Remarks about Mixed Discriminants and Volumes

S. Artstein-Avidan, D. Florentin, Y. Ostrover

Abstract

In this note we prove certain inequalities for mixed discriminants of positive semi-definite matrices, and mixed volumes of compact convex sets in $\mathbb{R}^n$. Moreover, we discuss how the latter are related to the monotonicity of an information functional on the class of convex bodies, which is a geometric analogue of the classical Fisher information.

1 Introduction and Results

Starting from the seminal works of A. D. Aleksandrov, the theory of mixed discriminants and volumes serves as a powerful tool for studying various quantities associated with convex bodies, such as volume, surface area, and mean width. In addition to their significant role in convex geometry, inequalities emanating from this theory - the most famous of which is probably the Alexandrov–Fenchel inequality - have numerous applications and deep connections to various fields, such as differential and algebraic geometry, probability theory, combinatorics, and more. We refer the reader to [7, 13], and the references therein, for a more detailed exposition of this subject.

In this paper we prove some inequalities for mixed discriminants of positive semi-definite matrices, and mixed volumes of compact convex sets in $\mathbb{R}^n$, denoted by $D(A_1, \dots, A_n)$ and $V(K_1, \dots, K_n)$ respectively (the precise definitions will be given in the following sections). Our work is partially motivated by a result of Hug and Schneider regarding a certain inequality for mixed volumes of zonoids (Theorem 2 in [10]), which is conjectured to hold (ibid., page 2643) for arbitrary convex bodies (cf. inequality (16) in [2]). Our first result is the following simple observation regarding mixed discriminants which seems to have been overlooked in the literature.

Theorem 1.1. *For any positive semi-definite $n \times n$ matrices A_1, A_2, A_3 one has:*

$$D(A_1, A_3[n-1])D(A_2, A_3[n-1]) \geq \frac{n-1}{n} D(A_1, A_2, A_3[n-2])D(A_3[n]). \quad (1)$$

Moreover, equality holds if and only if one of the following three cases occurs: (i) A_3 is invertible and $A_1 A_3^{-1} A_2 = 0$, (ii) A_3 is of rank at most $n-2$, (iii) A_3 is of rank $n-1$ and either A_1 or A_2 satisfies $\mathrm{Im}(A_i) \subset \mathrm{Im}(A_3)$.

Here, and in the following, the abbreviation $A[i]$ stands for i copies of the object A. The analogue of inequality (1) for mixed volumes of convex bodies is

$$V(K, A[n-1])V(T, A[n-1]) \geq \frac{n-1}{n} V(K, T, A[n-2])V(A[n]). \tag{2}$$

Note that if two of the bodies coincide, this inequality holds even without the $\frac{n-1}{n}$ factor due to Alexandrov–Fenchel inequality for $K = T$, or trivially for $A = K$ or $A = T$. Inequality (2) fails in general for $n \geq 3$ (see, e.g., Subsection 4.1 below). However, in the case where $A = B_2^n$ is the Euclidean unit ball and T is a zonoid, not only does inequality (2) hold, but actually a slightly stronger inequality is valid. Namely,

Theorem 1.2. *For every convex body $K \subset \mathbb{R}^n$ and every zonoid $Z \subset \mathbb{R}^n$*

$$V(K, B_2^n[n-1])V(Z, B_2^n[n-1]) \geq \frac{n-1}{n} \frac{\kappa_{n-1}^2}{\kappa_n \kappa_{n-2}} V(K, Z, B_2^n[n-2])V(B_2^n[n]), \tag{3}$$

where κ_n stands for the volume of the n-dimensional Euclidean unit ball. Moreover, equality holds if and only if K and Z lie in orthogonal (affine) subspaces of $\mathbb{R}^n$.

When both K and Z are zonoids, inequality (3) was proved by Hug and Schneider in [10], (cf. [2]), and was conjectured to hold for arbitrary convex bodies K and Z. Note that the constant $\frac{\kappa_{n-1}^2}{\kappa_n \kappa_{n-2}}$ in (3) is strictly greater than one, and approaches one as n tends to infinity. More precisely $1 < \frac{\kappa_{n-1}^2}{\kappa_n \kappa_{n-2}} < 1 + \frac{1}{n-1}$.

It turns out that inequality (2) fails for some triples (K, T, A), even in the case where $K = B_2^n$ is the Euclidean unit ball and T is an interval. This case is equivalent to an inequality which was conjectured by Giannopoulos, Hartzoulaki, and Paouris in [8], and then disproved by Fradelizi, Giannopoulos, and Meyer in [6], where also a positive result was proven which gives a special case of (2) with different constants. In Subsection 4.1 we give yet another example of the failure of (2) when $K = B_2^n$, T is an interval, and A is a certain truncated box. Any case where (2) fails with $K = B_2^n$ gives a negative answer to the question of the monotonicity of a certain geometric analogue of the Fisher information functional on the class of convex domains which was introduced in [4]. More precisely, denote by $\mathcal{K}^n$ the class of compact convex sets in $\mathbb{R}^n$, and for $K \in \mathcal{K}^n$ set $I(K) = |K|/|\partial K|$, where $|K|$ stands for the volume of K and $|\partial K|$ for its surface area.

