Four-point curve subdivision based on iterated chordal and centripetal parameterizations

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Abstract

Dubuc's interpolatory four-point scheme inserts a new point by fitting a cubic polynomial to neighbouring points over uniformly spaced parameter values. In this paper we replace uniform parameter values by chordal and centripetal ones. Since we update the parameterization at each refinement level, both schemes are non-linear. We prove convergence of the two schemes and bound the distance between the limit curve and the initial control polygon. Our numerical examples indicate that the limit curves are smooth and that the centripetal one is tighter, as suggested by our bounds.

Keywords: Non-linear subdivision, chordal curve parameterization, centripetal curve parameterization, cubic Lagrange interpolation.

1 Introduction

Dubuc's four-point subdivision scheme [3] is a method for generating a smooth curve passing through a sequence of points in $\mathbb{R}^d$. The algorithm is based on fitting cubic polynomials to local data, parameterized uniformly. This scheme was generalized by Daubechies, Guskov, and Sweldens [2] to allow non-uniform parameter values. Yet their scheme is linear in the data. Here we generalize further by determining the parameterization at each refinement level according to the geometry of the points at that level. We focus on the chordal and centripetal parameterizations [1, 6, 5]. The resulting two schemes are non-linear and cannot be analyzed by existing techniques.

Specifically, let $P_0 = \{\boldsymbol{p}_{0,k} : k \in \mathbb{Z}\}$ with $\boldsymbol{p}_{0,k} \in \mathbb{R}^d$ and $\boldsymbol{p}_{0,k+1} \neq \boldsymbol{p}_{0,k}$, be the initial set of control points, and let $P_j = \{\boldsymbol{p}_{j,k} : k \in \mathbb{Z}\}$ with $\boldsymbol{p}_{j,k} \in \mathbb{R}^d$ be the refined set of control points at level j. These points determine the set of parameter values $\{t_{j,k} : k \in \mathbb{Z}\}$ with $t_{j,0} = 0$ and $t_{j,k+1} - t_{j,k} = \|\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}\|^\alpha$ for $k \in \mathbb{Z}$, where $\alpha = 1$ gives chordal parameter values and $\alpha = 1/2$ gives centripetal ones. Note that $\alpha = 0$ corresponds to uniform parameterization. The refinement rule is then

$$\begin{aligned} \boldsymbol{p}_{j+1,2k} &= \boldsymbol{p}_{j,k}, \\ \boldsymbol{p}_{j+1,2k+1} &= \boldsymbol{\pi}_{j,k}(t_*), \end{aligned} \tag{1}$$

where $\boldsymbol{\pi}_{j,k}$ is the parametric cubic polynomial that interpolates $\boldsymbol{p}_{j,k-1}, \boldsymbol{p}_{j,k}, \boldsymbol{p}_{j,k+1}, \boldsymbol{p}_{j,k+2}$ at the values $t_{j,k-1}, t_{j,k}, t_{j,k+1}, t_{j,k+2}$ and $t_* = (t_{j,k} + t_{j,k+1})/2$; see Figure 1. We note that the four values $t_{j,k-1}, t_{j,k}, t_{j,k+1}, t_{j,k+2}$ must be distinct for the Lagrange interpolation to be well-defined. This in turn requires that each pair of consecutive points $\boldsymbol{p}_{j,k}$ and $\boldsymbol{p}_{j,k+1}$ be distinct. We assume this property holds for $j = 0$ and we prove that it holds for $j \geq 1$ for the chordal and centripetal schemes ($\alpha = 1$ and $\alpha = 1/2$, respectively).

We prove convergence of these two schemes and derive upper bounds on the distance between the limit curve and the initial control polygon. These schemes are very easy to implement and our numerical examples suggest that the limit curves are C^1, like Dubuc's scheme, but we have not so far been able to prove this. The numerical examples and our upper bounds indicate that the centripetal limit curve is tighter than the chordal and Dubuc's curves.

