Asymptotic Equivalence of Symplectic Capacities

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Abstract

A long-standing conjecture states that all normalized symplectic capacities coincide on the class of convex subsets of $\mathbb{R}^{2n}$. In this note we focus on an asymptotic (in the dimension) version of this conjecture, and show that when restricted to the class of centrally symmetric convex bodies in $\mathbb{R}^{2n}$, several symplectic capacities, including the Ekeland–Hofer–Zehnder capacity, the displacement energy capacity, and the cylindrical capacity, are all equivalent up to an absolute constant.

1 Introduction

Consider the space $\mathbb{R}^{2n}$ equipped both with the standard symplectic form $\omega = dp \wedge dq$, and with the standard inner product $\langle \cdot, \cdot \rangle$. Note that under the usual identification between $\mathbb{R}^{2n}$ and $\mathbb{C}^n$, these two structures are the real and the imaginary parts, respectively, of the standard Hermitian inner product in $\mathbb{C}^n$. Moreover, one has that $\omega(v, u) = \langle v, Ju \rangle$, where J is the standard complex structure in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$. Symplectic capacities, whose axiomatic definition below is due to Ekeland and Hofer [4], are numerical invariants which roughly speaking measure the symplectic size of sets. More precisely, let $B^{2n}(r)$ stand for the Euclidean open ball of radius r, and $Z^{2n}(r)$ for the cylinder $B^2(r) \times \mathbb{C}^{n-1}$.

Definition 1.1. *A symplectic capacity on $(\mathbb{R}^{2n}, \omega)$ associates to each subset $U \subset \mathbb{R}^{2n}$ a number $c(U) \in [0, \infty]$ such that the following hold:*

$(P1)$ $c(U) \leqslant c(V)$ *whenever* $U \subseteq V$ *(monotonicity),*

$(P2)$ $c\big(\psi(U)\big) = |\alpha|\, c(U)$ *for* $\psi \in \mathrm{Diff}(\mathbb{R}^{2n})$ *such that* $\psi^* \omega = \alpha\, \omega$ *(conformality),*

$(P3)$ $0 < c\big(B^{2n}(r)\big)$, *and* $c\big(Z^{2n}(r)\big) < \infty$ *(nontriviality).*

Moreover, a symplectic capacity is said to be normalized if in addition it satisfies

$(P4)$ $c\big(B^{2n}(r)\big) = c\big(Z^{2n}(r)\big) = \pi r^2$ *(normalization).*

Note that propery $(P2)$ implies that c is a symplectic invariant which scales like a two-dimensional invariant, and $(P3)$ that symplectic capacities significantly differ from any volume related invariants. The first examples of symplectic capacities were constructed by Gromov in [9], where he developed and used pseudoholomorphic curve techniques to prove a striking symplectic rigidity result, nowadays known as Gromov's "non-squeezing theorem". It states that one cannot map a ball inside a thinner cylinder by a symplectic embedding.

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More precisely, the theorem asserts that if $r < 1$, there is no symplectic embedding of the unit ball B^{2n} into the cylinder $Z^{2n}(r)$. This naturally leads to the definition of two normalized symplectic capacities: the Gromov width, given by $\underline{c}(U) = \sup\{\pi r^2 \,|\, B^{2n}(r) \overset{\text{s}}{\hookrightarrow} U\}$; and the cylindrical capacity, $\overline{c}(U) = \inf\{\pi r^2 \,|\, U \overset{\text{s}}{\hookrightarrow} Z^{2n}(r)\}$. Here $\overset{\text{s}}{\hookrightarrow}$ stands for symplectic embedding. It is not hard to verify that these two capacities are the smallest and largest possible normalized symplectic capacities, respectively.

Shortly after Gromov's work [9] many other symplectic capacities were constructed, reflecting different geometrical and dynamical properties. Among these are the Hofer–Zehnder capacity [13, 14], the Ekeland–Hofer capacities [4, 5], the displacement energy [11], the Floer–Hofer capacity [7, 8], spectral capacities [6, 19, 26], and more recently, Hutchings' embedded contact homology (ECH) capacities [15]. These quantities play an important role in symplectic geometry, and their properties, interrelations, and applications to symplectic topology and Hamiltonian dynamics are intensively studied (see e.g., [3] and [17] for two excellent surveys).