The functional I, which can be considered as a dual analogue of the Fisher information, was introduced by Dembo, Cover, and Thomas in [4]. In the same paper it was asked whether I satisfies a Brunn–Minkowski type inequality i.e., whether for any $K_1, K_2 \in \mathcal{K}^n$ one has $I(K_1 + K_2) \geq I(K_1) + I(K_2)$, or at least whether I is monotone with respect to Minkowski addition, namely, satisfies $I(K_1 + K_2) \geq I(K_1)$

for every $K_1, K_2 \in \mathcal{K}^n$. In [4] it was verified that I is monotone with respect to the addition of a Euclidean ball. This is a simple consequence of the Alexandrov–Fenchel inequality. It was also noted that without convexity the above mentioned Brunn–Minkowski type inequality cannot hold. In [6] it was shown that even for convex bodies, a counterexample to this inequality exists. In fact, the example given in [6] is also a counterexample for the monotonicity question above, although this was not pointed out explicitly in [6].

Our next observation regarding mixed volumes is the equivalence of the monotonicity property of I with a certain inequality for mixed volumes. More precisely,

Proposition 1.3. *Let $T \in \mathcal{K}^n$. The following two inequalities are equivalent:*

(i) $\forall A \in \mathcal{K}^n :\ V(B_2^n, A[n-1])V(T, A[n-1]) \geq \frac{n-1}{n} V(B_2^n, T, A[n-2])V(A[n])$;

(ii) $\forall A \in \mathcal{K}^n :\ I(A+T) \geq I(A)$.

In Section 5 we shall prove that in dimension 2 the information functional I is monotone with respect to Minkowski addition. In fact, inequality (2) holds for all K, T and A. As noted above, in any other dimension $n \geq 3$ both inequalities in Proposition 1.3 fail in general. In Subsection 4.1 we give for any $n \geq 3$ an explicit example of a pair of convex bodies $T, A \in \mathcal{K}^n$ for which the two inequalities in Proposition 1.3 fail to hold. It remains an interesting question to determine for which convex bodies T the inequality in Proposition 1.3 does hold, and monotonicity is satisfied (for example, the ball B_2^n is such a body).

The rest of the paper is organized as follows: in Sections 2 and 3 we prove Theorems 1.1 and 1.2, respectively. In Section 4 we discuss the relation between the information functional I and inequality (2), and prove Proposition 1.3. Finally, in Section 5 we prove inequality (2) in the two-dimensional case.

Notations: Throughout the text we shall use the following notations: By a convex body we shall mean a compact convex set with non-empty interior. The class of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. Given $K \in \mathcal{K}^n$, we denote by $h_K : \mathbb{R}^n \to \mathbb{R}$ its support function, given by $h_K(u) = \sup\{\langle x, u\rangle \ ;\ x \in K\}$. We set σ to be the normalized Haar measure on the sphere $S^{n-1} \subset \mathbb{R}^n$, and λ_n the standard n-dimensional Lebesgue measure. The volume of the n-dimensional Euclidean unit ball is denoted by κ_n. Finally, we denote $M^*(K) := \int_{S^{n-1}} h_K \, d\sigma$.

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2 Mixed Discriminants

Mixed discriminants were introduced by A. D. Aleksandrov as a tool to study mixed volumes of convex sets (see e.g., §25.4 in [3] and the references therein). They are the coefficients in the polynomial expansion of the determinant of a sum of matrices. More precisely, let $A_1, \dots, A_m$ be symmetric real $n \times n$ matrices .The determinant of the sum $\sum_{i=1}^m \lambda_i A_i$ is a homogeneous polynomial of degree n in $\lambda_1, \dots, \lambda_m$, and can be written as

$$\det\left(\sum_{i=1}^m \lambda_i A_i\right) = \sum_{i_1,\dots,i_n=1}^m \lambda_{i_1}\cdots\lambda_{i_n} D(A_{i_1}, \dots, A_{i_n}) \tag{4}$$

(see [3], or §2.5 in [13]). The quantity $D(A_1, \dots, A_n)$ is called the mixed discriminant of $A_1, \dots, A_n$.

In the following lemma we gather some basic well known facts regarding mixed discriminants (see e.g. [1]). Here, A^i stands for the i-th column of the matrix A, the notation $A \geq 0$ means that A is symmetric and positive semi-definite, and Π_n stands for the permutation group of n elements.

Lemma 2.1. *Let $A_1, \dots, A_n$ be symmetric real $n \times n$ matrices.*

(i) *If $A_i \geq 0$ for all i, then $D(A_1, \dots, A_n) \geq 0$;*

(ii) $D(BA_1, \dots, BA_n) = \det(B) D(A_1, \dots, A_n)$*, for any $n \times n$ matrix B;*

(iii) $D(A_1, \dots, A_n) = \frac{1}{n!}\sum_{\sigma\in\Pi_n} \det(A^1_{\sigma(1)}, \dots, A^n_{\sigma(n)})$.

