Set-Valued Approximations with Minkowski Averages – Convergence and Convexification Rates

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Abstract. Three approximation processes for set-valued functions (multifunctions) with compact images in $\mathbb{R}^n$ are investigated. Each process generates a sequence of approximants, obtained as finite Minkowski averages (convex combinations) of given data of compact sets in $\mathbb{R}^n$. The limit of the sequence exists and and is equal to the limit of the same process, starting from the convex hulls of the given data. The common phenomenon of convexification of the approximating sequence is investigated and rates of convergence are obtained. The main quantitative tool in our analysis is the Pythagorean type estimate of Cassels for the "inner radius" measure of nonconvexity of a compact set. In particular we prove the convexity of the images of the limit multifunction of set-valued spline subdivision schemes and provide error estimates for the approximation of set-valued integrals by Riemann sums of sets and for Bernstein type approximation to a set-valued function.

Key words: Minkowski addition, set-valued functions, convexity, measures of nonconvexity, Bernstein type approximation, spline subdivision schemes, set-valued integral, Riemann sums.

1 Introduction

Three limit processes involving Minkowski averages (Minkowski convex combinations) of compact (possibly nonconvex) sets, are studied in this work. Each process generates a sequence of approximants (sets or set-valued functions), obtained as finite Minkowski averages of given data of compact sets in $\mathbb{R}^n$. The limit of the sequence exists and and is equal to the limit of the same process, starting from the convex hulls of the given data. The common phenomenon of convexification of the approximating sequence is investigated and rates of convergence are obtained.

The estimates of the error in the approximation, in terms of the Hausdorff distance between the limit and the approximants comprise of two elements:

*Partially supported by the Israel Science Foundation – Center of Excellence Program and by the Hermann Minkowski Center for Geometry at Tel-Aviv University

(a) Convex case estimates – the distance between the convex limit and the approximants obtained from the convex hulls of the data;

(b) Convexification rates – the distance between the two approximants, one obtained from the given data, and one from their convex hulls.

The estimates of type (a) are mostly known and are obtained by a reduction of the problem to the approximation of support functions ([16, 6, 3, 8]). The estimates of type (b) provide the convexification rate and are obtained by the use of a nonconvexity measure of a set, ρ, called the "inner radius" of this set ([15], [1], [17]). A measure of nonconvexity of a compact set is a nonnegative number which is equal to zero iff the set is convex.

The main tool in our analysis is the Pythagorean type inequality,

$$\rho(\sum_{i=1}^{k} A_i) \le \sqrt{\sum_{i=1}^{k} \rho^2(A_i)}, \tag{1}$$

obtained in [4] for a nonconvexity measure which was later proved to be equal to the inner radius [17].

Another inequality we use to estimate the convexification rate is the following Shapley–Folkman–Starr type estimate, which is a consequence of (1) (cp. Appendix 2 in [15], Theorem 2 in [4] and Theorem 3.1.6 in [14]):

$$\rho(\sum_{i=1}^{k} \lambda_i A_i) \le \min\{\sqrt{k}, \sqrt{n}\} \max_{1\le i\le k} |\lambda_i| \rho(A_i). \tag{2}$$

The first example of a convexification process is the sequence of Riemann sums of a Riemann integrable multifunction [12], approximating its Aumann integral [2]. The convexity of this integral is well-known [2], and here we use the above technique to provide estimates of the Hausdorff distance between the integral and the Riemannian sums.

The second process is the sequence of positive operators called here Bernstein-type approximations of a given multifunction. This class of approximants extends the classical Bernstein operators for multifunctions [16]. We combine estimates of the approximation of scalar functions and the inequality (2) to obtain pointwise convergence estimates to a set-valued function with images which are the convex hulls of the images of the approximated multifunction.

The third process is a spline subdivision scheme for compact sets [8]. Here we establish the correctness of the conjecture formulated in [8] that the images of the limit multifunction are convex, and provide error estimates.

Note that for k tending to infinity and $\max_{1\le i\le k} |\lambda_i|$ tending to zero, the bound (2) reconfirms the well-known claim (see e.g. [14]) that limit processes involving convex Minkowski averages with an increasing number of summands and vanishing weights, are convexification processes. This is the case in Bernstein-type approximation and in set-valued numerical integraion for which (2) is applied. However, in cases of iterative averaging processes of recursive character, where at each step one constructs a fixed convex combination of a small number of summands, obtained in the previous step, (2) may not imply reduction in the nonconvexity measure. Yet, a repeated application of the sharp estimate (1) implies the convergence of the nonconvexity measure to zero. In particular, (1) is applied to subdivision schemes.

The paper is organized as follows: In the next section we present and compare some nonconvexity measures and survey their basic properties. The error estimates for set-valued numerical integration and Bernstein type approximation of multifunctions with compact images are presented in sections 3 and 4. In Section 5 we analyse the case of set-valued subdivision schemes.

