A Comparison of Hofer's Metrics on Hamiltonian Diffeomorphisms and Lagrangian Submanifolds

Yaron Ostrover*
School of Mathematical Sciences
Tel Aviv University 69978 Tel Aviv, Israel
yaronost@post.tau.ac.il

Abstract

We compare Hofer's geometries on two spaces associated with a closed symplectic manifold (M, ω). The first space is the group of Hamiltonian diffeomorphisms. The second space $\mathcal{L}$ consists of all Lagrangian submanifolds of $M \times M$ which are exact Lagrangian isotopic to the diagonal. We show that in the case of a closed symplectic manifold with $\pi_2(M) = 0$, the canonical embedding of $\mathrm{Ham}(M)$ into $\mathcal{L}$, $f \mapsto graph(f)$ is not an isometric embedding, although it preserves Hofer's length of smooth paths.

1 Introduction and Main Results

In this paper we compare Hofer's geometries on two remarkable spaces associated with a closed symplectic manifold (M, ω). The first space $\mathrm{Ham}(M, \omega)$ is the group of Hamiltonian diffeomorphisms. The second consists of all Lagrangian submanifolds of $(M \times M, -\omega \oplus \omega)$ which are exact Lagrangian isotopic to the diagonal $\triangle \subset M \times M$. Let us denote this second space by $\mathcal{L}$. The canonical embedding

$$j : \mathrm{Ham}(M, \omega) \to \mathcal{L}, \quad f \mapsto graph(f)$$

preserves Hofer's length of smooth paths. Thus, it naturally follows to ask whether j is an isometric embedding with respect to Hofer's distance. Here,

*This paper is a part of the author's Ph.D. thesis, being carried out under the supervision of Prof. Leonid Polterovich, at Tel-Aviv university.

we provide a negative answer to this question for the case of a closed symplectic manifold with $\pi_2(M) = 0$. In fact, our main result shows that the image of $\mathrm{Ham}(M,\omega)$ inside $\mathcal{L}$ is "strongly distorted" (see Theorem 1.1 below).

Let us proceed with precise formulations. Given a path $\alpha = \{f_t\},\ t \in [0,1]$ of Hamiltonian diffeomorphisms of (M,ω), define its Hofer's length (see [H]) as

$$\mathrm{length}(\alpha) = \int_0^1 \left\{ \max_{x\in M} F(x,t) - \min_{x\in M} F(x,t) \right\} dt$$

where $F(x,t)$ is the Hamiltonian function generating $\{f_t\}$. For two Hamiltonian diffeomorphisms ϕ and ψ, define the Hofer distance $d(\phi,\psi) = \inf \mathrm{length}(\alpha)$ where the infimum is taken over all smooth paths α connecting ϕ and ψ. For further discussion see e.g. [LM1],[MS], and [P1].

Hofer's metric can be defined in a more general context of Lagrangian submanifolds (see [C]). Let (P,σ) be a closed symplectic manifold, and let $\triangle \subset P$ be a closed Lagrangian submanifold. Consider a smooth family $\alpha = \{L_t\},\ t \in [0,1]$ of Lagrangian submanifolds, such that each L_t is diffeomorphic to $\triangle$. We call α an *exact path* connecting L_0 and L_1, if there exists a smooth map $\Psi : \triangle \times [0,1] \to P$ such that for every t, $\Psi(\triangle \times \{t\}) = L_t$, and in addition $\Psi^*\sigma = dH_t \wedge dt$ for some smooth function $H : \triangle \times [0,1] \to \mathbb{R}$. The Hofer length of an exact path is defined by

$$\mathrm{length}(\alpha) = \int_0^1 \left\{ \max_{x\in\triangle} H(x,t) - \min_{x\in\triangle} H(x,t) \right\} dt.$$

It is easy to check that the above notion of length is well-defined. Denote by $\mathcal{L}(P,\triangle)$ the space of all Lagrangian submanifolds of P which can be connected to $\triangle$ by an exact path. For two Lagrangian submanifolds L_1 and L_2 in $\mathcal{L}(P,\triangle)$, define the Hofer distance ρ on $\mathcal{L}(P,\triangle)$ as follows: $\rho(L_1,L_2) = \inf \mathrm{length}(\alpha)$, where the infimum is taken over all exact paths on $\mathcal{L}(P,\triangle)$ that connect L_1 and L_2.