Note that if $A_i = A$ for all i, then $D(A_1, \dots, A_n) = \det(A)$. We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. From property (i) of Lemma 2.1 it follows that inequality (1) holds trivially when $\det(A_3) = 0$. Thus, we can assume without loss of generality that A_3 is invertible. Hence, using property (ii) of Lemma 2.1, we conclude that in order to prove inequality (1) it suffices to show that:

$$D(X, I[n-1]) D(Y, I[n-1]) \geq \frac{n-1}{n} D(X, Y, I[n-2]), \tag{5}$$

where $X = A_3^{-1}A_1$, $Y = A_3^{-1}A_2$, and I is the $n \times n$ identity matrix. By property (iii) of Lemma 2.1 we have:

$$D(X, I[n-1]) = \frac{1}{n}\sum_{i=1}^n \det(e_1, \dots, e_{i-1}, X^i, e_{i+1}, \dots, e_n) = \frac{1}{n}\sum_{i=1}^n x_{ii} = \frac{\mathrm{tr}(X)}{n},$$

where $\{e_i\}_{i=1}^n$ stands for the i-th column of the identity matrix I, and

$$D(X,Y,I[n-2]) = \frac{1}{n(n-1)} \sum_{i \neq j} \det(Z(i,j)).$$

where $Z(i,j)$ denotes the identity matrix with the i^{th} column replaced by X^i and the j^{th} column replaced by Y^j. Separating into two sums we get

$$\begin{aligned} D(X,Y,I[n-2]) &= \frac{1}{n(n-1)} \sum_{i<j} \det(e_1,\ldots,X^i,\ldots,Y^j,\ldots,e_n) + \\ & \quad \frac{1}{n(n-1)} \sum_{j<i} \det(e_1,\ldots,Y^j,\ldots,X^i,\ldots,e_n) \qquad (6) \\ &= \frac{1}{n(n-1)} \sum_{i \neq j} (x_{ii}y_{jj} - x_{ji}y_{ij}) = \frac{1}{n(n-1)} \sum_{i,j=1}^{n} (x_{ii}y_{jj} - x_{ji}y_{ij}). \end{aligned}$$

Combining these relations we conclude that

$$\begin{aligned} & D(X,I[n-1])D(Y,I[n-1]) - \frac{n-1}{n} D(X,Y,I[n-2]) \\ &= \left(\frac{1}{n}\sum_{i=1}^{n} x_{ii}\right)\left(\frac{1}{n}\sum_{j=1}^{n} y_{jj}\right) - \frac{n-1}{n} \cdot \frac{1}{n(n-1)} \sum_{i,j=1}^{n} (x_{ii}y_{jj} - x_{ji}y_{ij}) \\ &= \frac{1}{n^2} \sum_{i,j=1}^{n} (x_{ii}y_{jj} - x_{ii}y_{jj} + x_{ji}y_{ij}) = \frac{1}{n^2} \sum_{i,j=1}^{n} x_{ji}y_{ij} = \frac{\operatorname{tr}(XY)}{n^2}. \end{aligned}$$

Note that $\operatorname{tr}(XY) = \operatorname{tr}(A_3^{-1}A_1A_3^{-1}A_2)$. Moreover, it is not hard to check that the matrix $A_3^{-1}A_1A_3^{-1}$ is also symmetric and positive semi-definite. Inequality (5) now immediately follows since the trace of the product of two symmetric positive semi-definite matrices is always non-negative[1]. Moreover, under the assumption that A_3 is invertible, equality in (5) holds if and only if $\operatorname{tr}(A_3^{-1}A_1A_3^{-1}A_2) = 0$, or equivalently (as it is the product of two positive definite matrices), $A_1A_3^{-1}A_2 = 0$. Moreover, it follows from [12] that for singular A_3 equality in (1) holds if and only if either A_3 is of rank at most $n-2$, or A_3 is of rank $n-1$ and either A_1 or A_2 satisfies $\operatorname{Im}(A_i) \subset \operatorname{Im}(A_3)$. This completes the proof of Theorem 1.1. □

3 Mixed Volumes

Arising from the classical works of Minkowski, Aleksandrov, Hadwiger, and many others, mixed volumes have been studied in a variety of contexts. In addition to having

[1]Indeed, for any two symmetric positive semi-definite matrices A and B one has $\operatorname{tr}(AB) = \operatorname{tr}(\sqrt{A}\sqrt{A}\sqrt{B}\sqrt{B}) = \operatorname{tr}(\sqrt{B}\sqrt{A}\sqrt{A}\sqrt{B}) = \operatorname{tr}((\sqrt{A}\sqrt{B})^*\sqrt{A}\sqrt{B})$, which is a sum of squares.

vast applications to convex geometry, mixed volumes provide geometric techniques to study sparse systems of polynomial equations - and thus serve as a bridge between algebraic and convex geometry, appear as intersection numbers in tropical geometry, and can be used as a powerful tool in combinatorics and computational geometry. For a detailed exposition and further information on the properties of mixed volumes we refer the reader to Chapter 5 of [13].