2 Nonconvexity Measures of a Compact Set

Throughout the paper we consider only compact sets in $\mathbb{R}^n$. First we introduce some notions and notation. $\mathcal{K}_n$ is the set of all nonempty compact subsets of $\mathbb{R}^n$, $\mathcal{C}_n$ is the set of all convex elements of $\mathcal{K}_n$, $\langle\cdot,\cdot\rangle$ is the inner product in R^n, $|x|$ is the Euclidean norm of $x \in \mathbb{R}^n$, $B_r(x)$ is the closed ball with center x and radius r, $S_r(x)$ is the sphere with center x and radius r. The Euclidean distance from a point $x \in \mathbb{R}^n$ to a set A is denoted by $\mathrm{dist}(x, A)$.The Hausdorff distance between the sets A and B is $\mathrm{haus}(A, B)$, ∂A is the boundary of A, $\mathrm{relint}(A)$ is the relative interior of A, $\mathrm{co}A$ denotes the convex hull of A, $ext(A)$ is the set of extremal points of A (the last two are nonempty, if A is nonempty). A k-dimensional simplex is a convex polytope of dimension k with $k+1$ vertices. $\mathbb{R}_+$ is the set of all nonnegative real numbers. A linear Minkowski combination of the sets A and B is

$$\lambda A + \mu B = \{ \ \lambda a + \mu b \ : \ a \in A, \ b \in B \ \},$$

with $\lambda, \mu \in \mathbb{R}$. In case $\lambda, \mu \in \mathbb{R}_+, \ \lambda + \mu = 1, \ \lambda A + \mu B$ is termed a convex Minkowski combination or a Minkowski average. We denote

$$\mathrm{rad}A = \inf_{x\in\mathbb{R}^n} \sup_{a\in A} |x - a|, \qquad \mathrm{diam}A = \sup_{x,y\in A} |x - y|,$$

a segment $[c, d]$ by $[c, d] = \{\lambda c + (1 - \lambda)d \ : \ 0 \le \lambda \le 1\}$.
We use in what follows the identity (see e.g. [14]) for $\lambda, \mu \in \mathbb{R}$

$$\mathrm{co}(\lambda A + \mu B) = \lambda \mathrm{co}A + \mu \mathrm{co}B. \tag{3}$$

Throughout the paper we study set limits in the sense of the Hausdorff metric. It is well-known ([14]) that $\mathcal{K}_n$ is a complete metric space with respect to this metric.

We define a (general) measure of nonconvexity of a set A, based on a local modulus of nonconvexity.

Definition 2.1 *Let $A \in \mathcal{K}_n$. The function $\rho(\cdot, A) : \mathrm{co}A \to \mathbb{R}_+$ satisfying*

(a) $\rho(x, A) = 0$ iff $x \in A$,

(b) for $D \subseteq A$ and $x \in \mathrm{co}D$, $\rho(x, D) \ge \rho(x, A)$,

is called local modulus of nonconvexity of A (ρ-distance from $x \in \mathrm{co}A$ to A).

There are many possibilities to define the local modulus of nonconvexity $\rho(x, A)$. Here we cite known moduli from the literature:

1. $\rho_1(x, A) = \inf \ \{ \ |x - a| \ : \ a \in A \ \}$ [4], [17];

2. $\rho_2(x,A) = \inf\left\{ \sqrt{\sum_{i=1}^{k} \alpha_i |x-a_i|^2} \ : \ a_i \in A, \ x = \sum_{i=1}^{k} \alpha_i a_i \ \sum_{i=1}^{k} \alpha_i = 1, \ \alpha_i \in \mathbb{R}_+ \right\}$ [4];

3. $\rho_3(x,A) = \inf \{ \ \mathrm{rad} S \ : \ S = \{a_i\}_{i=1}^k \subset A, \ x \in \mathrm{co} S \}$ [15, 1];

Clearly, all local nonconvexity moduli $\rho_j(\cdot, A), \ 1 \le j \le 3$, satisfy conditions (a), (b).

Remark 2.2 *In view of the Caratheodory theorem, the definitions of* $\rho_j(x,A), \ j = 2,3$ *consider only simplices* $S = \{a_i\}_{i=1}^k \subseteq A$ *of dimension* $k-1 \le n$. *Moreover, one can prove by standard compactness arguments that the infimum in every definition of* $\rho_j(x,A), \ 1 \le j \le 3$, *is a minimum.*

Definition 2.3 *The* **nonconvexity measure** *of a set* A *corresponding to the local nonconvexity modulus* $\rho_j(x,A)$ *is defined as*

$$\rho_j(A) = \sup\{ \ \rho_j(x,A) \ : \ x \in \mathrm{co} A \ \}. \tag{4}$$

Note that $\rho_1(A) = \mathrm{haus}(A, \mathrm{co}A)$. The measure $\rho_3(A)$ is called the "inner radius" of A [15], [1]. The measure ρ_2 is introduced and studied in [4]. Other measures of nonconvexity may be found in [17, 14].

