Analysis of Convergence and Smoothness by the Formalism of Laurent Polynomials

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Abstract. In order to design "good" subdivision schemes, tools for analyzing the convergence and smoothness of a scheme, given its mask, are needed.

A Laurent polynomial, encompassing all the available information on a subdivision scheme to be analysed, (a finite set of real numbers), is the basis of the analysis. By simple algebraic operations on such a polynomial, sufficient conditions for convergence of the subdivision scheme, and for the smoothness of the limit curves/surfaces generated by the subdivision scheme, can be checked rather automatically. The chapter concentrates on univariate subdivision schemes, (schemes for curve design) because of the simplicity of this case, and only hints on possible extensions to the bivariate case (schemes for surface design). The analysis is then demonstrated on schemes from the first two chapters [10,14].

1 Introduction

In this chapter, a procedure for analyzing the convergence of a subdivision scheme, based on the mask of the scheme, is presented. This procedure is derived and supported by mathematical analysis. The same mathematical tools lead also to a procedure for determining the smoothness of the limit functions generated by a convergent subdivision scheme. Our departure point is the following general form of one refinement step of the stationary subdivision scheme S with the mask $\mathbf{a} = \{a_i \ : \ i \in \mathbb{Z}^s\}$

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}^s} a_{i-2j} f_j^k, \quad i \in \mathbb{Z}^s, \tag{1}$$

with $s = 1$ for curves and $s = 2$ for surfaces.

For each scheme S with mask $\mathbf{a}$, we define the symbol

$$a(z) = \sum_{i \in \mathbb{Z}^s} a_i z^i, \tag{2}$$

with $z^i = z_1^{i_1} z_2^{i_2}$, in case $s = 2$. Since the schemes we consider have masks of finite support, the corresponding symbols are Laurent polynomials, namely polynomials in positive and negative powers of the variables. Any Laurent polynomial can be written as an algebraic polynomial (only non-negative powers) times a negative power.

Exercise 1.

1. Show that (1) corresponds to a univariate interpolatory scheme, whenever $s = 1$, $a_{2i} = 0,\ i \neq 0,\quad a_0 = 1$.
2. Show that (1) corresponds to a univariate m-th degree spline subdivision scheme, if $s = 1$ and $a(z) = 2^{-m}(1+z)^{m+1}$.
3. Show that for $s = 1$ there are two rules in (1), and four rules for $s = 2$.

The notion of Laurent polynomials enables us to write (1), in an algebraic form. Let $F(z; f) = \sum_{j\in\mathbb{Z}} f_j z^j$ be a formal generating function associated with the control points f. The relation (1) then becomes

$$F(z; S_a f) = a(z)F(z^2; f)\ . \tag{3}$$

Exercise 2. Show, by equating coefficients of the same power of the variables on both sides of the equality (3), that this equation is equivalent to (1).

Most of the procedures presented are for univariate schemes ($s = 1$). Only a special class of bivariate schemes is considered here, which includes the butterfly scheme. For tensor-product schemes on a regular quad-mesh, the convergence and smoothness follow from those of the corresponding univariate schemes.

2 Analysis of Univariate Schemes

Here we present several theorems, on univariate schemes, most of them without proofs, due to the limited scope of this chapter. Proofs can be found in [3].

Theorem 1. *Let S be a convergent subdivision scheme, with a mask* $\mathbf{a}$. *Then*

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1.$$

Exercise 3. Prove the above theorem.
Hint: Use (1) as the refinement step of S, and note that for k large enough, if the scheme is convergent then all f_j^k, which appear on both sides of the equality, are almost equal to each other.

It follows from Theorem 1 that the symbol of a convergent subdivision scheme satisfies,

$$a(-1) = 0, \quad a(1) = 2. \tag{4}$$

Thus the symbol factorizes into

$$a(z) = (1+z)q(z), \tag{5}$$

with $q(1) = 1$. The subdivision S_q with symbol $q(z)$ is related to S_a with symbol $a(z)$ by

Theorem 2. *Let S_a denote a subdivision scheme with symbol $a(z)$, and denote by $\Delta f = \{(\Delta f)_i = f_i - f_{i-1} \ : \ i \in \mathbb{Z}\}$, for $f = \{f_i \ : \ i \in \mathbb{Z}\}$. Then if (5) holds,*

$$\Delta(S_a f) = S_q \Delta f.$$

Proof. Recalling that $F(z; f)$ denotes the generating function of the control points f, we observe that

$$F(z; \Delta f) = (1-z)F(z; f) \ .$$

Thus, in view of (3) and (5)

$$\begin{aligned} F(z; \Delta S_a f) &= (1-z)F(z; S_a f) = (1-z)a(z)F(z^2; f) \\ &= q(z)(1-z^2)F(z^2; f) = q(z)F(z^2; \Delta f), \end{aligned} \tag{6}$$

which is equivalent to $\Delta(S_a f) = S_q \Delta f$. □

It is clear that if S_a is convergent then Δf^k tends to zero as $k \to \infty$. The opposite direction is also true.

Theorem 3. *The scheme S_a is convergent if and only if for all initial data f^0*

$$\lim_{k \to \infty} (S_q)^k f^0 = 0. \tag{7}$$

Proof. To prove convergence of the subdivision it is sufficient to show that the sequence $\{f^k(t) : k \in \mathbb{Z}_+\}$, where

$$f^k(t) \in \pi_1, \quad t \in (i2^{-k}, (i+1)2^{-k}), \quad f^k(2^{-k}i) = f_i^k, \quad i \in \mathbb{Z} \ ,$$

satisfies

$$\sup_{t \in \mathbb{R}} |f^{k+1}(t) - f^k(t)| \le C\eta^k, \quad |\eta| < 1 \ ,$$

since then this sequence is uniformly convergent. Observe that the maximum absolute value of the piecewise linear function $f^{k+1}(t) - f^k(t)$ is attained at its breakpoints. Thus

$$\sup_{t \in R} |f^{k+1}(t) - f^k(t)| = \max \left\{ \sup_{i \in \mathbb{Z}} |f_{2i}^{k+1} - f_i^k|, \ \sup_{i \in \mathbb{Z}} \left| f_{2i+1}^{k+1} - \frac{f_i^k + f_{i+1}^k}{2} \right| \right\} \ . \tag{8}$$