A classical result due to Minkowski states that the volume of a linear combination $\sum_{i=1}^m \lambda_i K_i$ of convex bodies K_i is a homogeneous polynomial of degree n in $\lambda_i \geq 0$, where $A+B$ stands for Minkowski addition, $A+B=\{a+b : a\in A, b\in B\}$. Mixed volumes are the coefficients in this polynomial expansion. More precisely

$$\mathrm{Vol}\left(\sum_{i=1}^m \lambda_i K_i\right) = \sum_{i_1,\dots,i_n=1}^m \lambda_{i_1}\cdots\lambda_{i_n} V(K_{i_1},\dots,K_{i_n}), \tag{7}$$

where $K_i \subset \mathbb{R}^n$ are compact convex sets and $\lambda_i \geq 0$. The coefficient $V(K_{i_1},\dots,K_{i_n})$ of the monomial $\lambda_{i_1}\cdots\lambda_{i_n}$ is called the mixed volume of $K_{i_1},\dots,K_{i_n}$, and it depends only on $K_{i_1},\dots,K_{i_n}$ and not on any of the other bodies. One may assume that the coefficients are symmetric with respect to permutations of the bodies. Mixed volumes are known to be non-negative, and are clearly translation invariant. Moreover, they are monotone with respect to set inclusion, additive in each argument with respect to Minkowski addition, continuous with respect to the Hausdorff topology, and positively homogeneous in each argument (see e.g. Section §5.1 of [13], and [5], Chapter 5). We now turn to the proof of Theorem 1.2.

Proof of Theorem 1.2. Since mixed volumes are continuous with respect to the Hausdorff topology on $\mathcal{K}^n$, in order to prove inequality (3) it suffices to assume that Z is a zonotope, i.e., a Minkowski sum of intervals. Moreover, using the additivity, translation invariance, and positive homogeneity of mixed volumes, we may further assume that Z is the interval $Z=[0,u]$, for some $u\in S^{n-1}$.

Let $u \in S^{n-1}$. We denote by σ' the normalized Haar measure on the sphere S^{n-2} which we identify with $S^{n-1}\cap u^{\perp}$, by ν the $(n-1)$-dimensional mixed volume functional in $\mathbb{R}^{n-1}$, and by $K|_u$ the orthogonal projection of K onto $u^{\perp}$. It is well known (see e.g., equation (A.43) in [7]) that:

$$V([0,u],K_2,\dots,K_n) = \frac{1}{n}\nu(K_2|_u,\dots,K_n|_u).$$

Combining this with the standard integral representation of quermassintegrals (that is, mixed volumes where only two bodies are mixed - some body K and the Euclidean ball B_2^n), see (5.1.18) in §5.3 of [13], we conclude that

$$V(K,B^n[n-1]) = \kappa_n \int_{S^{n-1}} h_K\, d\sigma, \qquad V(Z,B^n[n-1]) = \frac{\kappa_{n-1}}{n},$$

$$V(K,Z,B^n[n-2]) = \frac{1}{n}\nu(K|_u, B^{n-1}[n-2]) = \frac{\kappa_{n-1}}{n}\int_{S^{n-1}\cap u^\perp} h_K\, d\sigma',$$

where in the last equality we have used the fact that $h_K = h_{K|_u}$ on $u^\perp$. With these equalities at our disposal, inequality (3) reduces to showing

$$M^*(K) := \int_{S^{n-1}} h_K\, d\sigma \geq C_n \int_{S^{n-1}\cap u^\perp} h_K\, d\sigma', \tag{8}$$

where the constant C_n is the same as in (3), i.e., $C_n = \frac{n-1}{n}\frac{\kappa_{n-1}^2}{\kappa_n\kappa_{n-2}}$.

Denote by Π_u the reflection operator with respect to the hyperplane $u^\perp$, that is, $\Pi_u(x) = x - 2u\langle x, u\rangle$. Clearly $M^*(K) = M^*(\frac{K+\Pi_u K}{2})$. The body $\frac{K+\Pi_u K}{2}$ is called the Minkowski symmetrization of K in direction u. It is also clear that $\frac{K+\Pi_u K}{2} \supseteq K|_u$. Thus we get that $M^*(K) = M^*(\frac{K+\Pi_u K}{2}) \geq M^*(K|_u)$, where the second inequality is due to inclusion. Moreover, since $h_K = h_{K|_u}$ on $u^\perp$, the right-hand side of (8) is the same for K and for $K|_u$. Therefore, to prove inequality (8) it suffices to assume that $K \subset u^\perp$. On the other hand, in the case where $K \subset u^\perp$, the left and right-hand side of equation (8) are equal. Indeed, it is well known (see e.g. [7], page 404, equation (A.29)) that for a k-dimensional convex body in $\mathbb{R}^n$ with $k < n$ one has for any $0 \leq i \leq k$ that

$$\frac{1}{c_{i,k}}V(K[i], B_2^k[k-i]) = \frac{1}{c_{i,n}}V(K[i], B_2^n[n-i]), \text{ where } c_{i,k} = \frac{\kappa_{k-i}}{\binom{k}{i}},$$

where the mixed volume functional on the left-hand side is k-dimensional. Thus, we conclude that

$$\begin{aligned}\int_{S^{n-1}} h_K d\sigma &= \frac{1}{\kappa_n}V(K, B_2^n[n-1]) = \frac{1}{\kappa_n}\frac{c_{1,n}}{c_{1,n-1}}V(K, B_2^{n-1}[n-2]) \\ &= \frac{\kappa_{n-1}^2}{\kappa_n\kappa_{n-2}}\frac{n-1}{n}\int_{S^{n-1}\cap u^\perp} h_K d\sigma'.\end{aligned}$$