In the two-dimensional case, Siburg [23] showed that any symplectic capacity of a compact connected domain with smooth boundary $\Omega \subset \mathbb{R}^2$ equals its Lebesgue measure. In higher dimensions symplectic capacities do not coincide in general. A theorem by Hermann [10] states that for any $n \geqslant 2$ there is a bounded star-shaped domain $S \subset \mathbb{R}^{2n}$ with cylindrical capacity $\overline{c}(S) \geqslant 1$, and arbitrarily small Gromov width $\underline{c}(S)$. Still, for a large class of sets in $\mathbb{R}^{2n}$, including ellipsoids, polydiscs, and convex Reinhardt domains, all normalized symplectic capacities coincide [10]. In [25] Viterbo showed that for any bounded convex set K of $\mathbb{R}^{2n}$ one has $\overline{c}(K) \leqslant 4n^2 \underline{c}(K)$. Moreover, it was conjectured [10, 12, 25] that:

Conjecture 1.2. *For any convex body K in $\mathbb{R}^{2n}$ one has $\underline{c}(K) = \overline{c}(K)$.*

Here, by a convex body we mean a compact convex subset of $\mathbb{R}^{2n}$ with non-empty interior. The above conjecture is particularly challenging due to the scarcity of examples of convex domains for which capacities have been computed. Moreover, an affirmative answer to Conjecture 1.2 would in particular implies Viterbo's volume-capacity conjecture [25], and it was recently shown that the latter would in turn settle a 70-years old question in convex geometry known as the Mahler conjecture. For more information regarding these applications of Conjecture 1.2 see [1] and [20].

A somewhat more modest question in the same direction (c.f. Problem 1.4 in [10], Problem 8 in [3], and Section 5 in [20]) is whether Conjecture 1.2 above holds asymptotically in the dimension, i.e.,

Question 1.3. *Is there is an absolute constant $A > 0$ such that for every convex body K in $\mathbb{R}^{2n}$ one has*

$$\overline{c}(K) \leqslant A\underline{c}(K).$$

Here we will give a partial answer to this question. Before we state our main result we need to recall the definition of the Ekeland–Hofer–Zehnder capacity. The restriction of

the symplectic form ω to a smooth closed hypersurface $\mathcal{S} \subset \mathbb{R}^{2n}$ canonically defines a 1-dimensional subbundle, $\ker(\omega|\mathcal{S})$, whose integral curves comprise the characteristic foliation of $\mathcal{S}$. In other words, a closed characteristic of $\mathcal{S}$ is an embedded circle in $\mathcal{S}$ tangent to the canonical line bundle

$$\mathfrak{S}_{\mathcal{S}} = \{(x, \xi) \in T\mathcal{S} \,|\, \omega(\xi, \eta) = 0 \text{ for all } \eta \in T_x\mathcal{S}\}.$$

Recall that the symplectic action of a closed curve γ is defined by $A(\gamma) = \int_{\gamma} \lambda$, where $\lambda = pdq$ is the Liouville 1-form. The action spectrum of $\mathcal{S}$ is

$$\mathcal{L}(\mathcal{S}) = \{|A(\gamma)| \,;\, \gamma \text{ is a closed characteristic on } \mathcal{S}\}.$$

In [4] and [14] it was proved that for a smooth convex body $K \subset \mathbb{R}^{2n}$, the two aforementioned Hofer–Zehnder and Ekeland–Hofer capacities coincide, and are given by the minimal action over all closed characteristics on the boundary of the body K, i.e.,

$$c_{\mathrm{EH}}(K) = c_{\mathrm{HZ}}(K) = \min \mathcal{L}(\partial K). \tag{1}$$

We remark that although the above definition of closed characteristics, as well as the equalities in (1), were given only for the class of convex bodies with smooth boundary, they can naturally be generalized to the class of convex sets in $\mathbb{R}^{2n}$ with nonempty interior (see e.g., [2]). In what follows, we refer to the coinciding Ekeland–Hofer and Hofer–Zehnder capacities on this class as the Ekeland–Hofer–Zehnder capacity, and denote it by c_{EHZ}.

Our first result in the note is the following. Recall that a convex body $K \subset \mathbb{R}^n$ is said to be centrally symmetric if $K = -K$.

Theorem 1.4. *For every centrally symmetric convex body K in $\mathbb{R}^{2n}$,*

$$\overline{c}(K) \leqslant 4c_{\mathrm{EHZ}}(K).$$

Remark 1.5. Other symplectic capacities, like the spectral capacities c_σ, which are based on a choice of an action selector σ, and the displacement energy d, are known to be bigger than or equal to the Hofer–Zehnder capacity (see e.g., Section 2.3.4 in [3]). Thus, it follows from Theorem 1.4 that on the class of symmetric convex sets in $\mathbb{R}^{2n}$, the normalized symplectic capacities $c_{\mathrm{EHZ}}, d, c_\sigma$ and $\overline{c}$, are all coincide up to an absolute constant.