In what follows we choose $P = M \times M$, $\sigma = -\omega \oplus \omega$ and take $\triangle$ to be the diagonal of $M \times M$. We abbreviate $\mathcal{L} = \mathcal{L}(P,\triangle)$ as in the beginning of the paper. Based on a result by Banyaga [B], it can be shown that every smooth path on $\mathcal{L}(P,\triangle)$ is necessarily exact. Our main result is the following:

Theorem 1.1. *Let (M,ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Then there exist a family $\{\varphi_t\},\ t \in [0,\infty)$ in $\mathrm{Ham}(M,\omega)$ and a constant c such that:*

1. $d(\mathbb{1}, \varphi_t) \to \infty$ *as* $t \to \infty$.

2. $\rho(\mathrm{graph}(\mathbb{1}), \mathrm{graph}(\varphi_t)) = c$.

In fact, we construct the above family $\{\varphi_t\}$ explicitly:

Example 1.2. Consider an open set $B \subset M$. Suppose that there exists a Hamiltonian diffeomorphism h such that $h(B) \cap \text{Closure }(B) = \emptyset$. By perturbing h slightly, we may assume that all the fixed points of h are non-degenerate. Let $F(x,t)$, where $x \in M$, $t \in [0,1]$ be a Hamiltonian function such that $F(x,t) = c_0 < 0$ for all $x \in M \setminus B$, $t \in [0,1]$. Assume that $F(t,x)$ is normalized such that for every t, $\int_M F(t,\cdot)\omega^n = 0$. We define the family $\{\varphi_t\}$, $t \in [0,\infty)$ by $\varphi_t = hf_t$, where $\{f_t\}$ is the Hamiltonian flow generated by $F(t,x)$. As we'll see below, the family $\{\varphi_t\}$ satisfies the requirements of Theorem 1.1.

Theorem 1.1 has some corollaries:

1. The embedding of $\mathrm{Ham}(M,\omega)$ in $\mathcal{L}$ is not isometric, rather, the image of $\mathrm{Ham}(M,\omega)$ in $\mathcal{L}$ is highly distorted. The minimal path between two graphs of Hamiltonian diffeomorphisms in $\mathcal{L}$, might pass through exact Lagrangian submanifolds which are not the graphs of any Hamiltonian diffeomorphisms. Compare with the situation described in [M], where it was proven that in the case of a compact manifold, the space of Hamiltonian deformations of the zero section in the cotangent bundle is locally flat in the Hofer metric.

2. The group of Hamiltonian diffeomorphisms of a closed symplectic manifold with $\pi_2(M) = 0$ has an infinite diameter with respect to Hofer's metric.

3. Hofer's metric d on $\mathrm{Ham}(M,\omega)$ *does not* coincide with the Viterbo-type metric on $\mathrm{Ham}(M,\omega)$ defined by Schwarz in [S].

As a by-product of our method we obtain the following result (see Section 3 below):

Theorem 1.3. *Let* (M,ω) *be a closed symplectic manifold with* $\pi_2(M) = 0$. *Then there exists an element* φ *in* $(\mathrm{Ham}(M,\omega), d)$ *which cannot be joined to the identity by a minimal geodesic.*

The first example of this kind was established by Lalonde and McDuff [LM2] for the case of S^2.

Acknowledgment. I would like to express my deep gratitude to my supervisor, Professor Leonid Polterovich, for his encouragement and for many hours of extremely useful conversations. I would also like to thank Paul Biran and Felix Schlenk for many fruitful discussions.