Next we characterize the equality case. For $Z = [0,u]$, equality in $M^*(K) \geq M^*(K|_u)$, and hence in (3), holds if and only if $K \subset x_0 + u^\perp$ for some $x_0 \in \mathbb{R}^n$. Using the additivity of inequality (3) in the parameter Z, we conclude that for a zonotope Z (i.e., a finite Minkowski sum of line segments), equality in (3) holds if and only if, up to translations, $K \subset E$ and $Z \subset E^\perp$ for some linear subspace $E \subset \mathbb{R}^n$. Finally, for a general zonoid Z we argue as follows. Let $K \in \mathcal{K}^n$, and consider the function

$$I_K(Z) := V(K, B^n[n-1])V(Z, B^n[n-1]) - \frac{n-1}{n}\frac{\kappa_{n-1}^2}{\kappa_n\kappa_{n-2}}V(K,Z,B^n[n-2])V(B^n[n]),$$

defined on the class of zonoids in $\mathbb{R}^n$. Set E_K to be the subspace of $\mathbb{R}^n$ parallel to the minimal affine space containing K. In these notations the above argument implies that $I_K([-u,u]) = 0$ only if $u \in E_K^\perp$. Moreover, since any zonoid Z is given by $Z = \int_{S^{n-1}}[-u,u]d\mu_Z$, where μ_Z is an even measure on S^{n-1} (see §3.5 in [13]), it follows from the properties of mixed volume that $I_K(Z) = \int_{S^{n-1}} I([-u,u])d\mu_Z$. Hence, using the fact that $Z \subset E_K^{\perp}$ if and only if μ_Z is supported on $S^{n-1} \cap E_K^{\perp}$, we conclude that $I_K(Z) > 0$ if and only if $Z \not\subset E_K^{\perp}$, and the proof is now complete. □

Remark 3.1. Note that if $K = Z$ in Theorem 1.2, even without the assumption that Z is a zonoid (i.e., for any $Z \in \mathcal{K}^n$), inequality (3) holds without the $\frac{n-1}{n}\frac{\kappa_{n-1}^2}{\kappa_n \kappa_{n-2}}$ factor, thanks to Minkowski's first inequality (see Equation (6.2.3) in [13]).

Remark 3.2. As mentioned in the introduction, inequality (3) is conjectured to hold for arbitrary convex bodies $K, T \in \mathcal{K}^n$ (see [10]). This conjecture can be reformulated in the language of harmonic analysis, or, more precisely, as a conjecture on the rate of decay of the coefficients of a convex function with respect to its spherical harmonic decomposition. More precisely, consider the following integral formula for $V(K,T,B[n-2])$ (see (5.3.17) in [13]):

$$V(K,T,B[n-2]) = \kappa_n \int_{S^{n-1}} h_K(u)\Big(h_T(u) + \frac{1}{n-1}\triangle_S h_T(u)\Big) d\sigma(u),$$

where $\triangle_S$ stands for the spherical Laplace operator on S^{n-1}. Combining this with the fact that $V(K,B,\dots,B) = \kappa_n \int_{S^{n-1}} h_K d\sigma$ (see (5.3.12) in [13]), we conclude that for $K, T \in K^n$ inequality (3) can be written as:

$$\int_{S^{n-1}} h_K\, d\sigma \int_{S^{n-1}} h_T\, d\sigma \geq C_n \int_{S^{n-1}} h_K(u)\Big(h_T(u) + \frac{1}{n-1}\triangle_S h_T(u)\Big) d\sigma(u), \qquad (9)$$

where $C_n = \frac{n-1}{n}\frac{\kappa_{n-1}^2}{\kappa_n \kappa_{n-2}}$. Next, consider the harmonic expansion of the support functions h_K and h_T, given by

$$h_K = \sum_{m=0}^{\infty}\sum_{l=0}^{N(m,n)} k_{m,l} Y_m^l, \text{ and } h_T = \sum_{m=0}^{\infty}\sum_{l=0}^{N(m,n)} t_{m,l} Y_m^l,$$

where $\{Y_m^l\}_{l=0}^{N(m,n)}$ stands for a basis for the m-eigenspace of the spherical Laplace operator $\triangle_S$ on the space $L_2(S^{n-1})$ with eigenvalue $\triangle_S Y_m^l = -m(m+n-2)Y_m^l$ (see the Appendix of [13] for more details). It is well known that $k_{0,0} = \int_{S^{n-1}} h_K d\sigma$, and similarly $t_{0,0} = \int_{S^{n-1}} h_T d\sigma$. Thus, we conclude that inequality (3) is equivalent to

$$k_{0,0}t_{0,0} \geq C_n \sum_{m=0}^{\infty} \frac{(1-m)(n+m-1)}{n-1} \sum_{l=0}^{N(m,n)} k_{m,l}t_{m,l}, \qquad (10)$$

which after a suitable rearrangement can be written as

$$k_{0,0}t_{0,0} \geq \frac{C_n}{1-C_n} \sum_{m=1}^{\infty} \frac{(1-m)(n+m-1)}{n-1} \sum_{l=0}^{N(m,n)} k_{m,l}t_{m,l}. \qquad (11)$$