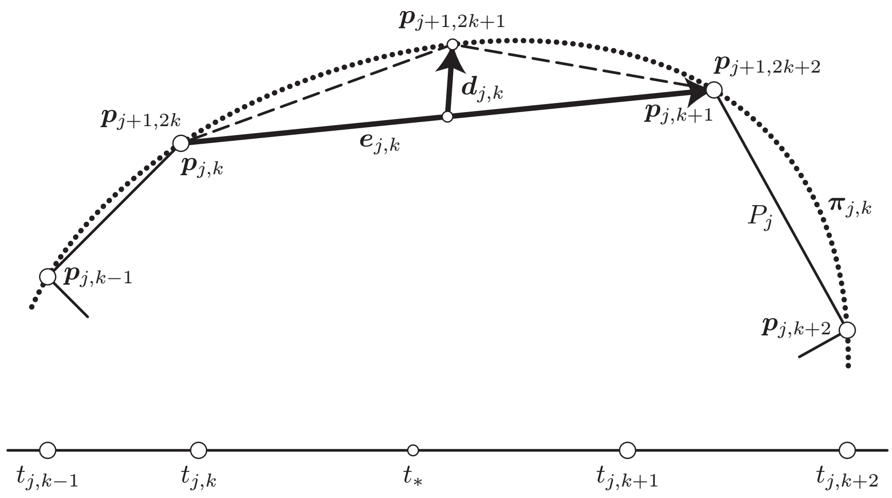

Figure 1: Insertion of a new point.

2 Cubic Lagrange interpolation

In order to analyze the schemes we need to establish some properties of cubic Lagrange interpolation. Consider functional data $f_0, f_1, f_2, f_3 \in \mathbb{R}$ given at the points t_0, t_1, t_2, t_3 and let f be the cubic polynomial satisfying $f(t_i) = f_i$ for $i = 0, 1, 2, 3$. Further let g be the linear polynomial that interpolates f_1 and f_2 at t_1 and t_2. Let $[s_0, s_1, \ldots, s_k]f$ denote the divided difference of f of order k at the points $s_0, s_1, \ldots, s_k$.

Lemma 1. *For $t \in \mathbb{R}$,*

$$f(t) - g(t) = \frac{(t - t_1)(t - t_2)}{t_3 - t_0} \big((t_3 - t)[t_0, t_1, t_2]f + (t - t_0)[t_1, t_2, t_3]f \big).$$

Proof. By inserting the recurrence formula

$$[t_0, t_1, t_2, t_3]f = ([t_1, t_2, t_3]f - [t_0, t_1, t_2]f)/(t_3 - t_0)$$

into the Newton form

$$f(t) = g(t) + (t - t_1)(t - t_2)[t_0, t_1, t_2]f + (t - t_0)(t - t_1)(t - t_2)[t_0, t_1, t_2, t_3]f,$$

the result follows. □

At the midpoint $t_* = (t_1 + t_2)/2$ of the interval $[t_1, t_2]$, Lemma 1 yields

$$f(t_*) - \frac{f_1 + f_2}{2} = -\frac{1}{4} \frac{(t_2 - t_1)^2}{t_3 - t_0} \big((t_3 - t_*)[t_0, t_1, t_2]f + (t_* - t_0)[t_1, t_2, t_3]f \big). \tag{2}$$

Consider now the subdivision scheme (1) and let $\boldsymbol{d}_{j,k}$ be the vector

$$\boldsymbol{d}_{j,k} = \boldsymbol{p}_{j+1,2k+1} - (\boldsymbol{p}_{j,k} + \boldsymbol{p}_{j,k+1})/2$$

depicted in Figure 1. Let $\boldsymbol{e}_{j,k} = \boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}$ and consider divided differences at level j,

$$\boldsymbol{p}_{j,k}^{[1]} = \frac{\boldsymbol{p}_{j,k+1} - \boldsymbol{p}_{j,k}}{t_{j,k+1} - t_{j,k}} = \frac{\boldsymbol{e}_{j,k}}{\|\boldsymbol{e}_{j,k}\|^\alpha}$$

$$\boldsymbol{p}_{j,k}^{[2]} = \frac{\boldsymbol{p}_{j,k+1}^{[1]} - \boldsymbol{p}_{j,k}^{[1]}}{t_{j,k+2} - t_{j,k}} = \left(\frac{\boldsymbol{e}_{j,k+1}}{\|\boldsymbol{e}_{j,k+1}\|^\alpha} - \frac{\boldsymbol{e}_{j,k}}{\|\boldsymbol{e}_{j,k}\|^\alpha} \right) \frac{1}{\|\boldsymbol{e}_{j,k+1}\|^\alpha + \|\boldsymbol{e}_{j,k}\|^\alpha}.$$

Combining Equation (2) with the subdivision rule in (1), we get

Lemma 2. *For all* $\alpha \in [0,1]$,

$$\boldsymbol{d}_{j,k} = -\frac{1}{4}\frac{(t_{j,k+1}-t_{j,k})^2}{a+b+1}\big((a+1/2)\boldsymbol{p}^{[2]}_{j,k} + (b+1/2)\boldsymbol{p}^{[2]}_{j,k-1}\big) \tag{3}$$

with $a = (t_{j,k}-t_{j,k-1})/(t_{j,k+1}-t_{j,k})$ *and* $b = (t_{j,k+2}-t_{j,k+1})/(t_{j,k+1}-t_{j,k})$.