Now, let

$$g_{2i}^{k+1} = f_i^k, \quad g_{2i+1}^{k+1} = \frac{f_i^k + f_{i+1}^k}{2}, \quad i \in \mathbb{Z} \ ,$$

then $G_{k+1}(z) = F(z; g^{k+1})$ is obtained from $F_k(z) = F(z; f^k)$ by a relation as (3) with the mask $d(z) = \frac{z^{-1}}{2} + 1 + \frac{z}{2} = \frac{z^{-1}}{2}(1+z)^2$,

$$G_{k+1}(z) = d(z)F_k(z^2) \ . \tag{9}$$

If we denote by $\|F(z;f)\|_\infty = \max_{i\in\mathbb{Z}} |f_i| = \|f\|_\infty$, then by (8)

$$\sup_{t\in\mathbb{R}} |f^{k+1}(t) - f^k(t)| = \|F_{k+1} - G_{k+1}\|_\infty \ . \tag{10}$$

Using the symbol $a(z) = (1+z)q(z)$, with $q(1) = 1$, we observe that by (9)

$$\begin{aligned} F_{k+1}(z) - G_{k+1}(z) &= \Big((1+z)q(z) - d(z)\Big)F_k(z^2) \\ &= (1+z)\Big(q(z) - \frac{z^{-1}}{2}(1+z)\Big)F_k(z^2) \\ &= (1+z)(1-z)r(z)F_k(z^2) = r(z)H_k(z^2) \ , \end{aligned} \tag{11}$$

with $H_k(z) = F(z;\Delta f^k)$, and where in the equality before the last we used the fact that $q(z) - \frac{z^{-1}}{2}(1+z)$ is divisible by $(z-1)$, since it vanishes at $z = 1$.

Combining (10) and (11), we finally obtain in view of Theorem 2

$$\begin{aligned} \sup_{t\in\mathbb{R}} |f^{k+1}(t) - f^k(t)| &= \|F_{k+1} - G_{k+1}\|_\infty \le R \max_i |f_i^k - f_{i-1}^k| \\ &= R\|\Delta f^k\|_\infty \le R\|S_q^k \Delta f^0\|_\infty \ , \end{aligned} \tag{12}$$

where $R = \sum_i |r_i|$.

Now, if (7) holds for any initial data f^0, then there exists $L > 0$, $L \in \mathbb{Z}$, such that the operator of L iterations of S_q, S_q^L, satisfies

$$\|S_q^L\|_\infty = \mu < 1$$

and we get from (12)

$$\sup_{t\in\mathbb{R}} |f^{k+1}(t) - f^k(t)| \le R\mu^{[\frac{k}{L}]} \max_{0\le j<L} \|\Delta f^j\|_\infty \le C\eta^k$$

with $\eta = \mu^{\frac{1}{L}} < 1$. □

A scheme S_q satisfying (7) for all initial data f^0 is termed "contractive". By Theorem 3, the check of the convergence of S_a is equivalent to checking whether S_q is contractive, which is equivalent to checking whether $\|S_q^L\|_\infty < 1$, for some $L \in \mathbb{Z}_+$, $L > 0$.

Now, from (1) with **q** replacing **a**

$$\|S_q\|_\infty = \max\left\{\sum_i |q_{2i}|,\ \sum_i |q_{2i+1}|\right\} \ ,$$

since in (1) there are two rules for computing the values at the next refinement level, one with the even, and one with the odd coefficients of the mask.

To compute $\|S_q^L\|_\infty$, we first observe that by (6), L iterations of S_q are given by the symbol

$$q_L(z) = q(z)q(z^2)\cdots q(z^{2^{L-1}}) \ ,$$

since

$$H_{k+L}(z) = q(z)H_{k+L-1}(z^2) = q(z)q(z^2)H_{k+L-2}(z^4) = \cdots = q_L(z)H_k(z^{2^L}) \ .$$

The relation

$$H_{k+L}(z) = q_L(z)H_k(z^{2^L})$$

with $q_L(z) = \sum q_i^{[L]} z^i$ is equivalent to the rules

$$(\Delta f^{k+L})_i = \sum_j q_{i-2^L j}^{[L]} (\Delta f^k)_j \ .$$

There are 2^L different subsets of $\{q_\ell^{[L]}\}$ used above, depending on the remainder in the division of i by 2^L. Thus

$$\|S_q^L\|_\infty = \max\left\{\sum_j |q_{i-2^L j}^{[L]}| : 0 \le i < 2^L\right\} \ . \tag{13}$$

The algorithm for verifying convergence, given the symbol $a(z)$ of the scheme S_a.

1. If $a(1) \neq 2$ the scheme does not converge. Stop!
2. If $a(-1) \neq 0$ the scheme does not converge. Stop!
3. Compute $q(z) = \frac{a(z)}{1+z}$.
4. Set $q_1(z) = \sum_i q_i^{[1]} z^i = q(z)$.
5. For $L = 1, \ldots, M$,
   (a) Compute $N_L = \max_{0\le i<2^L} \sum_j |q_{i-2^L j}^{[L]}|$.
   (b) If $N_L < 1$, S_a is convergent. Stop!
   (c) If $N_L \ge 1$ compute $q_{L+1}(z) = q(z)q_L(z^2) = \sum_i q_i^{[L+1]} z^i$.
6. S_q is not contractive after M iterations. Stop!

M is a parameter of the algorithm. If for small L, $\|S_q^L\|_\infty = \mu < 1$, then

$$\|\Delta f^k\|_\infty = \max_i |f_i^k - f_{i-1}^k| \le \mu^{[\frac{k}{L}]} \max_{0\le \ell < L} \|\Delta f^\ell\|_\infty$$

and the differences are small after a small number of iterations of S_a. In this case, it is enough to apply only a small number of refinement steps in order to "see" convergence. If $\|S_q^L\|_\infty \ge 1$ for $L = 1, \ldots, M-1$ and $\|S_q^M\|_\infty < 1$, with large M, then many iterations are needed, before the refined data "looks" continuous. Thus, from a practical point of view, $5 \le M \le 10$ is a reasonable choice of the parameter M.