2 Proof of The Main Theorem

In this section we prove Theorem 1.1. Throughout this section let (M, ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Let $\{\varphi_t\},\ t \in [0, \infty)$ the family of Hamiltonian diffeomorphisms defined in Example 1.2. We begin with the following lemma which states that Hamiltonian diffeomorphisms act as isometries on the space $(\mathcal{L}, \rho)$. The proof of the lemma follows immediately from the definitions.

Lemma 2.1. *Let $\Gamma : \triangle \times [0, 1] \to M \times M$ be an exact Lagrangian isotopy in $\mathcal{L}$ and let $\Phi : M \times M \to M \times M$ be a Hamiltonian diffeomorphism. Then*

$$\text{length}\{\Gamma\} = \text{length}\{\Phi \circ \Gamma\}.$$

In particular, $\rho(L_1, L_2) = \rho(\Phi(L_1), \Phi(L_2))$ for every $L_1, L_2 \in \mathcal{L}$.

Next, consider the following exact isotopy of the Lagrangian embeddings $\Psi : \triangle \times [0, \infty) \to M \times M,\ \Psi(x, t) = (x, \varphi_t(x))$. We denote by $L_t = \Psi(\triangle \times \{t\})$ the graph of $\varphi_t = hf_t$ in $M \times M$. The following proposition will be proved in Section 5 below.

Proposition 2.2. *For every $t \in [0, \infty)$ there exists a Hamiltonian isotopy $\{\Phi_s\},\ s \in [0, t]$ of $M \times M$, such that $\Phi_s(L_0) = L_s$ and such that for every s, $\Phi_s(\triangle) = \triangle$.*

Hence, it follows from Proposition 2.2 and Lemma 2.1, that the family $\{\varphi_t\},\ t \in [0, \infty)$ satisfies the second conclusion of Theorem 1.1 with constant $c = \rho(\triangle, L_0)$.

Let us now verify the first statement of Theorem 1.1. For this purpose we will use a theorem by Schwarz [S] stated below. First, recall the definitions of the action functional and the action spectrum. Consider a closed symplectic manifold (M, ω) with $\pi_2(M) = 0$. Let $\{f_t\}$ be a Hamiltonian path generated by a Hamiltonian function $F : [0, 1] \times M \to \mathbb{R}$. We denote by $\mathrm{Fix}^\circ(f_1)$ the set of fixed points, x, of the time-1-map f_1 whose orbits $\gamma = \{f_t(x)\},\ t \in [0, 1]$ are contractible. For $x \in \mathrm{Fix}^\circ(f_1)$, take any 2-disc $\Sigma \subset M$ with $\partial\Sigma = \gamma$, and define the symplectic action functional by

$$\mathcal{A}(F, x) = \int_\Sigma \omega - \int_0^1 F(t, f_t(x))dt.$$

The assumption $\pi_2(M) = 0$ ensures that the integral $\int_\Sigma \omega$ does not depend on the choice of Σ.

Remark 2.3. In the case of a closed symplectic manifold with $\pi_2(M) = 0$, a result by Schwarz [S], implies that for a Hamiltonian path $\{f_t\}$ with $f_1 \neq \mathbb{1}$ there exist two fixed points $x,\ y \in \mathrm{Fix}^\circ(f_1)$ with $\mathcal{A}(F, x) \neq \mathcal{A}(F, y)$. Moreover, the action functional does not depend on the choice of the Hamiltonian path generating f_1. Therefore, we can speak about the action of a fixed point of a Hamiltonian diffeomorphism, regardless of the Hamiltonian function used to define it.

Definition 2.4. *For each f in* $\mathrm{Ham}(M, \omega)$ *we define the action spectrum*

$$\Sigma_f = \{\mathcal{A}(f, x) \mid x \in \mathrm{Fix}^\circ(f)\} \subset \mathbb{R}.$$

The action spectrum Σ_f is a compact subset of $\mathbb{R}$ (see e.g. [S],[HZ]).

