A Construction of Bi-orthogonal Functions to B-splines with Multiple Knots

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Abstract

We present a construction of a refinable compactly supported vector of functions which is bi-orthogonal to the vector of B-splines of a given degree with multiple knots at the integers with prescribed multiplicity. The construction is based on Hermite interpolatory subdivision schemes, and on the relation between B-splines and divided differences. The bi-orthogonal vector of functions is shown to be refinable, with a mask related to that of the Hermite scheme. For simplicity of presentation the special (scalar) case, corresponding to B-splines with simple knots, is treated separately.

1. Introduction.

B-splines are important for applications [1], and those with integer knots provide explicit examples of a compactly supported refinable univariate function. Wavelets based on splines have received considerable treatment in the literature (see e.g. [4]). In this paper we show how to construct refinable compactly supported functions, that are bi-orthogonal to B-splines with simple or with multiple knots, using the refinable (scaling) functions generated by interpolatory subdivision schemes. Compactly supported bi-orthogonal functionals to B-splines built on an arbitrary sequence of knots (with possible multiplicities) are constructed in [2].

The case of B-splines with simple equidistant knots (the scalar case), is simpler and is treated here separately for the simplicity of the presentation. For this case the bi-orthogonal refinable function is generated by a "Lagrange type" interpolatory subdivision scheme [11], which uses function values at each control point. A different approach to the construction of bi-orthogonal refinable functions to B-splines with simple knots from the interpolatory schemes of Deslaurier and Dubuc [8] is mentioned in [14].

B-splines with multiple equidistant knots of the same multiplicity constitute a refinable vector of functions. Refinable vectors of spline functions are considered in [13]. For the case of B-splines with multiple knots we generate the bi-orthogonal vector of functions by "Hermite type" interpolatory subdivision schemes, which use values of a function and its derivatives at each control point to generate the same type of data in the next level [12]. For B-splines with knots of multiplicity μ, the corresponding Hermite interpolatory subdivision scheme is of order μ, namely a subdivision scheme which uses and computes the value of the function and the values of its first $\mu - 1$ derivatives at each control point. For information about subdivision schemes the reader is referred to [3] and [10].

The construction of the bi-orthogonal functions, is based on the relation between B-splines and divided differences. Information on B-spines, divided differences, and the relation between them can be found in [1].

First, we prove the bi-orthogonality property and then the refinability property. In both proofs we treat separately the case of B-splines with simple knots

The special case of splines which are not necessarily continuous, namely when μ is maximal (exceeds the degree of the B-spline by 1), is treated in [9]. The Hermite interpolatory schemes, in this case, are closely related to the "moments interpolating schemes".

Bi-orthogonal refinable functions of compact support are important in the construction of bi-orthogonal wavelets of compact support [5].

2. B-splines with simple knots.

Define

$$S_0(n) = 1 \ , \ \text{for } n > 0 \ , \quad S_0(n) = 0 \ , \ \text{for } n \leq 0 \ .$$

Then the sequence S_0 has the property that

$$[n, n+1]S_0 = \delta_{0,n} \ .$$

Here we use the notation $[t_0, \ldots, t_k]f$ for the divided difference of order k of f at the points $t_0, \ldots, t_k$, where f is a function defined on these points. Note that $S_0(n)$ is a polynomial of degree 0 in n for positive values of n. Define a sequence of sequences recursively by

$$[n, n+1]S_j = (j+1)S_{j-1}(n) \ ,$$

such that $S_j(n)$ for $n \leq 0$ is identically zero. ¿From the above definition it follows that

$$S_j(n) = (j+1)\sum_{\ell=1}^{n} S_{j-1}(\ell) \ . \tag{1}$$

With this definition, the sequences S_j have the following properties

$$[n, n+1, \cdots, n+j+1]S_j = \delta_{0,n} \ . \tag{2}$$

Also $S_j(n)$ is a polynomial in n of degree j for positive values of n, and $S_j(n)$ is identically equal to zero for nonpositive values of n. (See next section for a less explicit construction of the sequences S_j).

Let there be given an interpolatory subdivision scheme with a finite mask, which generates C^m limit functions. Denote by ϕ the basis limit function of the scheme, namely the limit function generated from the initial data $\Delta(n) = \delta_{n,0}$.

The main result of this section is

Theorem 1. *Let ϕ be as above. Then for all integers r, $1 \leq r \leq m$, the r-th derivative of the function*

$$\psi_r(x) = \sum_{n \in \mathbb{Z}} S_{r-1}(n)\phi(x-n) \ , \tag{3}$$

is bi-orthonormal to the B-spline of order r, (of degree $r-1$) with integer knots and support $[0, r]$, B_r, namely

$$\int B_r(t-n)\psi_r^{(r)}(t-\ell)dt = \delta_{n,\ell} \ , \ n, \ell \in \mathbb{Z} \ . \tag{4}$$

Moreover $\psi_r^{(r)}$ is a function of compact support, with support contained in the support of ϕ.