Lemma 3. *For* $\alpha = 0$ *(uniform parameterization),*

$$\|\boldsymbol{d}_{j,k}\| \le \frac{1}{8}\max\{\|\boldsymbol{e}_{j,k-1}\|, \|\boldsymbol{e}_{j,k+1}\|\}, \tag{4}$$

for $\alpha = 1/2$ *(centripetal parameterization),*

$$\|\boldsymbol{d}_{j,k}\| \le \frac{1}{4}\|\boldsymbol{e}_{j,k}\|, \tag{5}$$

and for $\alpha = 1$ *(chordal parameterization),*

$$\|\boldsymbol{d}_{j,k}\| \le \frac{3}{8}\max\{\|\boldsymbol{e}_{j,k-1}\|, \|\boldsymbol{e}_{j,k}\|, \|\boldsymbol{e}_{j,k+1}\|\}. \tag{6}$$

Proof. Consider first the case $\alpha = 0$. Then

$$\boldsymbol{p}^{[2]}_{j,k} = (\boldsymbol{e}_{j,k+1} - \boldsymbol{e}_{j,k})/2,$$

and since $a = b = 1$, Equation (3) reduces to

$$\boldsymbol{d}_{j,k} = -\frac{1}{16}(\boldsymbol{e}_{j,k+1} - \boldsymbol{e}_{j,k-1}),$$

so that the estimate (4) follows immediately.

In the case $\alpha = 1/2$, since $\|\boldsymbol{p}^{[1]}_{j,k}\| = \|\boldsymbol{e}_{j,k}\|^{1/2}$, we have

$$\|\boldsymbol{p}^{[2]}_{j,k}\| \le \frac{\|\boldsymbol{p}^{[1]}_{j,k+1}\| + \|\boldsymbol{p}^{[1]}_{j,k}\|}{\|\boldsymbol{e}_{j,k+1}\|^{1/2} + \|\boldsymbol{e}_{j,k}\|^{1/2}} = 1,$$

and using this inequality in (3) gives (5).

To prove (6) we write (3) as

$$\boldsymbol{d}_{j,k} = -\frac{1}{4}\frac{t_{j,k+1}-t_{j,k}}{a+b+1}\big(A(\boldsymbol{p}^{[1]}_{j,k+1} - \boldsymbol{p}^{[1]}_{j,k}) + B(\boldsymbol{p}^{[1]}_{j,k} - \boldsymbol{p}^{[1]}_{j,k-1})\big),$$

where

$$A = \frac{a+1/2}{b+1} \qquad \text{and} \qquad B = \frac{b+1/2}{a+1}.$$

Then, since $\|\boldsymbol{p}^{[1]}_{j,k}\| = 1$, we get

$$\|\boldsymbol{d}_{j,k}\| \le \frac{1}{4}\frac{\|\boldsymbol{e}_{j,k}\|}{a+b+1}(A + |A-B| + B).$$

Now suppose that $a \ge b$. Then $A \ge B$ and

$$\|\boldsymbol{d}_{j,k}\| \le \frac{1}{4}\frac{\|\boldsymbol{e}_{j,k}\|}{a+b+1}\frac{2a+1}{b+1} \le \frac{2a+1}{4(a+1)}\|\boldsymbol{e}_{j,k}\|. \tag{7}$$

For $a \le 1$, this immediately gives

$$\|\boldsymbol{d}_{j,k}\| \le \frac{3}{8}\|\boldsymbol{e}_{j,k}\|,$$

and for $a \ge 1$, since $a = \|\boldsymbol{e}_{j,k-1}\|/\|\boldsymbol{e}_{j,k}\|$, we have

$$\|\boldsymbol{d}_{j,k}\| \le \frac{2a+1}{4a(a+1)}\|\boldsymbol{e}_{j,k-1}\| \le \frac{3}{8}\|\boldsymbol{e}_{j,k-1}\|.$$

Since the opposite case $a \le b$ is similar with $\|\boldsymbol{e}_{j,k+1}\|$ replacing $\|\boldsymbol{e}_{j,k-1}\|$, Equation (6) follows. □

We are now able to show that the centripetal and chordal subdivision schemes are well-defined.