In fact, we prove a slightly stronger result than Theorem 1.4 which shows that for a centrally symmetric convex body $K \subset \mathbb{R}^{2n}$, the aforementioned symplectic capacities are all equivalent to yet another quantity associated with the body K. More precisely, for a convex body $K \subset \mathbb{R}^{2n}$ with $0 \in \mathrm{Int}(K)$, we denote by $K^\circ = \{y \in \mathbb{R}^{2n} \,|\, \langle x, y \rangle \leqslant 1, \text{ for every } x \in K\}$ the polar body of $K^\dagger$. Moreover, we denote

$$\|J\|_{K^\circ \to K} := \sup_{v, u \in K^\circ} \langle Jv, u \rangle.$$

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†As a matter of fact, the polar body K° should be defined as a subset of the dual space of $\mathbb{R}^{2n}$. However, since we have fixed a scalar product in our setting, we will identify the latter space with $\mathbb{R}^{2n}$ itself.

To explain the reason for this notation, we remark that when the convex body K is centrally symmetric, $\|J\|_{K^\circ \to K}$ is the operator norm of the complex structure J, when the latter is considered as a linear map between the normed spaces $J : (\mathbb{R}^{2n}, \|\cdot\|_{K^\circ}) \to (\mathbb{R}^{2n}, \|\cdot\|_K)$, i.e.,

$$\|J\|_{K^\circ \to K} = \sup_{v,u \in K^\circ} \langle Jv, u \rangle = \sup_{v : \|v\|_{K^\circ} \leqslant 1} \|Jv\|_K.$$

Here we use the standard identification between normed spaces and centrally symmetric convex bodies, i.e., for a non-empty centrally symmetric convex body K in $\mathbb{R}^{2n}$ we denote by $\|\cdot\|_K$ the norm on $\mathbb{R}^{2n}$ induced by K, that is, $\|\cdot\|_K = \inf\{r : x \in rK\}$.

Theorem 1.6. *For every centrally symmetric convex body K in $\mathbb{R}^{2n}$,*

$$\frac{1}{\|J\|_{K^\circ \to K}} \leqslant c_{\mathrm{EHZ}}(K) \leqslant \overline{c}(K) \leqslant \frac{4}{\|J\|_{K^\circ \to K}}. \tag{2}$$

Remark 1.7. In fact, in the proof of Theorem 1.6 we use the centrally symmetric assumption on the body K only for the right-most inequality of (2). The first two inequalities on the left-hand side hold for every convex body K in $\mathbb{R}^{2n}$.

Note that Theorem 1.4 follows immediately from Theorem 1.6. Moreover, we wish to emphasize that Theorem 1.6 provides in many cases an efficient way to approximate the numerical value of the capacities $c_{\mathrm{EHZ}}(K)$ and $\overline{c}(K)$ (for centrally symmetric convex bodies), as the quantity $\|J\|_{K^\circ \to K}$ is a-priori much easier to compute than the above mentioned symplectic capacities.

Another by-product of Theorem 1.6, which may be of independent interest, concerns the equivalence of the cylindrical capacity and the Gromov width capacity with their linearized versions $\overline{c}_{\mathrm{lin}}$ and $\underline{c}_{\mathrm{lin}}$ respectively. The definitions of these two quantities are given in Definitions 2.4 and 3.1 below. It turns out that for centrally symmetric convex bodies in $\mathbb{R}^{2n}$, the cylindrical capacity $\overline{c}$ is asymptotically equivalent to its linearized version $\overline{c}_{\mathrm{lin}}$, while surprisingly enough, this is false for the Gromov width capacity. More precisely,

Theorem 1.8. *For every centrally symmetric convex body K in $\mathbb{R}^{2n}$,*

$$\overline{c}(K) \leqslant \overline{c}_{\mathrm{lin}}(K) \leqslant 4\overline{c}(K).$$

On the other hand, there exist a centrally symmetric convex body $\widetilde{K}$ in $\mathbb{R}^{2n}$ such that

$$\underline{c}_{\mathrm{lin}}(\widetilde{K}) \leqslant \pi, \ \ while \ \ \underline{c}(\widetilde{K}) \geqslant \sqrt{\frac{n}{2}}.$$

Note that an immediate corollary from Theorem 1.8 is that the linearized versions of the Gromov width and the cylindrical capacity are not asymptotically equivalent.