The analysis of smoothness is similar. It is based on the following

Observation: It follows from Theorem 2 that if $a(z) = \frac{(1+z)^m}{2^m} b(z)$, then for $f^k = S_a f^{k-1}$

$$\frac{\Delta^m f^k}{(2^{-k})^m} = S_b \frac{\Delta^m f^{k-1}}{(2^{-(k-1)})^m} \ . \tag{14}$$

Since for data sampled from $f \in C^m(\mathbb{R})$, namely for

$$f_i^k = f(2^{-k} i), \quad i \in \mathbb{Z}, \quad k \in \mathbb{Z}_+$$

and for fixed $\ell \in \mathbb{Z}_+$

$$\lim_{\substack{k \to \infty \\ k > \ell}} (2^{mk} \Delta^m f^k)_{i 2^{k-\ell}} = f^{(m)}(i 2^{-\ell}) \ ,$$

the following result is plausible.

Theorem 4. *Let $a(z) = \frac{(1+z)^m}{2^m} b(z)$. If S_b is convergent then $S_a^\infty f^0 \in C^m(\mathbb{R})$ for any initial data f^0, and*

$$\frac{d^m}{dt^m} (S_a^\infty f^0)(t) = \big(S_b^\infty (\Delta^m f^0)\big)(t) \ , \tag{15}$$

where $\Delta^m f = \Delta(\Delta^{m-1} f)$ is defined recursively.

Thus the procedure for checking C^m smoothness of S_a is reduced to the verification of convergence of a scheme S_b, obtained from S_a by the factorization of the symbol $a(z)$ to

$$a(z) = \frac{(1+z)^m}{2^m} b(z) \ .$$

Example 1. The univariate spline schemes.
Univariate spline schemes are introduced in Section 2 of chapter [14]. The symbol of the subdivision scheme with basic limit function the B-spline (univariate box-spline) of degree m is

$$a^{[m]}(z) = \frac{(1+z)^{m+1}}{2^m}, \quad m \geq 0 \ . \tag{16}$$

To verify convergence consider the symbol

$$q^{[m]}(z) = \frac{a^{[m]}(z)}{1+z} = \frac{(1+z)^m}{2^m} = \frac{1}{2^m} \sum_{j=0}^{m} \binom{m}{j} z^j \ ,$$

$S_{a^{[m]}}$ is convergent if and only if $S_{q^{[m]}}$ is contractive. Now

$$\|S_{q^{[m]}}\|_\infty = \max \left\{ \frac{1}{2^m} \sum_{j=0}^{[\frac{m}{2}]} \binom{m}{2j}, \ \frac{1}{2^m} \sum_{j=0}^{[\frac{m}{2}]} \binom{m}{2j+1} \right\} \ .$$

But

$$\frac{1}{2^m}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{2j} = \frac{1}{2^m}\sum_{j=0}^{[\frac{m}{2}]}\binom{m}{2j+1} = \frac{(1+1)^m}{2^{m+1}}, \quad m \geq 1 \ .$$

Thus for $m \geq 1$, $\|S_{q^{[m]}}\|_\infty = \frac{1}{2}$, and $S_{q^{[m]}}$ is contractive, proving the convergence of $S_{a^{[m]}}$ for $m \geq 1$.

To obtain the smoothness of the limit functions generated by $S_{a^{[m]}}$, we show that the symbols $\frac{2^\ell a^{[m]}(z)}{(1+z)^\ell}$, $1 \leq \ell \leq m-1$ determine convergent schemes. It is sufficient to consider $b(z) = \frac{2^{m-1}a^{[m]}(z)}{(1+z)^{m-1}} = \frac{(1+z)^2}{2}$. Since $b(z) = a^{[1]}(z)$, S_b is convergent and so $S_{a^{[m]}}$ generates C^{m-1} limit functions for $m \geq 1$.

Example 2. The 4-point scheme
The insertion rule of the 4-point interpolatory subdivision scheme is

$$f_{2i+1}^{k+1} = -w(f_{i-1}^k + f_{i+2}^k) + (\tfrac{1}{2}+w)(f_i^k + f_{i+1}^k) \ .$$

Thus the mask of this scheme is (see Subsection 1.2 in [10]),

$$a_0 = 1, \quad a_{\pm1} = \tfrac{1}{2}+w, \quad a_{\pm2} = 0, \quad a_{\pm3} = -w \tag{17}$$

and the symbol is

$$a(z) = z^{-3}(1+z)^2\left(\tfrac{1}{2}z^2 - w(z-1)^2(1+z^2)\right) \ . \tag{18}$$

Now,

$$q(z) = \frac{a(z)}{1+z} = -wz^{-3} + wz^{-2} + \tfrac{1}{2}z^{-1} + \tfrac{1}{2} + wz - wz^2 \tag{19}$$

and $\|S_q\|_\infty = \frac{1}{2} + 2|w|$. S_a is convergent iff $\|S_q^L\|_\infty < 1$ for some $L \in \mathbb{Z}_+\backslash 0$. In case $L = 1$, $\|S_q\|_\infty < 1$ if $|w| < \frac{1}{4}$. Computing $q_2(z) = q(z)q(z^2)$ we get

$$\begin{aligned} q_2(z) = {} & w^2z^{-9} - w^2z^{-8} - (\tfrac{1}{2}w + w^2)z^{-7} + (w^2 - \tfrac{1}{2}w)z^{-6} \\ & -w^2z^{-5} + (w+w^2)z^{-4} + (\tfrac{1}{4} + w^2 - \tfrac{1}{2}w)z^{-3} + (\tfrac{1}{2}w + \tfrac{1}{4} - w^2)z^{-2} \\ & +(\tfrac{1}{2}w + \tfrac{1}{4} - w^2)z^{-1} + (\tfrac{1}{4} + w^2 - \tfrac{1}{2}w) + (w+w^2)z - w^2z^2 \\ & +(w^2 - \tfrac{1}{2}w)z^3 - (\tfrac{w}{2} + w^2)z^4 - w^2z^5 + w^2z^6 \ . \end{aligned}$$

This leads to

$$\begin{aligned} \|S_q^2\|_\infty = \max\Big\{ & |\tfrac{1}{2}+w||w| + |\tfrac{1}{4} + w^2 - \tfrac{1}{2}w| + |w||1+w| + w^2, \\ & |w||w - \tfrac{1}{2}| + |\tfrac{1}{4} + \tfrac{1}{2}w - w^2| + 2w^2\Big\} \ . \end{aligned} \tag{20}$$

Thus, for the case $L = 2$ we get from the requirement $\|S_q^2\|_\infty < 1$, the range $-\frac{3}{8} < w < \frac{-1+\sqrt{13}}{8} < \frac{1}{2}$ which is bigger than the range $|w| < \frac{1}{4}$, obtained from the case $L = 1$.