Proof: The function ϕ has the following two properties [11]

$$\phi(n) = \delta_{n,0} \ , \ n \in \mathbb{Z} \ , \tag{5}$$

For f a polynomial of degree not exceeding m

$$\sum_{n \in \mathbb{Z}} f(n)\phi(x-n) = f(x) \ .$$

Also, the support of ϕ is compact and equals the convex hull of the support of the mask of the subdivision scheme [3]. Let us assume, without loss of generality, that the support of ϕ is of the form $[0,\nu]$. Thus by (5) and (3),

$$\psi_r(n) = S_{r-1}(n) \ , \quad n \in \mathbb{Z} \ . \tag{6}$$

From (6) and the above properties of ϕ and from the properties of the sequences S_j, we conclude that $\psi_r(t)$, is identically equal to zero for t negative and is a polynomial of degree $r-1$ for $t > \nu$. Thus $\psi_r^{(r)}$ has a compact support contained in $[0,\nu]$. Moreover by (2) and (6)

$$[n, \cdots, n+r]\psi_r = \delta_{n,0} \ .$$

The above equality can be written in terms of B_r [1], as

$$\int B_r(t-n)\psi_r^{(r)}(t)dt = \delta_{n,0} \ .$$

This proves (4).

In the next section we prove a similar result for B-splines with multiple knots.

3. B-splines with multiple knots.

The case of multiple knots is treated very similarly to the case of simple knots.

Let $\mu \geq 1$ denote the multiplicity of the knots, which are all the integers. (The case $\mu = 1$ was treated separately in the previous section because of the simplicity of its presentation). Define sequences of vectors in $\mathbb{R}^\mu$

$$S_j^{\mu,\ell}(n) \ , \quad 1 \leq \ell \leq \mu \ , \quad n \in \mathbb{Z}$$

for $j \geq \mu - 1$ by

$$S_j^{\mu,\ell}(n) = 0 \in \mathbb{R}^\mu \ , \quad n \leq 0 \ . \tag{7}$$

For positive values of n each vector $S_j^{\mu,\ell}(n)$, consists of components of the following form,

$$p_{j,\mu,\ell}^{(s)}(n) \ , \quad s = 0, \cdots, \mu - 1 \ ,$$

where $p_{j,\mu,\ell}$ is a polynomial of degree j determined by the conditions,

$$\Delta_{j+1}^{\mu,s} S_j^{\mu,\ell}(m) = \delta_{0,m}\delta_{\ell,s} \ , \quad 1 \le s \le \mu \ , \quad m \in \mathbb{Z} \ . \tag{8}$$

In (8) $\Delta_j^{\mu,s} f(m)$ denotes the divided difference of order j of f at the following points: the integer m repeated s times, the consecutive next q integers repeated μ times and the integer $m+q+1$ repeated $(j+1-s-q\mu)_+$ times, where q is the quotient in the integer division of $j+1-s$ by μ. Here k_+ stands for k if $k > 0$ and for zero otherwise. These multiple-point divided differences involve values of the function and its derivatives up to order $t-1$ at a multiple point of multiplicity t. Due to the condition $j \ge \mu - 1$, the divided differences in (8) involve at least two different points, or components of at least two consecutive vectors of $S_j^{\mu,\ell}$. When applying $\Delta_{j+1}^{\mu,s}$ to $S_j^{\mu,\ell}$ in (8) we regard the components of the vectors $S_j^{\mu,\ell}(n)$ for $n > 0$, as the values of $p_{j,\mu,\ell}$ and its derivatives up to order $\mu - 1$ at the point n.

To determine the polynomial $p_{j,\mu,\ell}$, in the case $j \ge \mu - 1$, we consider the conditions in (8) which involve components of $S_j^{\mu,\ell}(n)$ with n positive and also nonpositive. These conditions constitute a linear system of order $j+1$ for the unknowns which are all the components of the vectors

$$S_j^{\mu,\ell}(m) \ , \quad m = 1, \cdots, q_{j,\mu}$$

and the first $(r_{j,\mu} - 1)_+$ components of the vector $S_j^{\mu,\ell}(q_{j,\mu}+1)$, where $q_{j,\mu}$ is the quotient and $r_{j,\mu}$ is the residual in the division of $j+2$ by μ. These $j+1$ unknowns determine $p_{j,\mu,\ell}$ uniquely, and are obtainable since the system we consider is triangular (each additional equation involves an additional unknown). The rest of the conditions in (8) for positive m are satisfied since these conditions do not involve components of the vectors $S_j^{\mu,\ell}(m)$ for $m \le 0$, and since the divided differences of order $j+1$ of a polynomial of degree j vanish. The rest of the conditions in (8) for m negative hold trivially by (7).