We thus arrive at the following conjecture

Conjecture 3.3. *Let $h_1, h_2 : S^{n-1} \to \mathbb{R}$ be two convex functions (that is, their homogeneous extension to $\mathbb{R}^n$ is assumed convex). Let their decomposition in the basis of spherical harmonics be given by*

$$h_1 = \sum_{m=0}^{\infty} \sum_{l=0}^{N(m,n)} a_{m,l} Y_m^l, \text{ and } \quad h_2 = \sum_{m=0}^{\infty} \sum_{l=0}^{N(m,n)} b_{m,l} Y_m^l.$$

Then

$$a_{0,0} b_{0,0} \geq D_n \sum_{m=1}^{\infty} \frac{(1-m)(n+m-1)}{n-1} \sum_{l=0}^{N(m,n)} a_{m,l} b_{m,l}, \tag{12}$$

where $D_n = \frac{(n-1)\kappa_{n-1}^2}{n\kappa_n \kappa_{n-2} - (n-1)\kappa_{n-1}^2}$.

4 Mixed Volumes and the Information Functional

Let $K \in \mathcal{K}^n$. Recall that the information functional I is defined by $I(K) = |K|/|\partial K|$, where $|K|$ stands for the volume of K, and $|\partial K|$ for its surface area. As mentioned in the introduction, the functional I was introduced and studied in [4]. In the same paper, it was asked whether I satisfies a Brunn–Minkowski type inequality, i.e., whether for any $K_1, K_2 \in \mathcal{K}^n$ one has $I(K_1 + K_2) \geq I(K_1) + I(K_2)$, or a weaker property, whether I is monotone with respect to Minkowski addition, namely, satisfies $I(K_1 + K_2) \geq I(K_1)$, for every $K_1, K_2 \in \mathcal{K}^n$. A counterexample to the above Brunn–Minkowski type inequality was given in [6]. Although this was not pointed out explicitly in [6], this example is also a counterexample for the monotonicity of the information functional I on the class $\mathcal{K}^n$.

We next turn to the proof of Proposition 1.3, and in the next subsection we shall give another example where both inequalities in Proposition 1.3 fail to hold.

Proof of Proposition 1.3. Let $T \in \mathcal{K}^n$. Both directions of the equivalence follow from the fact that inequality (i) is in a sense the "linearization" of inequality (ii). Indeed, assume (i) holds. For every $A \in \mathcal{K}^n$ and $\lambda \in [0,1]$, set $A_\lambda = A + \lambda T$ and $f_A(\lambda) = I(A_\lambda)$. We wish to show that $f_A(0) \leq f_A(1)$. To this end we shall prove that $f_A' \geq 0$ in the interval $[0,1]$. In fact, since $f_A(\lambda + h) = f_{A_\lambda}(h)$, and A is an arbitrary convex body, it is enough to prove $f_A'(0) \geq 0$ for every $A \in \mathcal{K}^n$. Denote $\mathcal{V}_i = V(T[i], A[n-i])$ for $1 \leq i \leq n$, and $\mathcal{W}_i = V(T[i], A[n-1-i], B_2^n)$ for $1 \leq i \leq n-1$. In particular $\mathcal{V}_0 = |A|$, and $n\mathcal{W}_0 = |\partial A|$. With these notation one has

$$|A_\lambda| = V(A_\lambda, \ldots, A_\lambda) = \sum_{i=0}^{n} \binom{n}{i} \lambda^i \mathcal{V}_i = |A| + n\mathcal{V}_1 \lambda + o(\lambda),$$

and similarly

$$|\partial A_\lambda| = nV(A_\lambda, \ldots, A_\lambda, B_2^n) = n\sum_{i=0}^{n-1}\binom{n-1}{i}\lambda^i\mathcal{W}_i = |\partial A| + n(n-1)\mathcal{W}_1\lambda + o(\lambda).$$

Combining these relations one has

$$\begin{aligned} I(A_\lambda) &= I(A)\,\frac{1+n\frac{\mathcal{V}_1}{\mathcal{V}_0}\lambda + o(\lambda)}{1+(n-1)\frac{\mathcal{W}_1}{\mathcal{W}_0}\lambda + o(\lambda)} \\ &= I(A)\left(1+\left(n\frac{\mathcal{V}_1}{\mathcal{V}_0} - (n-1)\frac{\mathcal{W}_1}{\mathcal{W}_0}\right)\lambda + o(\lambda)\right). \end{aligned}$$

Thus, we conclude that

$$f_A'(0) = I(A)\cdot\left(n\frac{\mathcal{V}_1}{\mathcal{V}_0} - (n-1)\frac{\mathcal{W}_1}{\mathcal{W}_0}\right). \tag{13}$$