By considering all $L \in \mathbb{Z}_+$ it was computed that the range for $w > 0$ is $0 < w < \frac{1}{2}$. The range $-\frac{1}{2} < w < 0$ is obtained from results on positive masks. Thus the exact range of w for S_a to be convergent is $-\frac{1}{2} < w < \frac{1}{2}$.

As for smoothness analysis, consider S_b with $b(z) = \frac{2a(z)}{1+z}$. Then S_b is convergent if and only if S_r is contractive, with

$$r(z) = \frac{b(z)}{1+z} = \frac{2a(z)}{(1+z)^2} = 2z^{-3}\left(\tfrac{1}{2}z^2 - w(z-1)^2(1+z^2)\right) .$$

But $\|S_r\|_\infty = \max\{8|w|, |1-4w| + 4|w|\} \geq 1$, and to see contractivity we consider $\|S_r^2\|_\infty$. The condition $\|S_r^2\|_\infty < 1$ gives the range $0 < w < \frac{-1+\sqrt{5}}{8} \cong 0.154$. Note that the special value $w = \frac{1}{16}$ is contained in this range. In this range of w, S_r is contractive implying that S_b is convergent and therefore S_a generates C^1 limit functions.

To check C^2 smoothness, we consider $w = \frac{1}{16}$. This is the only value of w for which the necessary condition of Theorem 1 in the previous chapter [10] is satisfied (the scheme is exact for cubics). In this case, the limit functions of S_a are C^2 if the scheme S_t is contractive, where

$$t(z) = \frac{4a(z)}{(1+z)^3} = \frac{z^{-1}}{4}(z^3 - 3z^2 - 3z + 1) .$$

But $\|S_t^L\|_\infty = 1$ for $L \in \mathbb{Z}_+ \backslash 0$ and our method of analysis fails to show that $S_a^\infty f^0 \in C^2$ for $w = \frac{1}{16}$. In fact it is possible to show by the Eigenanalysis, presented in the next chapter [15], that $S_a^\infty f^0$ does not have a second derivative at all dyadic points (see the solution of Exercise 3 in chapter [10]).

Exercise 4. Derive (18) from (17), and verify (19).

3 Analysis of Bivariate Schemes with Factorizable Symbols

Here we present similar analysis tools to those in the univariate case for bivariate subdivision schemes defined on regular quad-meshes and for subdivision schemes on regular triangulations.

3.1 Analysis of Schemes Defined on Regular Quad-Meshes

Theorem 5. *Let $a(z) = a(z_1, z_2) = \sum_{i,j} a_{ij} z_1^i z_2^j$ be the symbol of a bivariate subdivision scheme S, which is defined on quad-meshes. Then a necessary condition for the convergence of S is*

$$\sum_{\beta \in \mathbb{Z}^2} a_{\alpha - 2\beta} = 1, \quad \alpha \in \left\{(0,0), (0,1), (1,0), (1,1)\right\} . \tag{21}$$

The proof of this theorem is similar to the proof of Theorem 1.

Exercise 5. Show that condition (21) implies that

$$a(1,1)=4, \quad a(-1,1)=0, \quad a(1,-1)=0, \quad a(-1,-1)=0 \ . \tag{22}$$

In contrast to the univariate case ($s=1$), in the bivariate case ($s=2$) the necessary condition (21) and the derived conditions on $a(z)$, (22), do not imply a factorization of the mask.

We impose the following factorization

$$a(z)=(1+z_1)(1+z_2)b(z), \qquad z=(z_1,z_2) \ . \tag{23}$$

Theorem 6. *Suppose the schemes with the symbols*

$$a_1(z)=\frac{a(z)}{1+z_1}=(1+z_2)b(z),$$
$$a_2(z)=\frac{a(z)}{1+z_2}=(1+z_1)b(z)$$

are both contractive, namely

$$\lim_{k\to\infty}(S_{a_1})^k f^0=0, \quad \lim_{k\to\infty}(S_{a_2})^k f^0=0$$

for any initial data f^0, then the scheme S_a with the symbol (23) *is convergent. Conversely, if S_a is convergent then S_{a_1} and S_{a_2} are contractive.*

The proof of this theorem is similar to the proof of Theorem 3, due to the following observation: Define $\Delta_1 f=\{f_{i,j}-f_{i-1,j} : i,j\in\mathbb{Z}\}$, and $\Delta_2 f=\{f_{i,j}-f_{i,j-1} : i,j\in\mathbb{Z}\}$. Then $S_{a_\ell}\Delta_\ell f=\Delta_\ell S_a f$, $\ell=1,2$.

Thus convergence is checked in this case by checking the contractivity of two subdivision schemes S_{a_1}, S_{a_2}. If in (23) $b(z_1,z_2)=b(z_2,z_1)$, which is typical for schemes having the symmetry of the square grid (topologically equivalent rules for the computation of vertices corresponding to edges), then $a_1(z_1,z_2)=a_2(z_2,z_1)$, and the contractivity of only one scheme has to be checked.

For the smoothness result, we introduce the inductive definition of differences: $\Delta^{[i,j]}=\Delta_1\Delta^{[i-1,j]}$, $\Delta^{[i,j]}=\Delta_2\Delta^{[i,j-1]}$, $\Delta^{[1,0]}=\Delta_1$, $\Delta^{[0,1]}=\Delta_2$.