Notations: We denote by $\mathcal{K}^n$ the class of convex bodies of $\mathbb{R}^n$, i.e., compact convex sets with non-empty interior. For $K \in \mathcal{K}^n$, we denote by $h_K : \mathbb{R}^n \to \mathbb{R}$ its support function given by $h_K(u) = \sup\{\langle x, u \rangle : x \in K\}$. Also, we denote by $g_K : \mathbb{R}^n \to \mathbb{R}$ the gauge function $g_K(x) = \inf\{r | x \in rK\}$ associated with K. Note that when K is centrally symmetric, i.e.,

$K = -K$, the gauge function $g_K(x)$ is a norm, and is denoted by $\|x\|_K$. Furthermore, when $0 \in \text{int}(K)$, one has that $h_K = g_{K^\circ}$, where $K^\circ = \{y \in \mathbb{R}^n \mid \langle x, y \rangle \leqslant 1, \text{ for every } x \in K\}$ is the polar body of K. The Euclidean norm will be denoted by $|\cdot|$. Finally, we denote by $\mathbb{S}^n$ the unit sphere in $\mathbb{R}^{n+1}$, i.e., $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$.

2 Proof of Theorem 1.6

Note first that there is no loss of generality in assuming that in addition to being compact and with non-empty interior, all convex bodies considered also have a smooth boundary, and contain the origin in their interior. Indeed, affine translations in $\mathbb{R}^{2n}$ are symplectomorphisms, which accounts for the assumption that the origin is in the interior. Secondly, once Theorem 1.6 is proved for smooth convex domains, the general case follows by standard approximation arguments, as symplectic capacities are continuous on the class of convex bodies with respect to the Hausdorff distance (see e.g. [18] page 376).

Moreover, in what follows we will make repeated use of the following well-known geometric observation form convex geometry.

Lemma 2.1. *Let g_K be the gauge function associated with a smooth convex body K. Then, when restricted to the boundary ∂K, the gradient ∇g_K is a surjective map $\nabla g_K : \partial K \to \partial K^\circ$.*

A proof of Lemma 2.1 can be found e.g., in Subsection 1.7.1 of [22]. We turn now to the proof of Theorem 1.6, and start with the following proposition.

Proposition 2.2. *For every smooth convex body $K \in \mathcal{K}^{2n}$,*

$$\frac{1}{\|J\|_{K^\circ \to K}} \leqslant c_{\text{EHZ}}(K).$$

To prove Proposition 2.2 we first need some preparation. Recall (see e.g., Chapter 1 of [13]) that the classical geometric problem of finding closed characteristics on ∂K has the following dynamical interpretation. If the boundary ∂K is represented as a regular energy surface $\{x \in \mathbb{R}^{2n} \mid H(x) = 1\}$ of a smooth Hamiltonian function $H : \mathbb{R}^{2n} \to \mathbb{R}$, then the restriction to ∂K of the Hamiltonian vector field X_H, defined by $i_{X_H}\omega = -dH$, is a section of the line bundle $\mathfrak{S}_{\partial K}$. Thus, the images of the periodic solutions of the classical Hamiltonian equation $\dot{x} = X_H(x) = J\nabla H(x)$ on ∂K are precisely the closed characteristics of ∂K. In particular, the closed characteristics do not depend (up to parametrization) on the choice of the Hamiltonian function. Indeed, if the energy surface can be represented as a regular level set of some other function $F : \mathbb{R}^{2n} \to \mathbb{R}$, then $X_H = \alpha X_F$ on ∂K for some scalar function $\alpha \neq 0$, and the corresponding Hamiltonian equations have the same

solutions up to parametrization. Finally, note that for a smooth convex body K the gauge function g_K is a *defining function* for K, i.e., $K = g_K^{-1}([0,1]), \partial K = g_K^{-1}(1)$, and 1 is a regular value of g_K.

Lemma 2.3. *Let $\gamma : [0,T] \to \partial K$ be a solution of the Hamiltonian equation $\dot{\gamma} = J\nabla g_K(\gamma)$, with $\gamma(0) = \gamma(T)$. Then there exist $t_0 \in [0,T]$ such that $g_K(\gamma(t_0) - \gamma(0)) \geqslant 1$.*

Proof of Lemma 2.3. It follows immediately from the assumptions that

$$0 = \int_0^T \dot{\gamma}(t)dt = \int_0^T J\nabla g_K(\gamma(t))dt = J\int_0^T \nabla g_K(\gamma(t))dt.$$

From this one can conclude that

$$\int_0^T \langle \nabla g_K(\gamma(t)), \gamma(0) \rangle dt = 0.$$

In particular, this implies that there exists $t_0 \in [0,T]$ such that

$$\langle \nabla g_K(\gamma(t_0)), \gamma(0) \rangle \leqslant 0. \tag{3}$$

Next, from Lemma 2.1 it follows that $\nabla g_K(\gamma(t_0)) \in \partial K^\circ$, and we obtain that

$$g_K(\gamma(t_0) - \gamma(0)) = \sup\{\langle \gamma(t_0) - \gamma(0), u \rangle \mid u \in K^\circ\} \geqslant \langle \gamma(t_0) - \gamma(0), \nabla g_K(\gamma(t_0)) \rangle. \tag{4}$$