Theorem 2.5. [S]. *Let (M, ω) be a closed symplectic manifold with $\pi_2(M) = 0$. Then, for every f in* $\mathrm{Ham}(M, \omega)$

$$d(\mathbb{1}, f) \geq \min \Sigma_f.$$

Next, consider the family $\{\varphi_t\} = \{hf_t\},\ t \in [0, \infty)$. Note that $\mathrm{Fix}^\circ(h) = \mathrm{Fix}^\circ(\varphi_t)$ for every t. The following proposition shows that the action spectrum of φ_t is a linear translation of the action spectrum of h. Its proof is carried out in Section 4.

Proposition 2.6. *For every $t \in [0, \infty)$, and for every fixed point $z \in \mathrm{Fix}^\circ(\varphi_t) = \mathrm{Fix}^\circ(h)$,*

$$\mathcal{A}(\varphi_t, z) = \mathcal{A}(h, z) - tc_0$$

where c_0 is the negative (constant) value that F attains on $M \setminus B$ (see Example 1.2).

We are now in a position to complete the proof of Theorem 1.1. Indeed, the action spectrum is a compact subset of $\mathbb{R}$, hence its minimum is finite. By proposition 2.6 the minimum of Σ_{φ_t} tends to infinity as $t \to \infty$. Thus,

$$d(\mathbb{1}, \varphi_t) \to \infty \text{ as } t \to \infty$$

as follows from Theorem 2.5. This completes the proof of Theorem 1.1. □

3 Geodesics in Ham(M, ω) and Proof of Theorem 1.3

In this section we describe our result about geodesics in the group of Hamiltonian diffeomorphisms endowed with the Hofer metric d. We refer the reader to [BP], [LM1], [LM2], and [P2] for further details on this subject.

Let $\gamma = \{\phi_t\},\ t \in [0,1]$ be a smooth regular path in $\mathrm{Ham}(M, \omega)$, i.e. $\frac{d}{dt}\phi_t \neq 0$ for every $t \in [0,1]$. The path γ is called a *minimal geodesic* if it minimizes the distance between its end-points:

$$\mathrm{length}(\gamma) = d(\phi_0, \phi_1).$$

The graph of a Hamiltonian path $\gamma = \{\phi_t\}$ is the family of embedded images of M in $M \times M$ defined by the map $\Gamma : M \times [0,1] \to M \times M,\ (x,t) \mapsto (x, \phi_t(x))$. Next, consider the family $\{\varphi_t\},\ t \in [0, \infty)$ that was constructed in Example 1.2. We will show that there exists no minimal geodesic joining the identity and φ_{t_0}, for some t_0.

Proof of Theorem 1.3. Assume (by contradiction) that for every t, there exists a minimal geodesic in $\mathrm{Ham}(M, \omega)$ joining the identity with φ_t. Fix $t_0 \in [0, \infty)$. There exists a Hamiltonian path $\alpha = \{g_s\},\ s \in [0,1]$ in $\mathrm{Ham}(M, \omega)$ such that