In the rest of the paper we denote by $\rho(A)$ the inner radius $\rho_3(A)$ and by $\rho(\cdot, A)$ the corresponding local modulus. We also denote $\rho_H(A) = \rho_1(A)$.

It is known [17] that $\rho_2(A) = \rho(A)$. It is easy to see that $\rho_H(x,A) \le \rho_2(x,A)$, therefore

$$\rho_H(A) \le \rho_2(A) = \rho(A). \tag{5}$$

In [17] it is shown by an example that ρ_H is not equal to ρ_2. Indeed, the measures ρ_H and ρ are not topologically equivalent, as is demonstrated by the next example.

Example 2.4 *Let* $A_n \subset \mathbb{R}^2, \ n = 1, 2, \ldots,$ *be sets defined by* $A_n = \mathrm{co}\{ \ (-1, \frac{1}{n}), (1, \frac{1}{n})\} \cup \{(0,0)\}$ *and let* $A = \mathrm{co}\{(-1,0),(1,0)\}$. *Then* $\rho_H(A_n) \le \frac{1}{n}$, *and* $\lim_{n\to\infty} \mathrm{haus}(A_n, A) = 0$. *Yet,* $c_n = (\frac{1}{2}, \frac{1}{2n}) \in \mathrm{co}A_n$ *and* $\rho(A_n) \ge \rho(c_n, A_n) \ge \frac{1}{2}$.

Next we present properties of the measure $\rho(\cdot)$. First, basic properties following directly from the definition and then the Pythagorean-type inequality (1) proved in [4] for ρ_2.

Proposition 2.5 *Let* $A \in \mathcal{K}_n$, *then*

1. $\rho(\lambda A) = |\lambda| \rho(A)$ *for every* $\lambda \in \mathbb{R}$.

2. $\rho(A + \{b\}) = \rho(A)$ *for every* $b \in \mathbb{R}^n$.

3. $\rho(A) \le \mathrm{rad}(A)$.

Theorem 2.6 *([4]) Let* $A_i \in \mathcal{K}_n, \ i = 1, \ldots, k$. *Then*

$$\rho\left(\sum_{i=1}^{k} A_i\right) \le \sqrt{\sum_{i=1}^{k} (\rho(A_i))^2}. \tag{6}$$

We conjecture that (6) holds also for ρ_H.

The following example shows that (6) is sharp.

Example 2.7 *Let* $A = \{a_1, a_2\} \subset \mathbb{R}^n, B = \{b_1, b_2\} \subset \mathbb{R}^n$. *Then*

$$\rho(A + B) \leq \sqrt{\rho^2(A) + \rho^2(B)}. \tag{7}$$

with equality iff $\langle a_1 - a_2, b_1 - b_2 \rangle = 0$.

Indeed, for $A = \{a_1, a_2\}$ with $a_1 \neq a_2$, $\rho(A) = \frac{|a_1 - a_2|}{2}$. The parallelogram $\text{co}(A + B) = [a_1, a_2] + [b_1, b_2]$ can be represented as the union of two congruent nonintersecting triangles T_1, T_2, each of them with edges of length $|a_1 - a_2|$, $|b_1 - b_2|$ with an acute (or at most right) angle between them. Note that $\rho(ext(T_1)) = \rho(ext(T_2))$ Hence $\rho(A + B) = \rho(ext(T_1))$. If all the angles of T_1 do not exceed $\frac{\pi}{2}$, then the center of the circumscribed circle of T_1 is in T_1, and $\rho(ext(T_1)) = R$, where R is the radius of the circumscribed circle of T_1. In this case, it is an elementary fact that $2R \leq c$, where $c = \sqrt{|a_1 - a_2|^2 + |b_1 - b_2|^2}$, and there is an equality when the angle between the sides is $\frac{\pi}{2}$. If there is an obtuse angle in T_1 (not between the segments $[a_1, a_2]$ and $[b_1, b_2]$), then the maximal side of T_1 is of length $d = \max\{|a_1 - a_2|, |b_1 - b_2|\}$ and $\rho(ext(T_1)) = \frac{d}{2} \leq \frac{c}{2}$.

In case of a right angle between the two segments A and B, there is an equality in (7).

Note that the above proof establishes (7) also for ρ_H.

We apply in our analysis the following inequality which follows from (6) and Property 1 of Proposition 2.5.