Finally, from Euler's homogeneous function theorem it follows that for every $x \in \partial K$, one has $\langle x, \nabla g_K(x) \rangle = g_K(x) = 1$, and hence the combination of this fact together with inequalities (3) and (4) completes the proof of the lemma. $\qquad\square$

Proof of Proposition 2.2. Let $\gamma : [0,T] \to \partial K$ be a closed characteristic on the boundary ∂K, i.e., a solution of the Hamiltonian equation $\dot{\gamma} = J\nabla g_K(\gamma)$, with $\gamma(0) = \gamma(T)$. Note that

$$A(\gamma) = \frac{1}{2}\int_0^T \langle J\gamma(t), \dot{\gamma}(t) \rangle dt = \frac{1}{2}\int_0^T \langle \gamma(t), \nabla g_K(\gamma(t)) \rangle dt = \frac{T}{2}. \tag{5}$$

It follows from Lemma 2.3, the subadditivity property of g_K, and the definition of γ, that,

$$1 \leqslant g_K\left(\int_0^{t_0} \dot{\gamma}(t)dt\right) \leqslant \int_0^{t_0} g_K(\dot{\gamma}(t))dt = \int_0^{t_0} g_K(J\nabla g_K(\gamma(t)))dt. \tag{6}$$

On the other hand, it follows from the definition of an operator norm that

$$\int_0^{t_0} g_K(J\nabla g_K(\gamma(t)))dt \leqslant \int_0^{t_0} \|J\|_{K^\circ \to K}\, g_{K^\circ}(\nabla g_K(\gamma(t)))\, dt. \tag{7}$$

The combination of (6), (7), and Lemma 2.1 gives

$$1 \leqslant \int_0^{t_0} \|J\|_{K^\circ \to K}\, g_{K^\circ}(\nabla g_K(\gamma(t)))\, dt = \int_0^{t_0} \|J\|_{K^\circ \to K}\, dt, \tag{8}$$

and thus we obtain that

$$\frac{1}{\|J\|_{K^\circ \to K}} \leqslant t_0. \tag{9}$$

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Note that since $\gamma(0) = \gamma(T)$, repeating the same arguments as above (this time, integrating in (6), (7), and (8) between t_0 and T) we obtain also that

$$\frac{1}{\|J\|_{K^\circ \to K}} \leqslant T - t_0. \tag{10}$$

From (5) it follows that $\min\{t_0, T - t_0\} \leqslant T/2 = A(\gamma)$, and since, by definition, the capacity $c_{\mathrm{EHZ}}(K)$ is defined to be the minimal action of closed characteristics on the boundary ∂K, we conclude from (9) and (10) that,

$$\frac{1}{\|J\|_{K^\circ \to K}} \leqslant c_{\mathrm{EHZ}}(K).$$

This completes the proof of the proposition. □

To describe the second ingredient in the proof of Theorem 1.6 we need to introduce one more definition. It is known (see e.g., Appendix C in [21]) that for a Lebesgue measurable set $\mathcal{U} \subset \mathbb{R}^{2n}$,

$$\overline{c}(\mathcal{U}) = \inf_\varphi \mathrm{Area}\big(\pi(\varphi(\mathcal{U}))\big),$$

where π is the orthogonal projection to the complex line $E = \{z \in \mathbb{C}^n \mid z_j = 0 \text{ for } j \neq 1\}$, and the infimum is taken over all symplectic embeddings φ of $\mathcal{U}$ into $\mathbb{R}^{2n}$. Recall that with our notations, under the natural identification $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ one has that $z_j = q_j + ip_j$. Thus, a nature way to "linearize" the cylindrical capacity $\overline{c}$ is as follows. Let $\mathrm{ISp}(2n)$ be the affine symplectic group, defined as the semi-direct product $\mathrm{Sp}(2n) \ltimes \mathrm{T}(2n)$ of the linear symplectic group and the group of translations in $\mathbb{R}^{2n}$.

Definition 2.4. *The linearized cylindrical capacity $\overline{c}_{\mathrm{lin}}$ of a set $\mathcal{U} \subset \mathbb{R}^{2n}$ is defined as*

$$\overline{c}_{\mathrm{lin}}(\mathcal{U}) = \inf_S \mathrm{Area}\big(\pi(S(\mathcal{U}))\big),$$

where the infimum is taken over all affine symplectic maps $S \in \mathrm{ISp}(2n)$.