Theorem 1. *For $\alpha = 1/2$ and $\alpha = 1$ any two consecutive points $\boldsymbol{p}_{j,k}$ and $\boldsymbol{p}_{j,k+1}$ are distinct.*

Proof. It is sufficient to show that

$$\|\boldsymbol{d}_{j,k}\| < \frac{1}{2}\|\boldsymbol{e}_{j,k}\|.$$

In the centripetal case, $\alpha = 1/2$, this follows immediately from (5). In the chordal case, $\alpha = 1$, it follows from (7) if $a \geq b$ and similarly for $a \leq b$. $\square$

3 Convergence

In this section we prove the convergence of the centripetal and the chordal schemes. A key ingredient of the proof is the fact that the edge lengths $\|\boldsymbol{e}_{j,k}\|$ converge to zero as j increases, which follows directly from Lemma 3.

Lemma 4. *For $\alpha = 0$,*

$$\max\{\|\boldsymbol{e}_{j+1,2k}\|, \|\boldsymbol{e}_{j+1,2k+1}\|\} \leq \frac{5}{8}\max\{\|\boldsymbol{e}_{j,k-1}\|, \|\boldsymbol{e}_{j,k}\|, \|\boldsymbol{e}_{j,k+1}\|\},$$

for $\alpha = 1/2$,

$$\max\{\|\boldsymbol{e}_{j+1,2k}\|, \|\boldsymbol{e}_{j+1,2k+1}\|\} \leq \frac{3}{4}\|\boldsymbol{e}_{j,k}\|, \tag{8}$$

and for $\alpha = 1$,

$$\max\{\|\boldsymbol{e}_{j+1,2k}\|, \|\boldsymbol{e}_{j+1,2k+1}\|\} \leq \frac{7}{8}\max\{\|\boldsymbol{e}_{j,k-1}\|, \|\boldsymbol{e}_{j,k}\|, \|\boldsymbol{e}_{j,k+1}\|\}. \tag{9}$$

Proof. By the definition of $\boldsymbol{e}_{j,k}$ and $\boldsymbol{d}_{j,k}$ we have $\boldsymbol{e}_{j+1,2k} = \boldsymbol{e}_{j,k}/2 + \boldsymbol{d}_{j,k}$ and $\boldsymbol{e}_{j+1,2k+1} = \boldsymbol{e}_{j,k}/2 - \boldsymbol{d}_{j,k}$. The statement then follows by using the triangle inequality and the bounds on $\|\boldsymbol{d}_{j,k}\|$ from Lemma 3. $\square$

Next, we represent each polygon P_j parametrically as the continuous piecewise linear function $\boldsymbol{f}_j \colon \mathbb{R} \to \mathbb{R}^d$ that interpolates the data $(2^{-j}k, \boldsymbol{p}_{j,k})$ and show that the sequence $\boldsymbol{f}_0, \boldsymbol{f}_1, \ldots$ is a Cauchy sequence in the sup norm.

Theorem 2. *The centripetal and chordal subdivision schemes converge.*

Proof. Since

$$\|\boldsymbol{f}_{j+1} - \boldsymbol{f}_j\|_\infty = \sup_{t\in\mathbb{R}} \|\boldsymbol{f}_{j+1}(t) - \boldsymbol{f}_j(t)\| = \sup_{k\in\mathbb{Z}} \|\boldsymbol{d}_{j,k}\|,$$

it follows from Lemma 3 that

$$\|\boldsymbol{f}_{j+1} - \boldsymbol{f}_j\|_\infty \leq \frac{3}{8}\sup_{k\in\mathbb{Z}} \|\boldsymbol{e}_{j,k}\|.$$

Since by Lemma 4,

$$\sup_{k\in\mathbb{Z}} \|\boldsymbol{e}_{j,k}\| \leq \mu \sup_{k\in\mathbb{Z}} \|\boldsymbol{e}_{j-1,k}\| \leq \cdots \leq \mu^j \sup_{k\in\mathbb{Z}} \|\boldsymbol{e}_{0,k}\|, \tag{10}$$

with $\mu < 1$, the sequence $\{\boldsymbol{f}_j : j \in \mathbb{N}_0\}$ is a Cauchy sequence in the sup norm and therefore converges to a continuous limit

$$\boldsymbol{f} = \lim_{j\to\infty} \boldsymbol{f}_j.$$

$\square$

We note that the estimates (6) in Lemma 3 and (9) in Lemma 4 actually hold for all $\alpha \in [0, 1]$ if the scheme is well-defined, so that the above proof implies convergence in that case. However, we found examples for $\alpha \in (0, 1/2)$ where the scheme fails because it generates identical consecutive points.

