2k})) \ . \tag{5.5}$$

Recalling that g is bounded by 1 in $[c_k, d_k]$, we obtain,

$$\|\overline{g}\|_{[c_k,d_k],\infty} = O((b^2 - a^2)^N 2^{-(2k+3)N}) \ , \tag{5.6}$$

as k and $\ell = 2N$ increase. Defining $h^{[k]}(x) = 1 + \sum_{n=1}^N g_{2n}(z^n + z^{-n})$, we have

$$\|p^{[k]}\|_{2,\chi_\Omega} \leq \|h^{[k]}\|_{2,\chi_\Omega} = O((b^2 - a^2)^N 2^{-(2k+3)N}) \ . \tag{5.7}$$

6 Numerical Examples

Here we present numerical experiments for a family of signals with an almost compact Fourier transform

$$f(t) = cos(2\pi Ft + \beta sin(2\pi F_s t)) \ . \tag{6.1}$$

This is a widely used family of frequency modulated signals [1]. A function of the form (**??**) is essentially of a band limited support, centered at $\omega = 2\pi F$, with a bandwidth $\Delta\omega = 2\pi(\beta + 1)F_s$. We applied the spectral interpolation algorithm to refine equidistant data values measured from such functions, and we compared the results achieved with local $\ell - 1$ degree polynomial interpolation. In the following figures we present in the upper part the spectral density of the signal, in the middle part the percentage of the error in local polynomial interpolation, and in the lower part the percentage of the error by the spectral interpolation algorithm. In Figure 1 $F = 0.1$, $F_s = 0.0062$ and $\beta = 5.75$, in Fugure 2 $F = 0.3$, $F_s = 0.0062$ and $\beta = 5.75$, and in Figure 3 $F = 0.3$, $F_s = 0.0062$ and $\beta = 2.375$. In all cases the spectral interpolation give superior results, and the advantage is larger when the bandwidth is smaller. This is in accordance with the error estimates developed above and the bound (**??**).

References

[1] K. Clarke and D. Hess, Communication Circuits, Analysis and Design, Addison Wesley, 1978

[2] N. Dyn and D. Levin, Subdivision schemes in geometric modeling, Acta Numerica, 73-144,(2002).

[3] N. Dyn, D. Levin and A. Luzzatto, Non-stationary interpolatory subdivision schemes reproducing spaces of exponential polynomials, To appear.

[4] A. Luzzatto, Multi-Scale Signal Processing based on Non-Stationary Subdivision, PhD Thesis, Tel-Aviv University, 2000

[5] M. Schwartz : *Information, Transmission, Modulation and Noise,* McGraw-Hill, 1990.

Address(es):

Nira Dyn & David Levin
Tel-Aviv University
School of Mathematical Sciences
Tel-Aviv 69978, Israel
Ariel Luzzatto
Motorola Communications Israel
3 Kremenetzky st.
Tel-Aviv 67899, Israel