Now, the second main ingredient in the proof of Theorem 1.6 is the following:

Proposition 2.5. *For every centrally symmetric convex body $K \in \mathcal{K}^{2n}$,*

$$\overline{c}(K) \leqslant \overline{c}_{\mathrm{lin}}(K) \leqslant \frac{4}{\|J\|_{K^\circ \to K}}. \tag{11}$$

To establish Proposition 2.5 we shall need the following geometric observation. For $v \in \mathbb{R}^{2n}$, we denote by K_v the section $K \cap \{v\}^\perp$, and by $\|\cdot\|_{K_v^\circ}$ the semi-norm defined by

$$\|w\|_{K_v^\circ} = \sup\{\langle w, y \rangle \mid y \in K_v\}.$$

Lemma 2.6. *For a symmetric convex body $K \in \mathcal{K}^{2n}$, a linear symplectic map $S \in \mathrm{Sp}(2n)$, and the orthogonal projection π to the complex line $E = \{z \in \mathbb{C}^n \mid z_j = 0 \text{ for } j \neq 1\}$ defined above, one has*

$$\mathrm{Area}\big(\pi(S(K))\big) \leqslant 4 \|S^T e\|_{K^\circ} \|S^T J e\|_{K_v^\circ}, \tag{12}$$

where S^T stands for the transpose of the matrix S, e is a unit vector parallel to the q_1-axis, and $v = S^T e$.

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Proof of Lemma 2.6. The lemma follows from a much more general result by Rogers and Shephard [24], which states that for every symmetric convex body $K \subset \mathbb{R}^n$, one has

$$\mathrm{Vol}_n(K) \leqslant \left(\mathrm{Vol}_k(\pi_E(K)) \mathrm{Vol}_{n-k}(K \cap E^\perp) \right)^{1/k} \leqslant \binom{n}{k}^{1/k} \mathrm{Vol}_n(K), \tag{13}$$

for every k-dimensional subspace E of $\mathbb{R}^n$, where π_E stands for the orthogonal projection on the subspace E. We remark that we use only the case where $n = 2$ and $k = 1$, in which inequality (13) is an elementary geometric fact. It can be easily checked that the right-hand side of (12) exactly equals the product of the length of the projection of $\pi(SK)$ to the q_1-axis, and the length of the intersection of $\pi(SK)$ with the p_1-axis. $\qquad\square$

We are now in a position to prove Proposition 2.5.

Proof of Proposition 2.5. Note that, by definition, for every measurable set $\mathcal{U} \subset \mathbb{R}^{2n}$ one has $\overline{c}(\mathcal{U}) \leqslant \overline{c}_{\mathrm{lin}}(\mathcal{U})$, and hence the left-hand side inequality in (11) holds. Next, we recall the easily verified fact that for any $v, w \in \mathbb{R}^{2n}$ such that $\omega(v, w) = 1$, there exists a linear symplectic map $S \in \mathrm{Sp}(2n)$ such that $v = S^T e$ and $w = S^T J e$, where as before, e is a unit vector parallel to the q_1-axis. From this fact, Lemma 2.6, and Definition 2.4 it follows that for a centrally symmetric convex body $K \in \mathcal{K}^{2n}$

$$\overline{c}_{\mathrm{lin}}(K) \leqslant 4 \inf_{v \in \mathbb{S}^{2n-1}} \inf_{w : \langle Jv, w \rangle = 1} \|v\|_{K^\circ} \|w\|_{K_v^\circ} = 4 \inf_{v \in \mathbb{S}^{2n-1}} \|v\|_{K^\circ} \inf_{w : \langle Jv, w \rangle = 1} \|w\|_{K_v^\circ}. \tag{14}$$

We focus now on the second infimum on the right-hand side of (14). Note that for a fixed vector $v \in \mathbb{S}^{2n-1}$, the equality $\langle Jv, w \rangle = 1$ is equivalent to $\langle Jv, w - Jv \rangle = 0$. Denoting $z := w - Jv$, we can write

$$\inf_{w : \langle Jv, w \rangle = 1} \|w\|_{K_v^\circ} = \inf_{z : z \perp Jv} \|Jv + z\|_{K_v^\circ}. \tag{15}$$