Since inequality (i) states exactly that $\mathcal{W}_0\mathcal{V}_1 \geq \frac{n-1}{n}\mathcal{W}_1\mathcal{V}_0$, we conclude that $f_A'(0) \geq 0$, and hence inequality (ii) holds. Conversely, assume that inequality (i) does not hold, that is, there exists $A \in \mathcal{K}^n$ such that $\mathcal{W}_0\mathcal{V}_1 < \frac{n-1}{n}\mathcal{W}_1\mathcal{V}_0$. From (13) it follows that $f_A'(0) < 0$. From the continuity property of mixed volumes we conclude that there exists $\delta > 0$ such that for all $\lambda \in [0, \delta]$, one has $f_A'(\delta) < 0$, and hence f_A is strictly decreasing on $[0, \delta]$. From this it follows that $I(A + \delta T) < I(A)$, or equivalently, $I(\tilde{A} + T) < I(\tilde{A})$ for $\tilde{A} = A/\delta$. This completes the proof of Proposition 1.3. $\square$

4.1 An example without monotonicity

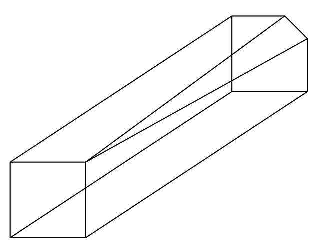

Figure 1: A counterexample to the monotonicity of the information functional I

In this subsection we provide a simple counterexample to the monotonicity property of the functional I for $n \geq 3$. More precisely, we give an example of a pair of convex bodies $T, A \in K^n$ for which the two equivalent inequalities in Proposition 1.3 fail to hold.

Proposition 4.1. *If $n \geq 3$, then there exist $A, T \in \mathcal{K}^n$ such that*

$$I(A+T) < I(A).$$

Proof of Proposition 4.1. It is enough to find A, T violating inequality (i) of Proposition 1.3. To do that, we use an interval $T = [0, u]$, say for $u = e_n$. In that case, inequality (i) of Proposition 1.3 becomes:

$$I(A|_u) \geq I(A). \tag{14}$$

Here, to ease notation, we use I to denote both the n-dimensional information of A, and the $(n-1)$-dimensional information of its projection to $u^\perp$, denoted by $A|_u$. Indeed, inequality (14) holds because $nV(B, T, A[n-2]) = \nu(B|_u, A|_u[n-2])$, $nV(T, A[n-1]) = \nu(A|_u[n-1])$, and the $(n-2)$-dimensional surface area of $A|_u$ is given by $\mathrm{Vol}_{n-2}(\partial A|_u) = (n-1)\nu(B|_u, A|_u[n-2])$.

For A we shall take a long cylindrical body and cut out a small piece of it by intersection with a half-space (see Figure 1), in such a way that the projection $A|_u$ is unchanged, but the information of A will slightly increase.

More precisely, let $0 < \varepsilon < 1 < M$, and denote by $Q = \sum_{i=1}^{n-1}[0, e_i]$ the $(n-1)$-dimensional unit cube. Let D be the $(n+1)$-simplex with vertices $0, \varepsilon e_1, \ldots, \varepsilon e_{n-1}, M e_n$, and opposite facets $F_0, \ldots, F_n$, respectively. Finally, set

$$A := (Q + M[0, e_n]) \setminus D.$$

Note that $A|_u = Q$ and hence $I(A|_u) = \frac{1}{2(n-1)}$. Moreover, a direct computation gives $|F_n| = \frac{\varepsilon^{n-1}}{(n-1)!}$, $|F_i| = \frac{M\varepsilon^{n-2}}{(n-1)!}$ for $1 \leq i \leq n-1$, and

$$\begin{aligned} |F_0| &= |(\mathrm{diag}\{1, \ldots, 1, M/\varepsilon\})\, \mathrm{conv}\{\varepsilon e_i\}_{i=1}^n| < \frac{M}{\varepsilon} |\mathrm{conv}\{\varepsilon e_i\}_{i=1}^n| \\ &= M\varepsilon^{n-2} |\mathrm{conv}\{e_i\}_{i=1}^n| = \frac{M\varepsilon^{n-2}\sqrt{n}}{(n-1)!}. \end{aligned}$$

From this we conclude that

$$|\partial(A)| = 2M(n-1) + 2 - \sum_{i=1}^n |F_i| + |F_0| < 2M(n-1) + 2 - \frac{M\varepsilon^{n-2}}{(n-2)!}\left(1 - \frac{\sqrt{n}}{n-1}\right).$$

Since $|D| = \frac{M\varepsilon^{n-1}}{n!}$, one has

$$\frac{I(A)}{I(A|_u)} > \frac{2M(n-1)\left(1 - \frac{\varepsilon^{n-1}}{n!}\right)}{2M(n-1)\left(1 + \frac{1}{M(n-1)} - \frac{\varepsilon^{n-2}}{2(n-1)!}\left(1 - \frac{\sqrt{n}}{n-1}\right)\right)}.$$

Thus, by choosing ε and M such that:

$$\frac{1}{M(n-1)} + \frac{\varepsilon^{n-1}}{n!} < \frac{\varepsilon^{n-2}}{2(n-1)!}\left(1 - \frac{\sqrt{n}}{n-1}\right),$$

we have shown that $I(A) > I(A|_u)$. This choice is indeed possible since for $n \geq 3$ the ε^{n-2} coefficient is positive. The proof of Proposition 4.1 is now complete. $\square$

5 Mixed Volumes in the Plane

In this section we show that in the 2-dimensional case inequality (2) for mixed volumes always holds (cf. [2] for a speical case). In particular, it follows from this and Proposition 1.3 that for $n = 2$ the information functional I is monotone with respect to Minkowski addition.