Corollary 2.8 *Let* $A_i \in \mathcal{K}_n, \ \lambda_i \in \mathbb{R}, \ i = 1, ..., k$. *Then*

$$\rho(\sum_{i=1}^{k} \lambda_i A_i) \leq \mu \max_{1 \leq i \leq k} \{\rho(A_i)\}, \qquad \textit{with} \quad \mu = \sqrt{\sum_{i=1}^{k} \lambda_i^2}. \tag{8}$$

Moreover, $\mu < 1$ *if*

$$k \geq 2, \quad \sum_{i=1}^{k} \lambda_i = 1, \ \lambda_i > 0, i = 1, 2, ..., k. \tag{9}$$

By (8) and (9), repeated Minkowski averaging with fixed weights produces a sequence of sets with the measure of nonconvexity $\rho(\cdot)$, decreasing at least geometrically.

On the other hand, by (2), averaging with an increasing number of summands k and with equal weights ($\lambda_i = \frac{1}{k}$ in (2)), is a convexification process with rate $\mathcal{O}(\frac{1}{k})$. Indeed, let us apply (2) to the simple averaging process with one set.

Example 2.9 *(cp. [10]) Let* $A \in \mathcal{K}_n$, *and define* $A_k = \frac{1}{k} \sum_{i=1}^{k} A$. *The inclusion* $A \subseteq A_k \subseteq \text{co}A$, *implies* $\text{co}A_k = \text{co}A$, *and one easily gets by (5) and (2) for* $k \geq n$,

$$\text{haus}(A_k, \text{co}A) = \text{haus}(A_k, \text{co}A_k) \leq \rho(A_k) \leq \sqrt{n} \frac{\rho(A)}{k}. \tag{10}$$

Therefore $\lim_{k \to \infty} A_k = \text{co}A$. *An estimate of the form* $\text{haus}(A_k, \text{co}A) = \mathcal{O}(\frac{1}{k}\rho(A))$ *with a larger constant than* $\sqrt{n}$ *is obtained in [10]. It is not hard to see that in* $\mathbb{R}^2$ *the sharp constant is not* $\sqrt{2}$, *but* 1.

3 Set-Valued Numerical Integration

By application of the estimate (2) we obtain estimates of the rate of approximation of the set-valued integral of a multifunction with compact images by its Riemann sums of sets. We note that numerical integration and numerical solution of diffrential inclusions for set-valued functions with convex compact images are studied in [7, 6, 3].

We consider a multifunction $F : [a,b] \to \mathcal{K}_n$ which is globally bounded on $[a,b]$. Recall that the integral of a set-valued function $F : [a,b] \to \mathcal{K}_n$ is defined as the set of all integrals of integrable selections of F [2], namely

$$I(F) = \int_a^b F(x)\,dx = \{ \int_a^b f(x)\,dx \ : \ f(x) \in F(x) \text{ a.e. in } (a,b) \ , \ f \text{ is integrable} \}. \quad (11)$$

The Riemann sum of F is defined in the standard way.
Let $\Delta = \{ x_i \ : \ x_i \in [a,b], \ 0 \le i \le N, \ x_{i-1} < x_i, \ 1 \le i \le N \ , \ x_0 = a, x_N = b \}$ be a partition of $[a,b]$, $h_i = x_i - x_{i-1}$. The Riemann sum of F relative to this partition is defined as $S^\Delta(F) = \sum\limits_{i=1}^{N} h_i F(c_i)$, where $c_i \in [x_{i-1}, x_i]$. Recall that F is integrable in the sense of Riemann if its Riemann sums tend in the Hausdorff metric to $I(F)$, when the maximal subinterval length $h(\Delta) = \max\limits_{1\le i\le N} h_i$ tends to zero (see e.g. [12]). It is well known that $I(F)$ is convex and compact [2]. In particular,

$$I(F) = I(\text{co}F). \quad (12)$$

For any Riemann integrable multifunction $F : [a,b] \to \mathcal{K}_n$, $I(F) = \lim\limits_{h(\Delta)\to 0} S^\Delta(F)$. Moreover, $F(\cdot)$ is Riemann integrable if and only if $\text{co}F(\cdot)$ is Riemann integrable [12]. Here we provide an estimate of the approximation error $\text{haus}(S^\Delta(F), I(F))$ based on the estimate (2), and on known estimates for $\text{haus}(S^\Delta(\text{co}F), I(F))$. In particular, if F is Hölder continuous with an exponent α (hence, so is $\text{co}F$), then $\text{haus}(S^\Delta(\text{co}F), I(F)) = \mathcal{O}(h^\alpha)$ (see e.g. [7, 6, 3]).