4 Distance bounds

In a similar way that Lemma 3 led to the convergence proof in the previous section, the same lemma can also be used to derive upper bounds on the Hausdorff distance d_{H} between the piece of the limit curve $\{\boldsymbol{f}(s) : s \in [k, k+1]\}$ and the line segment $[\boldsymbol{p}_{0,k}, \boldsymbol{p}_{0,k+1}]$. In order to prove these bounds, let us first establish a local variant of the estimate in Equation (10).

Lemma 5. *For* $\alpha = 0$*,*

$$\max_{2^j k-2 \le i \le 2^j(k+1)+1} \|\boldsymbol{e}_{j,i}\| \le \left(\frac{5}{8}\right)^j \max_{k-2 \le \ell \le k+2} \|\boldsymbol{e}_{0,\ell}\|,$$

for $\alpha = 1/2$*,*

$$\max_{2^j k \le i \le 2^j(k+1)-1} \|\boldsymbol{e}_{j,i}\| \le \left(\frac{3}{4}\right)^j \|\boldsymbol{e}_{0,k}\|,$$

and for $\alpha = 1$*,*

$$\max_{2^j k-2 \le i \le 2^j(k+1)+1} \|\boldsymbol{e}_{j,i}\| \le \left(\frac{7}{8}\right)^j \max_{k-2 \le \ell \le k+2} \|\boldsymbol{e}_{0,\ell}\|.$$

Proof. Since all the control points at level j between $\boldsymbol{p}_{0,k} = \boldsymbol{p}_{j,2^j k}$ and $\boldsymbol{p}_{0,k+1} = \boldsymbol{p}_{j,2^j(k+1)}$ depend only on the six initial points $\boldsymbol{p}_{0,k-2}, \boldsymbol{p}_{0,k-1}, \dots, \boldsymbol{p}_{0,k+3}$, the first and third inequalities follow from Lemma 4 by induction on j. The second inequality also follows by induction on j from (8) in Lemma 4. □

The upper bound on the Hausdorff distance now follows from this lemma and Lemma 3.

Theorem 3. *For* $\alpha = 0$*,*

$$d_{\mathrm{H}}\big(\boldsymbol{f}([k, k+1]), [\boldsymbol{p}_{0,k}, \boldsymbol{p}_{0,k+1}]\big) \le \frac{3}{13} \max_{k-2 \le \ell \le k+2} \|\boldsymbol{e}_{0,\ell}\|,$$

for $\alpha = 1/2$*,*

$$d_{\mathrm{H}}\big(\boldsymbol{f}([k, k+1]), [\boldsymbol{p}_{0,k}, \boldsymbol{p}_{0,k+1}]\big) \le \frac{5}{7} \|\boldsymbol{e}_{0,k}\|,$$

and for $\alpha = 1$*,*

$$d_{\mathrm{H}}\big(\boldsymbol{f}([k, k+1]), [\boldsymbol{p}_{0,k}, \boldsymbol{p}_{0,k+1}]\big) \le \frac{11}{5} \max_{k-2 \le \ell \le k+2} \|\boldsymbol{e}_{0,\ell}\|.$$

Proof. Let $s_{j,i} = 2^{-j} i$ and consider the difference between $\boldsymbol{f}_{j+2}$ and $\boldsymbol{f}_j$. Since

$$\boldsymbol{f}_{j+2}(s) - \boldsymbol{f}_j(s) = \begin{cases} \boldsymbol{0}, & s = s_{j+2,4i}, \\ \boldsymbol{d}_{j,i}/2 + \boldsymbol{d}_{j+1,2i}, & s = s_{j+2,4i+1}, \\ \boldsymbol{d}_{j,i}, & s = s_{j+2,4i+2}, \\ \boldsymbol{d}_{j,i}/2 + \boldsymbol{d}_{j+1,2i+1}, & s = s_{j+2,4i+3}, \end{cases}$$

we have

$$\sup_{s_{j,i} \le s \le s_{j,i+1}} \|\boldsymbol{f}_{j+2}(s) - \boldsymbol{f}_j(s)\| \le \max\{\|\boldsymbol{d}_{j,i}\|/2 + \|\boldsymbol{d}_{j+1,2i}\|, \|\boldsymbol{d}_{j,i}\|, \|\boldsymbol{d}_{j,i}\|/2 + \|\boldsymbol{d}_{j+1,2i+1}\|\}.$$