Theorem 7. *Let*

$$a(z)=(1+z_1)^m(1+z_2)^m b(z) \ . \tag{24}$$

If the schemes with the masks

$$a_{i,j}(z)=\frac{2^{i+j}a(z)}{(1+z_1)^i(1+z_2)^j}, \quad i,j=0,\ldots,m \tag{25}$$

are convergent, then S_a generates C^m limit functions. Moreover,

$$\frac{\partial^{i+j}}{\partial t_1^i \partial t_2^j}S_a^\infty f^0=S_{a_{i,j}}^\infty \Delta_1^i\Delta_2^j f^0 \ , \quad i,j=0,\ldots,m. \tag{26}$$

To verify that a scheme S_a generates C^1 limit functions, with the aid of the last two theorems, we have to assume that

$$a(z) = (1+z_1)^2(1+z_2)^2 b(z) \ ,$$

and to check the contractivity of the three schemes with the symbols

$$2(1+z_1)(1+z_2)b(z), \quad 2(1+z_2)^2 b(z), \quad 2(1+z_1)^2 b(z) \ .$$

Exercise 6. Verify the last statement.

This analysis applies also to tensor-product schemes, but is not needed, since if $a(z) = a_1(z_1)a_2(z_2)$ is the symbol of a tensor-product scheme, then the basic limit function of S_a, ϕ_a, is $\phi_a(t_1,t_2) = \phi_{a_1}(t_1) \cdot \phi_{a_2}(t_2)$, and its smoothness properties are derived from those of ϕ_{a_1}, ϕ_{a_2}.

3.2 Analysis of Schemes Defined on Regular Triangulations

For the topology of a regular triangulation, we regard the subdivision scheme as operating on the 3-directional grid. (The vertices of $\mathbb{Z}^2$ with edges in the directions $(1,0),(0,1),(1,1)$.)

Since the 3-directional grid can be regarded also as $\mathbb{Z}^2$, (21) and (22) hold for convergent schemes on this grid.

A scheme with a mask which treats each edge in the 3-directional grid in the same way with respect to the topology of the grid, is a scheme for regular triangulations. The symbol of such a scheme, when being factorizable, has the form

$$a(z) = (1+z_1)^m(1+z_2)^m(1+z_1z_2)^m b(z) \ , \tag{27}$$

where $b(z)$ has some symmetries, e.g. $b(z_1,z_2) = b(z_2,z_1)$.

Example 3. The symbol of the butterfly scheme on this grid has the form

$$a(z) = \frac{1}{2}(1+z_1)(1+z_2)(1+z_1z_2)(1-wc(z_1,z_2))(z_1z_2)^{-1} \tag{28}$$

with

$$\begin{aligned} c(z_1,z_2) = {} & 2z_1^{-2}z_2^{-1} + 2z_1^{-1}z_2^{-2} - 4z_1^{-1}z_2^{-1} - 4z_1^{-1} - 4z_2^{-1} \\ & + 2z_1^{-1}z_2 + 2z_1z_2^{-1} + 12 - 4z_1 - 4z_2 - 4z_1z_2 + 2z_1^2z_2 + 2z_1z_2^2 \ . \end{aligned} \tag{29}$$

Exercise 7.

1. Derive the mask of the butterfly scheme from the insertion rule of the scheme. *Hint:* Consult the solution of Exercise 11 in chapter [10].
2. Derive the symbol of the butterfly scheme.
3. Verify (28) and (29).

Convergence analysis for schemes with factorizable symbols of the form (27) is similar to that for schemes defined on quad-meshes.

Theorem 8. *Let S_a have the symbol*

$$a(z) = (1+z_1)(1+z_2)(1+z_1 z_2)b(z) \ . \tag{30}$$

S_a is convergent if and only if the schemes with symbols

$$a_1(z) = \frac{a(z)}{1+z_1}, \quad a_2(z) = \frac{a(z)}{1+z_2}, \quad a_3(z) = \frac{a(z)}{1+z_1 z_2} \tag{31}$$

are contractive. If any two of these schemes are contractive then the third is also contractive.

Note that

$$S_{a_3}\Delta_3 f = \Delta_3 S_a f \ ,$$

where $(\Delta_3 f)_{i,j} = f_{i,j} - f_{i-1,j-1}$.

If two of the schemes S_{a_i}, $i = 1,2,3$, are contractive then the differences in two linearly independent directions tend to zero as $k \to \infty$, which implies as in the proof of Theorem 3, that the bilinear interpolants $\{f^k(t)\}_{k\in\mathbb{Z}_+}$, where

$$\left.\begin{aligned} & f_k(t)\Big|_{t\in[i,i+1]\times[j,j+1]} \in \pi_1 \times \pi_1, \\ & f_k(i,j) = f^k_{i,j}, \quad (i,j)\in\mathbb{Z}^2 \ , \end{aligned}\right. \qquad k \in \mathbb{Z}_+,$$

is a Cauchy sequence of continuous functions, with a continuous limit.

The smoothness analysis for a scheme with a symbol (30) is different from that for schemes defined on regular quad meshes.

Theorem 9. *Let S_a have the symbol* (30)*. Then S_a generates C^1 limit functions if the schemes with the symbols $2a_i(z)$, $i = 1,2,3$, are convergent. If any two of these schemes are convergent then the third is also convergent. Moreover,*

$$\begin{aligned} \frac{\partial}{\partial t_i} S_a^\infty f^0 &= S_{2a_i}\Delta_i f^0, \quad i = 1,2 \\ \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right) S_a^\infty f^0 &= S_{2a_3}\Delta_3 f^0 \ . \end{aligned}$$

The verification that the scheme S_a with symbol (30) generates C^1 limit functions, based on Theorems 8 and 9, requires checking the contractivity of the three schemes with symbols,

$$2(1+z_1)b(z), \quad 2(1+z_2)b(z), \quad 2(1+z_1 z_2)b(z) \ .$$

If these three schemes are contractive, then S_a generates C^1 limit functions.

Exercise 8. Verify the last statement.