Proposition 3.1 *If the multifunction $F : [a,b] \to \mathcal{K}_n$ is globally bounded, then*

$$\text{haus}(S^\Delta(F), I(F)) \le \sqrt{n}h(\Delta) \sup_{a\le t\le b} \text{rad}(F(t)) + \text{haus}(S^\Delta(\text{co}F), I(\text{co}F)). \quad (13)$$

Proof: In view of (3), $S^\Delta(\text{co}F) = \text{co}S^\Delta(F)$. Thus, by the triangle inequality and by (12), we get

$$\text{haus}(S^\Delta(F), I(F)) \le \text{haus}(S^\Delta(F), \text{co}S^\Delta(F)) + \text{haus}(S^\Delta(\text{co}F), I(\text{co}F)). \quad (14)$$

By (5), (2) and by Property 3 in Proposition 2.5, the first summand in the right-hand side of (14) is bounded by

$$\text{haus}(S^\Delta(F), \text{co}S^\Delta(F)) \le \rho(S^\Delta(F)) \le \sqrt{n}h(\Delta) \sup_{a\le t\le b} \text{rad}(F(t)). \quad (15)$$

The proof is completed by combining (14) and (15). ∎

4 Bernstein-Type Approximation of Set-Valued Functions

In this section we consider a class of positive linear operators called here Bernstein-type approximations, applied to set-valued functions. This class of approximations contains the Bernstein approximation of multifunctions [16].

We consider approximations to a set-valued function with compact images $F : [0,1] \to \mathcal{K}_n$, of the form:

$$A_m(F,x) = \sum_{k=0}^{m} C_{k,m}(x) F(\frac{k}{m}), \quad x \in [0,1], \tag{16}$$

where $C_{k,m}(x) \geq 0$ for any $x \in [0,1]$. We study the convergence of $A_m(F,x)$ to co$F(x)$ as $m \to \infty$. Our basic assumption on (16) is that this approximation sequence converges for scalar functions.

The most interesting example is the Bernstein approximating polynomial of degree m of F, $B_m(F,\cdot) : [0,1] \to \mathcal{K}_n$ defined by (16) with the binomial distribution probabilities as coefficients,

$$C_{k,m}(x) = \binom{m}{k} x^k (1-x)^{m-k}. \tag{17}$$

The Bernstein set-valued approximation is studied by Vitale [16].

Another example is the polynomial approximation $H_m^{(N)}(F,x)$ defined with the hypergeometric distribution probabilities,

$$C_{k,m}^{(N)}(x) = \frac{\binom{Nx}{k}\binom{N(1-x)}{m-k}}{\binom{N}{m}}, \tag{18}$$

with the parameters $N, m,\ N > m$ and the real binomial coefficients $\binom{y}{k} = \frac{y(y-1)\dots(y-k+1)}{k!}$.

First we prove that if F has convex images, then for each $x \in [0,1]$, $\lim\limits_{m\to\infty} A_m(F,x) = F(x)$ in the Hausdorff metric. We also give error estimates. After that we show that if the images of F are only compact, then, for a fixed $x \in (0,1)$, the limit of $A_m(F,x)$ when m tends to infinity is the convex hull of $F(x)$. Estimates of haus(co$F(x), A_m(F,x)$) are provided.

For F with convex images, we apply the following results for approximations of scalar functions by linear positive operators. An operator $L(f,x)$ is a linear positive operator if it is linear with respect to f and if for non-negative f, $L(f,x)$ is non-negative.

Result A ([5], Chapter 9) *Let $L_m(f,x), m = 1,2,\dots$ be a sequence of linear positive operators on the space of continuous functions defined on $[0,1]$. Denote*

$$\lambda_m = \max_{k=0,1,2} \|L_m(x^k,\cdot) - x^k(\cdot)\|_{\infty,[0,1]}$$

Then, for any continuous function $f : [0,1] \to \mathbb{R}$,

$$\|f(\cdot) - L_m(f,\cdot)\|_{\infty,[0,1]} \leq \|f\|_{\infty,[0,1]}\lambda_m + C_1\omega(f,\sqrt{\lambda_m}), \tag{19}$$

where $\omega(f,\cdot)$ is the modulus of continuity of f and $C_1 = 3$ if L_m reproduces constants.

Note that $\lim_{m\to\infty} \lambda_m = 0$ for any sequence of positive operators $\{L_m(f,x)\}_{m=1}^{\infty}$ which converges to f.

For the Bernstein approximation $\lambda_m = \frac{1}{m}$ and we have for scalar functions [5]:

Result AB *Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz continuous function, then*

$$|f(x) - B_m(f,x)| \le C\frac{1}{\sqrt{m}}, \tag{20}$$

where the constant C depends on the Lipschitz constant of the function f. Moreover, if f has a Lipschitz continuous first derivative, then

$$|f(x) - B_m(f,x)| \le \tilde{C}\frac{1}{m}. \tag{21}$$

For the "hypergeometric" approximation (16), (18), as in the case of Bernstein approximation, using the mean and the variance of the hypergeometric distribution [11], one may show that $\lambda_m = O(\frac{1}{m})$.