With this construction we obtain for each ℓ , $1 \le \ell \le \mu$ a sequence of sequences $S_j^{\mu,\ell}$, $j+1 \ge \mu$, with the properties

$$\Delta_{j+1}^{\mu,s} S_j^{\mu,\ell}(m) = \delta_{m,0}\delta_{s,\ell} \ , \quad m \in \mathbb{Z} \ , \ 1 \le s \le \mu \ . \tag{9}$$

The above construction of the sequences $S_j^{\mu,\ell}$, is also applicable in the context of the previous section ($\mu = 1$), and in the case of unequally spaced knots.

Let there be given a Hermite interpolatory subdivision scheme of order μ , with a finite mask, which generates C^m limit functions, with $m \geq \mu - 1$. The existence of Hermite schemes of any order μ generating C^m limit functions for any $m \geq \mu - 1$ has not been proven yet in general. It is already known for the case $\mu = 1$ and $m \geq 0$ [7], while for various values of μ and $m \geq \mu - 1$ it was checked numerically in [9]. We think it is true in general, and assume it here. Denote by ϕ_i the function generated from the initial data $\Delta_i(n) = \delta_{0,n} e^{(i)}$ by the given Hermite scheme, where

$$e^{(i)} \ , \quad i = 1, \cdots, \mu$$

is the standard basis of $\mathbb{R}^\mu$. Denote by Φ the vector of functions with components

$$\phi_i \ , \quad 1 \leq i \leq \mu \ ,$$

and denote by $[0, \nu]$ the support of Φ . Then we have

Theorem 2. *Let Φ , be as above. For $\mu \leq r \leq m$, define the functions*

$$\psi_{\ell,r}(x) = \sum_{n \in \mathbb{Z}} \Phi(x - n) \cdot S_{r-1}^{\mu,\ell}(n) \ , \tag{10}$$

where in the above definition the dot denotes the usual inner- product in $\mathbb{R}^\mu$. Then the r-th derivative of $\psi_{\ell,r}$ is of compact support, contained in $[0, \nu]$, and the vector of functions Ψ_r with components

$$\psi_{\ell,r}^{(r)} \ , \quad \ell = 1, \cdots, \mu \ ,$$

is bi-orthogonal to the vector of μ B-splines of order r (degree $r - 1$), which constitute together with their integer shifts a basis of the space of splines of order r, with knots at the integers of multiplicity μ . The bi-orthogonality can be formulated as

$$\int B_{r,j}^{\mu}(x - n) \psi_{\ell,r}^{(r)}(x - m) dx = \delta_{n,m} \delta_{j,\ell} \ , \quad 1 \leq j, \ell \leq \mu \ , \quad m, n \in \mathbb{Z} \ . \tag{11}$$

In (11) $B_{r,j}^{\mu}$, $j = 1, \ldots, \mu$, is the B-spline of order r, corresponding to the following $r + 1$ "active" knots: the knot 0 repeated j times, the knots $1, 2, \cdots, q$ repeated μ times and *the knot $q+1$ repeated $(r+1-j-q\mu)_+$ times, where q is the quotient in the division of $r+1-j$ by μ.*

Proof: By the interpolatory nature of the Hermite scheme, and by the structure of the initial conditions generating the limit functions ϕ_i , $1 \le i \le \mu$, the vector of functions Φ satisfies

$$\Phi^{(s-1)}(n) = \delta_{n,0} e^{(s)} \ , \quad n \in \mathbb{Z} \ , \quad 1 \le s \le \mu \ . \tag{12}$$

Thus, by (9) and (10)

$$\Delta_r^{\mu,j} \psi_{\ell,r}(m) = \delta_{m,0}\delta_{j,r} \ . \tag{13}$$

Equation (13) is equivalent to the claim (11), in view of the definition of $\Delta_r^{\mu,j}$ and of $B_{r,j}^{\mu}$ [1].

Since the Hermite scheme generates C^m limit functions it reproduces polynomials of degree up to m [11], namely for any f a polynomial of degree not exceeding m,

$$\sum_{n \in \mathbb{Z}} f(n) \cdot \Phi(x-n) = f(x) \ , \tag{14}$$

where $f(n) \in \mathbb{R}^{\mu}$ is a vector with components

$$f^{(s)}(n) \ , \quad s = 0, \ldots, \mu - 1 \ .$$

Thus by the structure of the sequences $S_j^{\mu,s}$, by (10) and by (12), $\psi_{\ell,r}(n)$ is identically equal to zero for nonpositive n , and is a polynomial of degree not exceeding $r-1$ for $n > \nu$. This proves that $\psi_{\ell,r}^{(r)}$ has compact support contained in $[0,\nu]$.