This quantity measures the distance, with respect to the semi-metric induced by $\| \cdot \|_{K_v^\circ}$, between the vector Jv and the subspace $\{Jv\}^\perp$ orthogonal to it. Using the Hahn–Banach theorem we obtain

$$\inf_{z : z \perp Jv} \|Jv + z\|_{K_v^\circ} = \mathrm{dist}_{\|\cdot\|_{K_v^\circ}}(Jv, \{Jv\}^\perp) = \sup_u \langle u, Jv \rangle, \tag{16}$$

where the supremum is taken over all vectors u such that $u \in \mathrm{span}\{Jv\}$ and $\|u\|_{K_v} \leqslant 1$. Note that we have used the fact that $(K_v^\circ)^\circ = K_v$. Next, we use the fact that Jv is orthogonal to v (and hence in particular $\|Jv\|_{K_v} < \infty$) to deduce from (15) and (16) that

$$\inf_{w : \langle Jv, w \rangle = 1} \|w\|_{K_v^\circ} = \sup_{u \in \mathrm{span}\{Jv\}, \, \|u\|_{K_v} \leqslant 1} \langle u, Jv \rangle \leqslant \frac{\langle Jv, Jv \rangle}{\|Jv\|_{K_v}} = \frac{1}{\|Jv\|_K}. \tag{17}$$

From the combination of (14) and (17) we obtain that

$$\overline{c}_{\mathrm{lin}}(K) \leqslant 4 \inf_{v \neq 0} \frac{\|v\|_{K^\circ}}{\|Jv\|_K} = \frac{4}{\|J\|_{K^\circ \to K}}, \tag{18}$$

which completes the proof of the proposition. $\qquad\square$

Remark 2.7. For a general convex body K in $\mathbb{R}^{2n}$ (not necessarily centrally symmetric), the same proof as the one above will give the following bound:

$$\frac{1}{4}\,\overline{c}_{\mathrm{lin}}(K-K) \leqslant \overline{c}_{\mathrm{lin}}(K) \leqslant \frac{1}{\|J\|_{(K-K)^\circ \to (K-K)}}. \tag{19}$$

On the other hand, from Proposition 2.2 it follows that

$$c_{\mathrm{EHZ}}(K) \geqslant \sup_v \frac{1}{\|J\|_{(K-v)^\circ \to (K-v)}}, \tag{20}$$

where the supremum is taken over all $v \in \mathbb{R}^{2n}$ such that $v \in \mathrm{Int}(K)$. We remark that although the upper bound for $\overline{c}_{\mathrm{lin}}(K)$ in (19), and the lower bound for $c_{\mathrm{EHZ}}(K)$ in (20) seem not too far away, we do not expect them to be asymptotically equivalent in general.

Proof of Theorem 1.6. For a smooth symmetric convex body K, the proof follows immediately from Propositions 2.2 and 2.5. The general case (i.e., without the smoothness assumption) follows by a standard approximation argument, as indicated at the beginning of this section. $\qquad\square$

3 Linearized Symplectic Capacities

In this section we prove Theorem 1.8. We recall first the following definition.

Definition 3.1. *The linearized Gromov width $\underline{c}_{\mathrm{lin}}$ of a set $\mathcal{U} \subset \mathbb{R}^{2n}$ is defined as*

$$\underline{c}_{\mathrm{lin}}(\mathcal{U}) = \sup_S \{\pi r^2 \,|\, SB^{2n}(r) \subset \mathcal{U}\},$$

where the supremum is taken over all affine symplectic maps $S \in \mathrm{ISp}(2n)$.

The following is the main ingredient in the proof of Theorem 1.8.

Proposition 3.2. *Let $Q = [-1,1]^{2n}$ be the standard cube in $\mathbb{R}^{2n}$. Then, for every orthogonal transformation $O \in \mathrm{O}(2n)$ one has $\underline{c}_{\mathrm{lin}}(OQ) \leqslant \pi$. Moreover, there is a rotation $\widetilde{O} \in \mathrm{O}(2n)$ for which $\underline{c}(\widetilde{O}Q) \geqslant \sqrt{n/2}$.*

Proof of Proposition 3.2. Note first that for every orthogonal transformation $O \in \mathrm{O}(2n)$,

$$\underline{c}_{\mathrm{lin}}(OQ) \leqslant \sup_L \{\pi r^2 \,|\, LB^{2n}(r) \subseteq Q\}, \tag{21}$$

where the supremum is taken over all affine volume-preserving linear maps L of $\mathbb{R}^{2n}$. It is straightforward to check that the largest ellipsoid contained in the cube Q is the unit-ball $B^{2n}(1)$, and hence $\underline{c}_{\mathrm{lin}}(OQ) \leqslant \pi$ for every orthogonal transformation $O \in \mathrm{O}(2n)$.