Since the technique of support functions is central to the case of convex-valued multifunctions, we recall the definition of these functions (see e.g. [13]). For a set $A \in \mathcal{C}_n$ its support function $\delta^*(A,\cdot) : \mathbb{R}^n \to \mathbb{R}$ is defined as follows:

$$\delta^*(A,l) = \max_{a\in A}\langle l, a\rangle, \quad l \in \mathbb{R}^n.$$

Note that for any fixed $l \in \mathbb{R}^n$, $\delta^*(A,l)$ is finite.

The following properties of the support functions δ^* ([13]) are important for our analysis.

For $A, B \in \mathcal{C}_n$,

1. $\delta^*(A+B,\cdot) = \delta^*(A,\cdot) + \delta^*(B,\cdot)$,
2. $\delta^*(\lambda A,\cdot) = \lambda\delta^*(A,\cdot), \quad \lambda \ge 0$,
3. $\mathrm{haus}(A,B) = \max_{l\in S_{n-1}} |\delta^*(A,l) - \delta^*(B,l)|$, where S_{n-1} is the unit sphere in $\mathbb{R}^n$.

We consider now the set-valued Bernstein-type approximations of a multifunction defined on $[0,1]$, with convex images, which is Lipschitz continuous:

$$\mathrm{haus}(F(x), F(y)) \le L|x-y|, \quad x, y \in [0,1].$$

Proposition 4.1 *If $F : [0,1] \to \mathcal{C}_n$ is Lipschitz continuous, then*

$$\mathrm{haus}(A_m(F,x), F(x)) \le C\sqrt{\lambda_m}, \tag{22}$$

where the constant C depends only on the Lipschitz constant of F.

Proof: For a fixed vector $l \in \mathbb{R}^n$, by (16), and by Properties 1, 2 of support functions, the support function of the set $A_m(F,x)$ is a Bernstein-type approximation of the support function of $F(\cdot)$, namely

$$\delta^*(A_m(F,x),l) = \sum_{k=0}^{m} C_{k,m}\delta^*(F(\frac{k}{m}),l) = A_m(\delta^*(F(\cdot),l),x). \tag{23}$$

Since by Property 3 of the support functions, $\delta^*(F(\cdot),l)$ is Lipschitz continuous with the same Lipschitz constant as F, uniformly in $l \in S_{n-1}$, we may apply Result A for the approximation of the scalar function $\delta^*(F(\cdot),l)$. This gives

$$|A_m(\delta^*(F(x),l)) - \delta^*(F(x),l)| \leq C\sqrt{\lambda_m},$$

with C depending on the Lipschitz constant of F. Taking the supremum over l in S_{n-1}, we obtain (22), in view of (23) and Property 3 of support functions.

■

Corollary 4.2 *For* $\{B_m(F,x)\}_m$ *and* $\{H_m^{(N)}(F,x)\}_m$, $\lambda_m = O(\frac{1}{m})$, *and the rate of approximation is* $O(\frac{1}{\sqrt{m}})$. *Moreover, if the support functions* $\delta^*(F(\cdot),l)$ *have first derivatives, which are uniformly Lipschitz in* $l \in S_{n-1}$, *then*

$$\mathrm{haus}(B_m(F,x),F(x)) \leq C\ m^{-1}. \tag{24}$$

In [16] the convergence of $\{B_m(F,x)\}_m$ is proved for a set-valued fuction with convex compact images by reducing the problem to Bernstein approximation of support functions, but without explicit error estimates of the form (20), (21). Also the shape-preserving properties and the convexification property of $\{B_m(F,x)\}_m$ are studied there.

To study the convergence of $\{A_m(F,x)\}_m$ to $\mathrm{co}F(x)$ for F with compact (not necessarily convex) images, we introduce the quantity $Q_m(x) = \max\limits_{0\leq k\leq m} C_{k,m}(x)$.

Theorem 4.3 *If* $F:[0,1]\to\mathcal{K}_n$ *is Lipschitz continuous, then*

$$\mathrm{haus}(A_m(F,x),\mathrm{co}F(x)) \leq C\sqrt{\lambda_m} + \sqrt{n}Q_m(x)\max_{t\in[0,1]}\mathrm{rad}F(t), \quad x\in[0,1]. \tag{25}$$

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Proof: By (3), $A_m(\mathrm{co}F,x) = \mathrm{co}A_m(F,x)$. From the triangle inequality, we get

$$\mathrm{haus}(A_m(F,x),\mathrm{co}F(x)) \leq \mathrm{haus}(A_m(F,x),\mathrm{co}A_m(F,x)) + \mathrm{haus}(A_m(\mathrm{co}F,x),\mathrm{co}F(x)). \tag{26}$$