4. Refinability of the bi-orthogonal functions.

First the refinability of the bi-orthogonal functions to B-splines with simple knots is proved. Also in this case the proof is simpler, and therefore given separately, although the proof for the multiple knot case, which is more involved, applies also for the simple knot case.

Theorem 3. *Under the conditions of Theorem 1, the function* $\psi_r^{(r)}$, *with* ψ_r , *defined in (3), for* $1 \leq r \leq m$, *is refinable.*

Proof: The interpolatory subdivision scheme, I, with ϕ its basis limit function, generates C^m limit functions, and therefore there exists a subdivision scheme I_r, for the divided differences of order r , $1 \leq r \leq m$, of the control points of I. The scheme I_r generates from the divided differences of any set of control points of I, a limit function which is the r-th derivative of the limit function generated by I from the given control points [11].

In view of (6) the function ψ_r in (3) can be regarded as generated by I from the initial data S_{r-1} . By (2) the divided differences of order r of this initial data, vanish at all the integers, except at zero, where the divided difference of order r is equal to 1. Thus the initial data for I_r is Δ and therefore $\psi_r^{(r)}$ is the basis limit function of I_r . A basis limit function of a subdivision scheme is refinable, with the mask of the scheme as the coefficients of the refinement equation [3]. This establishes the claim of the theorem.

The case of multiple knots is treated similarly.

Theorem 4. *Under the assumptions of Theorem 2, the vector of functions* Ψ_r *is refinable for* $\mu \leq r \leq m$.

Proof: Let H denote the Hermite interpolatory subdivision scheme which generates the limit vector of functions Φ satisfying (12). The function $\psi_{\ell,r}$, $\ell = 1, \ldots, \mu$, defined in (10), can be regarded as generated by H from the initial data $S_{r-1}^{\mu,\ell}(n)$, $n \in \mathbb{Z}$, in view of (12).

Since H generates C^m limit functions, there exist subdivision schemes, H_r, for the divided differences of order r , $\mu \leq r \leq m$, of the vectors of data (in $\mathbb{R}^\mu$) generated by H. These divided differences regard the control points as having multiplicity μ. The limit of the subdivision scheme for the divided differences of order r is the r-th derivative of the corresponding limit function generated by H [12].

Thus, H_r , $\mu \leq r \leq m$, generates the limit function $\psi_{\ell,r}^{(r)}$ from the initial data which for each $n \in \mathbb{Z}$ is a vector with components consisting of the divided differences of order r of the initial data from which H generates the function $\psi_{\ell,r}$, namely

$$\Delta_r^{\mu,s} S_{r-1}^{\mu,\ell}(n) \ , \quad s = 1, \ldots, \mu \ . \tag{15}$$

Note that for each r , $r \geq \mu$, there are μ different divided differences of order r, according to the multiplicity of the first point, which can be $1, \ldots, \mu$. The subdivision scheme for the divided differences of order r , $r \geq \mu$, is a matrix subdivision scheme (the coefficients of the mask are matrices [6]) which maps vectors of control points in $I\!R^\mu$ to such vectors in the next level, and which generates in the limit a vector of functions. In this vector of functions all the functions are identical, in case the initial data consists of divided differences of order r of some initial data for H.

Let the mask of H_r be denoted by

$$A_i^{(r)} \ , \ \ i \in Z\!\!\!Z \ , \ \ A_i^{(r)} \neq 0 \ , \ \ i \in I_r \ , \tag{16}$$

where I_r is a finite set of integers, and $A_i^{(r)}$ are $\mu \times \mu$ matrices. Then the scheme H_r operates on vectors of data at level k , $f_j^k \in I\!R^\mu$, $j \in Z\!\!\!Z$, to generate the data at level $k+1$, in the following way

$$f_i^{k+1} = \sum_{j \in Z\!\!\!Z} A_{i-2j}^{(r)} f_j^k \ . \tag{17}$$

From the above discussion we conclude that the vector of functions Ψ_r, consists of components which are generated by H_r from the initial data (15) for $\ell = 1, \ldots, \mu$. This vector of functions satisfies, in view of (9), the following refinement equation [6],

$$\Psi_r(2\cdot) = \sum_{i \in I_r} (A_i^{(r)})^T \Psi_r(\cdot - i) \ . \tag{18}$$

where the matrices $A_i^{(r)}$ are the ones in (16) and (17). Thus, the vector of functions Ψ_r is refinable.

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