Example 4. Continuation of Example 3.
To verify that the butterfly scheme generates C^1 limit functions, we use the fact that the symbol $a(z)$ of the butterfly scheme, given in (28) is factorizable. In view of the observation following Theorem 9, we have to check the contractivity of the schemes with the symbols

$$\begin{aligned} q_1(z) &= (1+z_1)\big(1-wc(z_1,z_2)\big)(z_1z_2)^{-1} \\ q_2(z) &= (1+z_2)\big(1-wc(z_1,z_2)\big)(z_1z_2)^{-1} \\ q_3(t) &= (1+z_1z_2)\big(1-wc(z_1,z_2)\big)(z_1z_2)^{-1} \ . \end{aligned}$$

Noting that

$$c(z_1,z_2) = c(z_2,z_1) = c(z_1z_2, z_1^{-1}) \ ,$$

and that the factor $(z_1z_2)^{-1}$ in a symbol does not affect the norm of the corresponding subdivision operator, it is sufficient to verify the contractivity of S_r, where

$$r(z) = (z_1z_2)q_1(z) = (1+z_1)\big(1-wc(z_1,z_2)\big) = \sum_{\alpha\in\mathbb{Z}^2} r_\alpha z^\alpha \ .$$

Now

$$\|S_r\|_\infty = \max_{0\le k,\ell\le 1}\Big(\sum_{i,j\in\mathbb{Z}} |r_{k+2i,\ell+2j}|\Big)$$

and since

$$\sum_{i,j\in\mathbb{Z}} |r_{2i,2j}| = |1-8w| + |8w|$$

$\|S_r\|_\infty \ge 1$ for all values of w.

Next, we show that there exists an interval $(0,w_0)$, such that we have for $w \in (0,w_0)$, $\|S_r^2\|_\infty < 1$. The value of w_0 is not computed (for a computed value of w_0 see the Bibliographical notes).

The following computation is based on the fact that w is small. Computing only terms which are bigger than $\mathcal{O}(w)$, we get

$$\begin{aligned} r^{[2]}(z) = r(z)r(z^2) &= (1+z_1+z_1^2+z_1^3)\big(1-wc(z_1,z_2)-wc(z_1^2,z_2^2)+\mathcal{O}(w^2)\big) \\ &= \sum_{i,j\in\mathbb{Z}} r_{ij}^{[2]} z_1^i z_2^j \ . \end{aligned}$$

Thus for $j \neq 0$, $r_{i,j}^{[2]} = \mathcal{O}(w)$ while $r_{i,0}^{[2]} = 1+\mathcal{O}(w)$, $i = 0,1,2,3$. From this we conclude that it is sufficient to show that for small enough w

$$\sum_{i,j\in\mathbb{Z}} \big|r_{\ell+4i,4j}^{[2]}\big| < 1, \quad \ell = 0,1,2,3 \ .$$

This requires considering terms of order $\mathcal{O}(w)$, and to ignore terms of order $\mathcal{O}(w^2)$. In case $\ell = 0$, all the non-zero coefficients $\{r^{[2]}_{4i,4j}\}$ are

$$r^{[2]}_{0,0} = 1-16w+\mathcal{O}(w^2), \quad r^{[2]}_{4,0} = 8w+\mathcal{O}(w^2), \quad r^{[2]}_{4,4} = r^{[2]}_{0,-4} = -2w+\mathcal{O}(w^2) \; .$$

Hence for $w > 0$ small enough

$$\sum_{i,j\in\mathbb{Z}} \left|r^{[2]}_{4i,4j}\right| = |1-16w| + 12|w| + \mathcal{O}(w^2) < 1 \; .$$

In case $\ell = 1$, the relevant coefficients are

$$r^{[2]}_{1,0} = 1-12w+\mathcal{O}(w^2), \quad r^{[2]}_{5,0} = 4w+\mathcal{O}(w^2), \quad r^{[2]}_{5,4} = r^{[2]}_{1,-4} = -2w+\mathcal{O}(w^2) \; .$$

Hence for $w > 0$ small enough,

$$\sum_{i,j} |r_{1+4i,4j}| = |1-12w| + 8|w| + \mathcal{O}(w^2) < 1 \; .$$

The cases $\ell = 2$ and $\ell = 3$ are similar to the cases $\ell = 1$ and $\ell = 0$, respectively. Thus for $w > 0$ small enough the limit surfaces/functions generated by the butterfly scheme on regular triangulations are C^1.

This establishes that the butterfly scheme generates C^1 surfaces/functions on regular grids, and in the vicinities of vertices of valency 6 in general triangulations, due to the locality of the insertion rule.

4 Bibliographic Notes

A detailed presentation of the analysis tools of this section is given in [3], with complete mathematical justifications, and with a detailed extension to the general bivariate case on quad-meshes.

The formalism of Laurent polynomials was introduced in [1]. For univariate schemes, the analysis of convergence, based on the contractivity of the corresponding difference scheme, was introduced in [5], together with the smoothness analysis based on the divided difference schemes. The ranges of the parameter w for the convergence of the 4-point scheme were computed in [4,5,9]. The full range $|w| < \frac{1}{2}$, is obtained for $w < 0$ from results on positive masks [13], while for $w > 0$, was computed by M.J.D. Powell. The analysis of the smoothness of the butterfly scheme on regular grids for small values of w, as presented here, is done in [9]. In [11] it is shown that for $0 < w \leq \frac{1}{12}$, the butterfly scheme generates C^1 limit functions/surfaces on regular grids.

The formalism of Laurent polynomials is inadequate for the analysis of non-stationary or non-uniform schemes. For a certain class of non-stationary schemes the analysis is done by relations to stationary schemes [6], [8], [7], while for certain non-uniform schemes the analysis is done by divided difference schemes [2].

To prove negative results (such as the result that the 4-point scheme generates functions without a second derivative at the dyadic points), one can show that certain necessary conditions are violated. Such necessary conditions are obtained by the Eigenanalysis presented in the next chapter [15]. This approach is explained in details in [3].