The second term in the right-hand side of (26) is estimated in Proposition 4.1. The first term in the right-hand side of (26) is estimated in view of (5), by the measure of nonconvexity of $A_m(F,x)$,

$$\mathrm{haus}(A_m(F,x),\mathrm{co}A_m(F,x)) \leq \rho(A_m(F,x)).$$

By (2), (16) and Proposition 2.5,

$$\rho(A_m(F,x)) \leq \sqrt{n}Q_m(x)\max_{t\in[0,1]}\mathrm{rad}F(t).$$

This, together with the estimate (22) applied to the second term in the right-hand side of (26), completes the proof. ■

To obtain the rate of convergence in (25), we need an upper bound of $Q_m(x)$ and of λ_m. In the case of Bernstein approximations the bound of $Q_m(x)$ is obtained easily from the local Central Limit Theorem for the binomial distribution (see e.g. [9], Sections 11, 12).

Result B *If* $Q_m(x) = \max_{0 \le k \le m} \binom{m}{k} x^k (1-x)^{m-k}, \ m \in \mathbb{N}, \ x \in (0,1)$, *then*

$$\lim_{m\to\infty} \sqrt{mx(1-x)}\, Q_m(x) = \frac{1}{\sqrt{2\pi}}. \tag{27}$$

It follows directly from Result B that there exists a constant $c(x)$, depending on x, such that

$$Q_m(x) \le \frac{c(x)}{\sqrt{m}}, \qquad m > 0, \quad x \in (0,1). \tag{28}$$

This together with Theorem 4.3 leads to

Corollary 4.4 *For a Lipschitz continuous multifunction* $F : [0,1] \to \mathcal{K}_n$,

$$\text{haus}(B_m(F,x), \text{co}F(x)) \le C(x)\ m^{-\frac{1}{2}}, \qquad x \in (0,1), \tag{29}$$

where the constant $C(x)$ *depends on* x *and on* F.

In the case of $\{H_m^{(N)}(F,x)\}$, $lim_{m\to\infty} Q_m(x) = 0$ since the hypergeometric distributions tend to the continuous Pearson distributions when $m \to \infty$ ([11]).

Corollary 4.5 *Let* $F : [0,1] \to \mathcal{K}_n$ *be Lipschitz continuous. Then for* $x \in (0,1)$,

$$\lim_{m\to\infty} \text{haus}(H_m^{(N)}(F,x), \text{co}F(x)) = 0. \tag{30}$$

Remark 4.6 *Note that by (27) and (28),* $C(x)$ *in (29) is unbounded in* $[0,1]$. *The non-uniformity of the estimate (29) reflects the fact that for all* m, $B_m(F,0) = F(0)$, $B_m(F,1) = F(1)$, *while at any* $x \in (0,1)$, *(29) holds.*

5 Convexification in Subdivision of Compact Sets

Spline subdivision schemes applied to compact convex sets are considered in [8]. For initial data $\{F_i^0,\ i \in \mathbb{Z}\}$, with $F_i^0,\ i \in \mathbb{Z}$ convex compact sets, the subdivision scheme of order $m \ge 2$ consists of a sequence of refinement steps,

$$F_i^{k+1} = \sum_{j\in\mathbb{Z}} a_{i-2j}^{[m]} F_j^k, \quad i \in \mathbb{Z}, \quad k = 0,1,2,... \tag{31}$$

with $a^{[m]} = \{a_i^{[m]} = \binom{m}{i}/2^{m-1},\ i = 0,1,...,m,\ a_i^{[m]} = 0,\ i \in \mathbb{Z} \setminus \{0,1,...,m\}\}$, a finitely supported mask.

Note that $\sum_i a_{2i}^{[m]} = \sum_i a_{2i+1}^{[m]} = 1$, and that for $m \ge 2$

$$\sum_i (a_{2i}^{[m]})^2 = \nu_m < 1\ ,\ \sum_i (a_{2i+1}^{[m]})^2 = \mu_m < 1\ . \tag{32}$$

Also note that convex compact sets are generated at each step of (31), if $F_i^0,\ i \in \mathbb{Z}$ are compact and convex. At the $k-$th refinement level one defines the piecewise linear multifunction

$$F^{[k]}(t) = \sum_{i\in\mathbb{Z}} F_i^k h(2^k t - i), \quad t \in \mathbb{R}, \tag{33}$$

where $h(\cdot)$ is the hat function

$$h(t) = \begin{cases} 1 - |t| & \text{for } |t| \leq 1, \\ 0 & \text{otherwise}, \end{cases} \tag{34}$$

and $\{F_i^k,\ i \in \mathbb{Z}\}$ are the sets generated by the subdivision scheme at refinement level k. It is shown in [8] that for initial convex sets $\{F_i^0,\ i \in \mathbb{Z}\}$, there exists a set-valued spline function $F_m^\infty : \mathbb{R} \to \mathcal{C}_n$ which is the uniform limit of the subdivision scheme, namely

$$\lim_{k\to\infty} \sup_{t\in\mathbb{R}} \text{haus}(F_m^\infty(t), F^{[k]}(t)) = 0. \tag{35}$$

The proofs in [8] are based on the support functions parametrization of convex compact sets. It is shown there that

$$F_m^\infty(t) = \sum_{i\in\mathbb{Z}} F_i^0 B_m(t-i) \quad \text{for each } t \in \mathbb{R}, \tag{36}$$

with B_m the scalar B-spline of order m (degree $m-1$) with integer knots and support $[0, m]$.