Proposition 5.1. *Let $K, T, A \in \mathcal{K}^2$. Then*

$$V(K, A)V(T, A) \geq \frac{1}{2}V(K, T)V(A, A). \tag{15}$$

Moreover, equality holds if and only if one of the following cases holds: (i) K and T are intervals, and A is a parallelogram whose edges are parallel to K and T (ii) K and A, or T and A (or all) are contained in parallel intervals (iii) A is a singleton.

We start with some preparations. Denote the inner and outer radii of T with respect to A by $r = r_A(T)$, and $R = R_A(T)$ respectively. These are the optimal numbers satisfying that $rA + x \subset T \subset RA + y$ for some $x, y \in \mathbb{R}^n$. It is well known that for $A, T \in \mathcal{K}^2$, the polynomial $P(\lambda) = V(A, A)\lambda^2 + 2V(T, A)\lambda + V(T, T)$ has only real roots (e.g., by Minkowski's inequality) which are clearly non-positive. Moreover, a Bonnesen-type inequality (see pages 323-324 in [13]) states that

$$\lambda^- \leq -R_A(T) \leq -r_A(T) \leq \lambda^+,$$

where $\lambda^\pm$ are the roots of $P(\lambda)$. In particular

$$P(-R) \leq 0, \qquad P(-r) \leq 0.$$

We are now in a position to prove Proposition 5.1.

Proof of Proposition 5.1. If $V(A, A) = 0$, or T is a singleton, we are done. Otherwise, we have $0 < R_A(T) < \infty$. Since the inequality is homogeneous in each of the bodies, we may assume $R_A(T) = 1$. By the remark preceding the proof:

$$0 \geq P(-1) = V(A, A) - 2V(T, A) + V(T, T) \geq V(A, A) - 2V(T, A),$$

and hence

$$V(T, A) \geq \frac{1}{2}V(A, A). \tag{16}$$

Combining the latter with $V(K, A) \geq V(K, T)$ (which is due to inclusion), proves (15). Next, we characterize the equality case. In the case where $V(A, A) = 0$, either A is a singleton, and there is equality, or A is contained in an interval and at least one of $V(K, A)$,$V(K, T)$ must equal 0, that is, we are in equality case (ii). Assume $V(A, A) > 0$. A necessary condition for equality in (16) is that $V(T, T) = 0$, i.e. T

is contained in an interval. By symmetry, K must be contained in an interval too. Next, let P be the minimal parallelogram containing A, which has edges parallel to K and T. Since $V(K,A)=V(K,P)$ and $V(T,A)=V(T,P)$, we have:

$$V(K,A)V(T,A)=V(K,P)V(T,P)=\frac{1}{2}V(T,K)V(P,P)\geq\frac{1}{2}V(T,K)V(A,A).$$

Equality holds above only if $V(P,P)=V(A,A)$, which implies $A=P$. □

Corollary 5.2. *For every $A,T\in\mathcal{K}^2$, $I(A+T)>I(A)$.*

Proof of Corollary 5.2. Combining Proposition 1.3 with Proposition 5.1 proves $I(A+T)\geq I(A)$. To show strict inequality we only note that the derivative f'_A is strictly positive, since $K=B_2$ does not qualify for the equality case of Theorem 5.1. □

Finally, still in the 2-dimensional case, we prove inequality (2) with an improved constant in the case where $A=B_2^2$ is the Euclidean disk. In fact, the following theorem amounts to Theorem 1.2 in dimension 2 without the assumption that one of the bodies is a zoonoid.

Proposition 5.3. *Let $K,T\in\mathcal{K}^2$. Then:*

$$V(T,B_2^2)V(K,B_2^2)\geq\frac{2}{\pi}V(T,K)V(B_2^2,B_2^2), \tag{17}$$

with equality if and only if K and T are orthogonal intervals.

Proof of Proposition 5.3. Denote by $R(T)$ the circumradius of T i.e., the smallest radius of a disc containing T, and by $L(T)$ its perimeter. Note that $V(B_2^2,B_2^2)=\pi$, $V(T,B_2^2)=\frac{L(T)}{2}$, and from the monotonicity of mixed volume one has

$$V(T,B_2^2)V(K,B_2^2)\geq\frac{L(T)V(K,T)}{2R(T)}.$$

Inequality (17) now follows from that fact that $L(T)\geq 4R(T)$ (see [11]). Moreover, $L(T)=4R(T)$ if and only if T is an interval. Since K and T play a symmetric role in (17), a necessary condition for equality in (17) is that both K and T are intervals. In that case, a direct computation shows that equality holds if and only if K and T are orthogonal. The proof of Proposition 5.3 is now complete. □

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Shiri Artstein-Avidan, Dan Florentin, Yaron Ostrover
School of Mathematical Science, Tel Aviv University, Tel Aviv, Israel.
e-mails: shiri@post.tau.ac.il, danflore@post.tau.ac.il , ostrover@post.tau.ac.il.