References

1. A. S. Cavaretta, W. Dahmen, and C. A. Micchelli. Stationary subdivision. Memoirs of AMS **93**, 1991.
2. I. Daubechies, I. Guskov, and W. Sweldens. Regularity of irregular subdivision. Constr. Approx. **15**, 1999, 381–426.
3. N. Dyn. Subdivision schemes in Computer-Aided Geometric Design. *Advances in Numerical Analysis, Vol. II, Wavelets, Subdivision Algorithms and Radial Basis Functions*, W. Light (ed.), Clarendon Press, Oxford, 1992, 36–104.
4. N. Dyn, J. A. Gregory, and D. Levin. A four-point interpolatory subdivision scheme for curve design. Comput. Aided Geom. Design **4**, 1987, 257–268.
5. N. Dyn, J. A. Gregory, and D. Levin. Analysis of linear binary subdivision schemes for curve design. Constr. Approx. **7**, 1991, 127–147.
6. N. Dyn and D. Levin. Analysis of asymptotically equivalent binary subdivision schemes. J. Math. Anal. Appl. **193**, 1995, 594–621.
7. N. Dyn and D. Levin. Subdivision schemes in geometric modeling. To appear in Acta Numerica, 2002.
8. N. Dyn, D. Levin, and A. Luzzatto. Non-stationary interpolatory subdivision schemes reproducing spaces of exponential polynomials. Preprint.
9. N. Dyn, D. Levin, and C. A. Micchelli. Using parameters to increase smoothness of curves and surfaces generated by subdivision. Comput. Aided Geom. Design **7**, 1990, 129–140.
10. N. Dyn. Interpolatory subdivision schemes. This volume.
11. J. A. Gregory. An introduction to bivariate uniform subdivision. *Numerical Analysis 1991*, D. F. Griffiths and G. A. Watson (eds.), Pitman Research Notes in Mathematics, Longman Scientific and Technical, 1991, 103–117.
12. A. Luzzatto. *Multi-Scale Signal Processing, Based on Non-Stationary Subdivision*. PhD Thesis, Tel-Aviv University, 2000.
13. C. A. Micchelli and H. Prautzsch. Uniform refinement of curves. Linear Algebra Appl. **114/115**, 1989, 841–870.
14. M. Sabin. Subdivision of box-splines. This volume.
15. M. Sabin. Eigenanalysis and artifacts of subdivision curves and surfaces. This volume.

Solutions of Selected Exercises

Exercise 1.

1. Show that (1) corresponds to a univariate interpolatory scheme, whenever $s = 1$, $a_{2i} = 0,\ i \neq 0,\quad a_0 = 1$.
2. Show that (1) corresponds to a univariate m-th degree spline subdivision scheme, if $s = 1$ and $a(z) = 2^{-m}(1+z)^{(m+1)}$.
3. Show that for $s = 1$ there are two rules in (1), and four rules for $s = 2$.

Solution 1.

1. The refinement step (1)

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}^s} a_{i-2j} f_j^k, \quad i \in \mathbb{Z}^s ,$$

with $s = 1,\ a_{2i} = 0,\ i \neq 0,\ a_0 = 1$, has the form

$$\begin{aligned} f_{2\ell}^{k+1} &= f_\ell^k, & i &= 2\ell, \\ f_{2\ell+1}^{k+1} &= \textstyle\sum_{j \in \mathbb{Z}} a_{2\ell+1-2j} f_j^k = \sum_{\nu \in \mathbb{Z}} a_{2\nu+1} f_{\ell-\nu}^k, & i &= 2\ell+1. \end{aligned}$$

These two rules are the same as in Definition 2 in the previous chapter [10] with $\alpha_j = a_{2j+1},\ j \in \mathbb{Z}$.

2. Expanding $a(z)$ in powers of z, we observe that the coefficients of the mask are $a_i = 2^{-m}\binom{m+1}{i},\ i = 0, \ldots, m+1$, as obtained from Pascal's triangle in Section 2 of chapter [14].
3. For $s = 1$ the two rules encompassed in (1) are for i even and for i odd. Let $i = 2\ell$, then

$$f_{2\ell}^{k+1} = \sum_{j \in \mathbb{Z}} a_{2\ell-2j} f_j^k = \sum_{\nu \in \mathbb{Z}} a_{2\nu} f_{\ell-\nu}^k .$$

To get the second rule, assume $i = 2\ell + 1$. Then

$$f_{2\ell+1}^{k+1} = \sum_{j \in \mathbb{Z}} a_{2\ell+1-2j} f_j^k = \sum_{\nu \in \mathbb{Z}} a_{2\nu+1} f_{\ell-\nu}^k .$$

Thus one rule is based on the even coefficients of the mask, and the other on the odd coefficients.

In the case $s = 2$ there are four rules depending on the parity of each component in the multi-index $i = (i_1, i_2)$. Writing all the multi-indices by components, we get the four rules

$$\begin{aligned}
f^{k+1}_{(2i_1,2i_2)} &= \sum_{j_1,j_2\in\mathbb{Z}} a_{(2i_1-2j_1,2i_2-2j_2)} f^k_{(j_1,j_2)} \\
&= \sum_{\nu_1,\nu_2\in\mathbb{Z}} a_{(2\nu_1,2\nu_2)} f^k_{(i_1-\nu_1,i_2-\nu_2)} \\
f^{k+1}_{(2i_1+1,2i_2)} &= \sum_{\nu_1,\nu_2\in\mathbb{Z}} a_{(2\nu_1+1,2\nu_2)} f^k_{(i_1-\nu_1,i_2-\nu_2)} \\
f^{k+1}_{(2i_1,2i_2+1)} &= \sum_{\nu_1,\nu_2\in\mathbb{Z}} a_{(2\nu_1,2\nu_2+1)} f^k_{(i_1-\nu_1,i_2-\nu_2)} \\
f^{k+1}_{(2i_1+1,2i_2+1)} &= \sum_{\nu_1,\nu_2\in\mathbb{Z}} a_{(2\nu_1+1,2\nu_2+1)} f^k_{(i_1-\nu_1,i_2-\nu_2)}
\end{aligned}$$

Exercise 2. Show, by equating coefficients of the same power of the variables on both sides of

$$F(z;S_a f) = a(z)F(z^2;f) \ ,$$

that this equation is equivalent to (1).

Solution 2. The above equality is in the sense that coefficients of the same power of z on both sides of the equality are the same. The coefficient of z^i on the left-hand side is $(S_a f)_i$. The cofficient of z^i on the right-hand side is $\sum_{j\in\mathbb{Z}} a_{i-2j} f_j$, since

$$a(z)F(z^2;f) = \sum_{i\in\mathbb{Z}} a_i z^i \sum_{j\in\mathbb{Z}} f_j z^{2j} \ .$$

Thus

$$(S_a f)_i = \sum_{j\in\mathbb{Z}} a_{i-2j} f_j \ ,$$

which is equivalent to (1), since there $f^{k+1} = S_a f^k$.