Using Lemma 3 and Lemma 4 we get the estimates

$$\max\{\|\boldsymbol{d}_{j+1,2i}\|, \|\boldsymbol{d}_{j+1,2i+1}\|\} \le \begin{cases} \frac{5}{64} \max_{i-2 \le \ell \le i+2} \|\boldsymbol{e}_{j,\ell}\|, & \alpha = 0, \\ \frac{3}{16} \|\boldsymbol{e}_{j,i}\|, & \alpha = 1/2, \\ \frac{21}{64} \max_{i-2 \le \ell \le i+2} \|\boldsymbol{e}_{j,\ell}\|, & \alpha = 1, \end{cases}$$

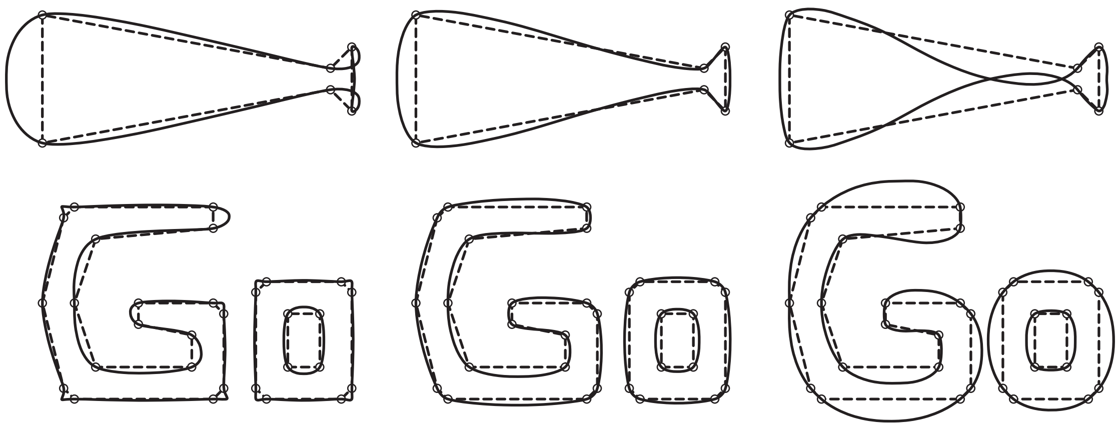

Figure 2: Examples of the four-point schemes.

and conclude that

$$\sup_{s_{j,i}\le s\le s_{j,i+1}} \|\boldsymbol{f}_{j+2}(s)-\boldsymbol{f}_j(s)\| \le \begin{cases} \frac{9}{64}\max_{i-2\le\ell\le i+2}\|\boldsymbol{e}_{j,\ell}\|, & \alpha=0,\\ \frac{5}{16}\|\boldsymbol{e}_{j,i}\|, & \alpha=1/2,\\ \frac{33}{64}\max_{i-2\le\ell\le i+2}\|\boldsymbol{e}_{j,\ell}\|, & \alpha=1. \end{cases}$$

Considering now all intervals $[s_{j,i}, s_{j,i+1}]$ between k and $k+1$, that is, $2^j k \le i \le 2^j(k+1)-1$, and taking Lemma 5 into account, we have

$$\sup_{k\le s\le k+1} \|\boldsymbol{f}_{j+2}(s)-\boldsymbol{f}_j(s)\| \le \begin{cases} \frac{9}{64}\left(\frac{5}{8}\right)^j\max_{k-2\le\ell\le k+2}\|\boldsymbol{e}_{0,\ell}\|, & \alpha=0,\\ \frac{5}{16}\left(\frac{3}{4}\right)^j\|\boldsymbol{e}_{0,k}\|, & \alpha=1/2,\\ \frac{33}{64}\left(\frac{7}{8}\right)^j\max_{k-2\le\ell\le k+2}\|\boldsymbol{e}_{0,\ell}\|, & \alpha=1. \end{cases}$$