It is conjectured there that for initial data consisting of compact (not necessarily convex) sets, $F^0 = \{F_i^0 : i \in \mathbb{Z}\}$, any spline subdivision scheme converges, and the limit multifunction is identical with the one obtained by the same subdivision scheme applied to initial data consisting of the convex hulls of the sets F^0. In this section we prove this conjecture by using Corollary 2.8.

Theorem 5.1 *The spline subdivision scheme of order m when applied to initial data consisting of compact sets F^0, converges uniformly in the Hausdorff metric to the spline multifunction with convex images*

$$F_m^\infty(t) = \sum_{i\in\mathbb{Z}} coF_i^0 B_m(t-i). \tag{37}$$

Moreover,

$$\text{haus}(F^{[k]}(t), F_m^\infty(t)) \leq (\alpha_m)^k \|\rho(F^{[0]})\| + \text{haus}(\text{co}F^{[k]}(t), F_m^\infty(t)), \tag{38}$$

where $\alpha_m = \max\{\sqrt{\mu_m}, \sqrt{\nu_m}\}$ and $\|\rho(F^{[k]}(\cdot))\| = \sup_t \rho(F^{[k]}(t))$.

Proof: By (33), for $2^{-k}i \leq t \leq 2^{-k}(i+1)$, $F^{[k]}(t)$ is a convex combination of F_i^k and F_{i+1}^k, hence by (8), $\|\rho(F^{[k]}(\cdot))\| = \max_i \rho(F_i^k)$. Thus, using (8), (31)and (32), we obtain that

$$\|\rho(F^{[k]}(\cdot))\| \leq \max_i \rho(F_i^k) \leq \alpha_m \max_i \rho(F_i^{k-1}) \leq ... \leq (\alpha_m)^k \max_i \rho(F_i^0).$$

The above inequality gives the convexification rate in the subdivision process.

To see that the limit multifunction is given by (37), we denote by $G^{[k]}(t)$ the functions as in (33), obtained from the initial data $G^0 = \{coF_i^0 : i \in \mathbb{Z}\}$, and by G^∞ the corresponding limit. It is proved in [8] that $G^\infty(t) = F_m^\infty(t)$ (with $F_m^\infty(\cdot)$ defined in (37)). Observe that by (3), $G^{[k]}(t) = \mathrm{co}F^{[k]}(t)$ for each t, and that

$$\mathrm{haus}(F^{[k]}(t), G^\infty(t)) \leq \mathrm{haus}(F^{[k]}(t), G^{[k]}(t)) + \mathrm{haus}(G^{[k]}(t), G^\infty(t))$$

$$= \mathrm{haus}(F^{[k]}(t), coF^{[k]}(t)) + \mathrm{haus}(G^{[k]}(t), F_m^\infty(t))$$

Using (5) and Corollary 2.8, we finally get

$$\mathrm{haus}(F^{[k]}(t), F_m^\infty(t)) \leq \rho(F^{[k]}(t)) + \mathrm{haus}(G^{[k]}(t), F_m^\infty(t))$$

$$\leq \|\rho(F^{[k]})\| + \mathrm{haus}(G^{[k]}(t), F_m^\infty(t)) \leq (\alpha_m)^k \|\rho(F^{[0]})\| + \mathrm{haus}(G^{[k]}(t), F_m^\infty(t)),$$

which implies (38). By the convergence of the subdivision scheme for convex sets [8], and since $|\alpha_m| < 1$, the proof of the theorem is completed. Estimates of $\mathrm{haus}(G^{[k]}(t), F_m^\infty(t))$, are obtained in [8]. ∎

Remark 5.2 *The proof of Theorem 5.1 applies also for the general class of subdivision schemes with non-negative masks $a = \{a_i \geq 0 : i \in \mathbb{Z}\}$, which converge uniformly for scalar data.*

Acknowledgement. We are thankful to I. Meilijson, for bringing to our attention some results in probability theory and to V. Milman and D. Schmeidler for pointing out references on nonconvexity measures. We also thank the Israel Science Foundation – Center of Excellence Program, and the Hermann Minkowski Center for Geometry at Tel-Aviv University for their support.

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