Exercise 3. Prove that the mask $\mathbf{a}$ of a convergent univariate subdivision scheme, satisfies

$$\sum_j a_{2j} = \sum_j a_{2j+1} = 1 \ .$$

Solution 3. Let S denote the convergent subdivision scheme. By the definition of convergence (see Definition 1 in the previous chapter [10]), there exists f^0 and x_0 such that $S^\infty f^0(x_0) \neq 0$. Also by Remark 1 in chapter [10] for any given small $\epsilon > 0$ there exists $K = K(\epsilon)$, such that for all $k \geq K$

$$\sup_{i\in 2^k(x_0-1,x_0+1)} |(S^k f^0)_i - S^\infty f^0(2^{-k}i)| \leq \epsilon \ .$$

Let $k \geq K(\epsilon)$ and ℓ be such that $\ell 2^{-k} \leq x_0 \leq (\ell+1)2^{-k}$. Denote $\xi_0 = 2^{-k}\ell$. Then $f_{2\ell}^{k+1} = S^\infty f^0(\xi_0) + \epsilon^*$ with $|\epsilon^*| \leq \epsilon$. Similarly, since the support of the mask is finite,

$$f_{\ell-j}^k = S^\infty f^0(\xi_0 - 2^{-k}j) + \epsilon_j^k \ , \quad j \in \mathrm{supp}(\mathbf{a}) \ ,$$

with $|\epsilon_j^k| \leq \epsilon$. Using (1) with $i = 2\ell$ (see the corresponding refinement rule in part 3 of the solution of Exercise 1), we get

$$S^\infty f^0(\xi_0) + \epsilon^* = \sum_{j\in\mathbb{Z}} a_{2j} \left(S^\infty f^0(\xi_0 - 2^{-k}j) + \epsilon_j^k \right) \ . \tag{32}$$

Let $\mathrm{supp}(\mathbf{a}) \subset [-M, M]$, then by the uniform continuity of $S^\infty f^0$ in the interval $[x_0 - 1, x_0 + 1]$, there exists K^* such that $|S^\infty f^0(x) - S^\infty f^0(y)| \leq \epsilon$ for $|x - y| \leq (M+1)2^{-K^*}$, $x, y \in [x_0 - 1, x_0 + 1]$.

Returning to (32) obtained from (1) with $i = 2\ell$ and $k \geq \max\{K, K^*\}$, we get

$$S^\infty f^0(x_0) + \tilde{\epsilon} = \sum_{j\in\mathbb{Z}} a_{2j} [S^\infty f^0(x_0) + \epsilon_j] \ ,$$

with $|\tilde{\epsilon}| \leq 2\epsilon$ and $|\epsilon_j| \leq 2\epsilon$. This leads to

$$\left| S^\infty f^0(x_0) \left[1 - \sum_{j\in\mathbb{Z}} a_{2j} \right] \right| \leq C\epsilon \ ,$$

with C a constant independent of ϵ.

Since ϵ can be chosen arbitrarily small, and since $S^\infty f^0(x_0) \neq 0$, we finally get

$$\sum_{j\in\mathbb{Z}} a_{2j} = 1 \ .$$

A similar analysis leads to $\sum_{j\in\mathbb{Z}} a_{2j+1} = 1$.

Exercise 5. Show that condition (21) implies that

$$a(1,1) = 4, \quad a(-1,1) = 0, \quad a(1,-1) = 0, \quad a(-1,-1) = 0 \ . \tag{33}$$

Solution 5. We show that the conditions

$$\sum_{\beta\in\mathbb{Z}^2} a_{\alpha-2\beta} = 1, \quad \alpha \in \big\{(0,0),(0,1),(1,0),(1,1)\big\} \ ,$$

imply $a(1,1) = 4$, $a(-1,1) = 0$. The other two implications are similar. By the definition of a symbol

$$\begin{aligned} a(1,1) = \sum_{j\in\mathbb{Z}^2} a_j &= \sum_{j_1,j_2\in\mathbb{Z}} a_{(2j_1,2j_2)} + \sum_{j_1,j_2\in\mathbb{Z}} a_{(2j_1+1,2j_2)} \\ &+ \sum_{j_1,j_2\in\mathbb{Z}} a_{(2j_1,2j_2+1)} + \sum_{j_1,j_2\in\mathbb{Z}} a_{(2j_1+1,2j_2+1)} \\ &= 1+1+1+1 = 4 \ . \end{aligned}$$

Similarly

$$a(-1,1) = \sum_{j_1,j_2\in\mathbb{Z}} a_{(2j_1,2j_2)} - \sum_{j_1,j_2\in\mathbb{Z}} a_{(2j_1+1,2j_2)} - \sum_{j_1,j_2\in\mathbb{Z}} a_{(2j_1,2j_2+1)} + \sum_{j_1,j_2\in\mathbb{Z}} a_{(2j_1+1,2j_2+1)} = 1-1-1+1 = 0 \ .$$

Exercise 6. Show that a bivariate scheme with a symbol

$$a(z) = (1+z_1)^2(1+z_2)^2 b(z) \ ,$$

generates C^1 limit functions, if the three schemes with symbols

$$2(1+z_1)(1+z_2)b(z), \quad 2(1+z_2)^2 b(z), \quad 2(1+z_1)^2 b(z) \ .$$

are contractive.

Solution 6. To verify that a scheme generates C^1 limit functions, by the tools of Theorem 7, we have to verify that the schemes corresponding to the symbols

$$\frac{2a(z)}{1+z_1} \ , \quad \frac{2a(z)}{1+z_2} \ , \tag{34}$$

are convergent.

To show by the tools of Theorem 6 that the first scheme in (34) is convergent, we have to show that the schemes with symbols

$$\frac{2a(z)}{(1+z_1)^2} \ , \frac{2a(z)}{(1+z_1)(1+z_2)} \ ,$$

are contractive. Similarly, for the second scheme in (34), we have to show the contractivity of the schemes with symbols

$$\frac{2a(z)}{(1+z_2)^2} \ , \frac{2a(z)}{(1+z_1)(1+z_2)} \ .$$

By the form of $a(z)$, we get that the contractivity of the three schemes with symbols

$$2(1+z_1)(1+z_2)b(z), \quad 2(1+z_2)^2 b(z), \quad 2(1+z_1)^2 b(z) \ ,$$

has to be verified.