For the second part of the proposition, consider the Lagrangian splitting $\mathbb{R}^n(q) \times \mathbb{R}^n(p)$ of $\mathbb{R}^{2n}$, and the following configuration: $B^n_\infty(\alpha) \times B^n_1(\beta) \subset \mathbb{R}^{2n}$, where

$$
\begin{aligned}
B^n_\infty(\alpha) &= \{(q_1,\ldots,x_q) \in \mathbb{R}^n(q) \,|\, \max\{|q_1|,\ldots,|q_n|\} < \alpha\}, \\
B^n_1(\beta) &= \{(p_1,\ldots,p_n) \in \mathbb{R}^n(p) \,|\, \sum_{i=1}^n |p_i| < \beta\}.
\end{aligned}
$$

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Note that $B_\infty^n(1) \times B_1^n(1)$ is the product of a hypercube and its dual body, the cross-polytope. It is known (see e.g., §4 of [16]) that for every $\varepsilon > 0$, the ball $B^{2n}(r)$ symplectically embeds (via a non-linear symplectomorphism) into the product $B_\infty^n(1) \times B_1^n(\beta(1+\varepsilon))$, for a parameter β such that $\mathrm{Vol}(B_\infty^n(1) \times B_1^n(\beta)) = \mathrm{Vol}(B^{2n}(r))$. In particular, this implies that for an orthogonal transformation O of $\mathbb{R}^{2n}$, one has the following lower bound

$$c(OQ) \geqslant \sup\{4r \,|\, B_\infty^n(1) \times B_1^n(r) \subseteq OQ\}.$$

Thus, to complete the proof of the proposition it is enough to find an orthogonal transformation $O \in \mathrm{O}(2n)$ such that
$$O(B_\infty^n(1) \times B_1^n(\widetilde{r})) \subseteq Q,$$

where $\widetilde{r} > \sqrt{n/2}$. In particular, it is enough to find an orthogonal transformation O' of $\mathbb{R}^n(p)$ such that $O'(B_1^n(\widetilde{r})) \subseteq [-1,1]^n \subset \mathbb{R}^n(p)$, with $\widetilde{r}$ as above. The fact that such a transformation exists is well known to experts. For completeness we will give an explicit construction[‡]. We define the elements O'_{kj} of the matrix O' by

$$\sqrt{n}\,O'_{kj} = \begin{cases} \sqrt{2}\sin\left(\frac{kj}{n}2\pi\right) & \text{for } 1 \leqslant k < \frac{n}{2} \text{ and } 1 \leqslant j \leqslant n, \\ (-1)^j & \text{for } k = \frac{n}{2} \text{ and } 1 \leqslant j \leqslant n, \\ \sqrt{2}\cos\left(\frac{kj}{n}2\pi\right) & \text{for } \frac{n}{2} < k < n \text{ and } 1 \leqslant j \leqslant n, \\ 1 & \text{for } k = n \text{ and } 1 \leqslant j \leqslant n. \end{cases} \tag{22}$$

It is a straightforward computation (based on the orthonormality of the standard Fourier basis) to check that the matrix O' defined by (22) is indeed an orthogonal transformation. Moreover, denote by $\{e_i\}_{i=1}^n$ the standard basis of $\mathbb{R}^n(p)$. Note that $B_1^n(1) = \mathrm{Conv}\{\pm e_i\}$. It follows immediately from the definition of the matrix O' that

$$\|O'e_i\|_\infty := \max_{1 \leqslant j \leqslant n} |(O'e_i)_j| \leqslant \frac{\sqrt{2}}{\sqrt{n}},$$

where $(O'e_i)_j$ stands for the j-th component of the vector $O'e_i \in \mathbb{R}^n(p)$. This implies in particular that

$$O'(B_1^n(\sqrt{n})) = O'(\mathrm{Conv}\{\pm\sqrt{n}e_i\}) = \mathrm{Conv}\{\pm O'\sqrt{n}e_i\} \subseteq [-\sqrt{2},\sqrt{2}]^n,$$

which completes the proof of Proposition 3.2. $\qquad\square$

Proof of Theorem 1.8. Note that an immediate corollary from Propositions 2.2 and 2.5 is that the cylindrical capacity $\overline{c}$ is asymptotically equivalent to its linearized version $\overline{c}_{\mathrm{lin}}$ for symmetric convex domains in $\mathbb{R}^{2n}$, i.e., for every symmetric convex body $K \in \mathcal{K}^{2n}$,

$$\overline{c}(K) \leqslant \overline{c}_{\mathrm{lin}}(K) \leqslant 4c_{\mathrm{EHZ}}(K) \leqslant 4\overline{c}(K).$$

This establishes the first part of Theorem 1.8. The second part follows from Proposition 3.2.
$\qquad\square$

[‡]The first named author learned this example from R.S. Ismagilov around 1976.

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