The statement then follows because the Hausdorff distance is clearly bounded from above by the parametric distance between $\boldsymbol{f}$ and $\boldsymbol{f}_0$,

$$d_{\mathrm{H}}\big(\boldsymbol{f}([k,k+1]),[\boldsymbol{p}_{0,k},\boldsymbol{p}_{0,k+1}]\big) \le \sup_{k\le s\le k+1}\|\boldsymbol{f}(s)-\boldsymbol{f}_0(s)\| \le \sum_{j=0}^{\infty}\sup_{k\le s\le k+1}\|\boldsymbol{f}_{2j+2}(s)-\boldsymbol{f}_{2j}(s)\|,$$

and by noticing that $\frac{9}{64}\sum_{j=0}^{\infty}\left(\frac{5}{8}\right)^{2j}=\frac{3}{13}$, $\frac{5}{16}\sum_{j=0}^{\infty}\left(\frac{3}{4}\right)^{2j}=\frac{5}{7}$, and $\frac{33}{64}\sum_{j=0}^{\infty}\left(\frac{7}{8}\right)^{2j}=\frac{11}{5}$. □

5 Numerical examples

We have implemented Dubuc's scheme and its non-linear siblings corresponding to $\alpha=1/2$ and $\alpha=1$ in *C++*. Figure 2 shows the different limit curves for several initial control polygons. The plots confirm the well-known effect that Dubuc's scheme tends to give curves that are very tight to long edges and overshoot at short ones, often leading to unwanted cusps and loops. On the other hand, the non-linear chordal scheme leads to very roundish shapes that closely follow the short edges and have relatively large distance to the long ones. The limit curves of the centripetal scheme nicely mediate between these

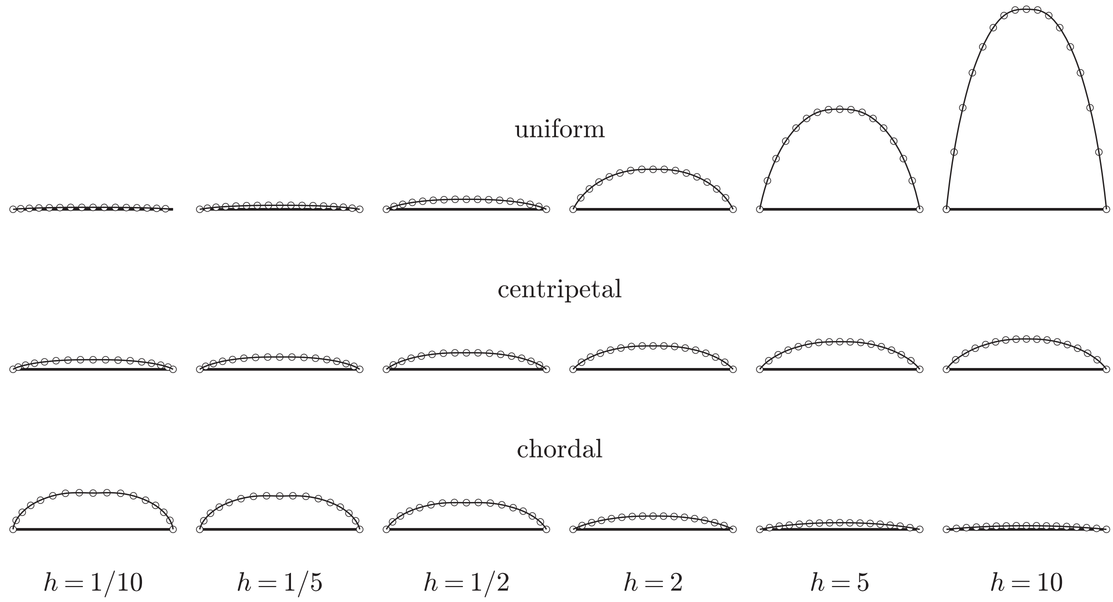

Figure 3: Shape effect over a rectangular control polygon.

two extremes: they are relatively close to all initial edges and still have a pleasing shape. Similar effects are known for cubic spline interpolation with uniform, centripetal, and chordal parameterization [4].

Another example that illustrates these shape effects is given in Figure 3 which shows the local behaviour of the limit curve over the top edge of a rectangle with fixed width 1 and varying height h. The dots mark the vertices of the refined polygon after four subdivision steps.

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