$$d_{t_0} := d(\mathbb{1}, \varphi_{t_0}) = \mathrm{length}(\alpha).$$

Expressed in Lagrangian submanifolds terms, $\Psi = \{\mathrm{graph}(g_s)\},\ s \in [0,1]$ is an exact path in $M \times M$ joining the diagonal with $\mathrm{graph}(\varphi_{t_0})$. By Proposition 2.2, there exists a Hamiltonian isotopy Φ such that for every t, $\Phi_t(\mathrm{graph}(\varphi_{t_0})) = \mathrm{graph}(\varphi_t)$, and $\Phi_t(\triangle) = \triangle$. We choose t_1 to be sufficiently close to t_0 so as to ensure that $\{\Phi_{t_1}(\mathrm{graph}(g_s))\},\ s \in [0,1]$ is the graph of some Hamiltonian path γ in $\mathrm{Ham}(M, \omega)$. Indeed, this can be done since it follows from the proof of Proposition 2.2, that the Hamiltonian diffeomorphism Φ is C^1-close to the identity in a small neighborhood of graph(φ_{t_0}). Moreover, using a compactness argument, we can choose a finite number of points $S = \{s_1 < \ldots < s_n\}$ in $[0, 1]$, and repeat the construction of Φ in a small neighborhood of graph(g_{s_i}) for $i = 1, \ldots, n$. Then, by smoothly patching together those Hamiltonian flows, we conclude that for every $s \in [0, 1]$, $\Phi_{t_1}(\text{graph}(g_s))$ is the graph of some Hamiltonian diffeomorphism. Next, we claim the following

$$d_{t_1} \leq \text{length}(\gamma) = \text{length}\{\text{graph}(\gamma)\} = \text{length}\{\text{graph}(\alpha)\} = \text{length}\{\alpha\} = d_{t_0}.$$

Indeed, a straightforward computation yields that the embedding $f \mapsto graph(f)$ preserves Hofer's length, and from Lemma 2.1, length$\{\text{graph}(\alpha)\} =$ length$\{\text{graph}(\gamma)\}$. We have shown that for every t_0 there exists $\varepsilon > 0$ such that if $|t - t_0| \leq \varepsilon$ then $d_t \leq d_{t_0}$. Since d_t is a continuous function, we conclude that d_t is a constant function. On the other hand, by Theorem 1.1, $d_t = d(\mathbb{1}, \varphi_t) \to \infty$ as $t \to \infty$. Hence there is a contradiction. □

4 Proof of Proposition 2.6

We investigate the expression $\mathcal{A}(\varphi_t, z)$ for some fixed t. Since the action functional does not depend on the choice of the Hamiltonian path generating the time-1-map (see Remark 2.3), we consider the following path generating φ_t.

$$\gamma(s) = \begin{cases} f_{2st} & , \quad s \in [0, \frac{1}{2}] \\ h_{2s-1} f_t & , \quad s \in (\frac{1}{2}, 1]. \end{cases}$$

Note that since $h(B) \cap B = \emptyset$ and f_t is supported in B, then for $z \in \mathrm{F}ix^\circ(\varphi_t) = \mathrm{F}ix^\circ(h)$ the path $\{\gamma_s(z)\}, s \in [0, 1]$ coincides with the path $\{h_s(z)\}$, $s \in [0, 1]$. Denote by α the loop $\{\gamma_s(z)\}$, $s \in [0, 1]$ and let Σ be any 2-disc with $\partial\Sigma = \alpha$. The details of the calculation of $\mathcal{A}(\varphi_t, z)$ are as follows:

$$\mathcal{A}(\varphi_t, z) = \int_\Sigma \omega - \int_0^1 tF(s, z)ds - \int_0^1 H(s, h_s(z))ds,$$

where F and H are the Hamiltonian functions generating $\{h_t\}$ and $\{f_t\}$ respectively. Recall that by definition, F is equal to a constant c_0 in $M \setminus B$. This implies that

$$\mathcal{A}(\varphi_t, z) = \int_\Sigma \omega - \int_0^1 H(s, h_s(z))ds - tc_0.$$

The right hand side is exactly $\mathcal{A}(h, z) - tc_0$. Hence, the proof is complete.

5 Extending the Hamiltonian Isotopy

In this section we prove Proposition 2.2. Let us first recall some relevant notations. Let $\{\varphi_t\},\ t \in [0, \infty)$ the family of Hamiltonian diffeomorphisms defined in Example 1.2. Consider the following exact isotopy of Lagrangian embeddings $\Psi : \triangle \times [0, \infty) \to M \times M,\ \Psi(x, t) = (x, \varphi_t(x))$. We denote by $L_t = \Psi(\triangle \times \{t\})$ the graph of $\varphi_t = hf_t$ in $M \times M$, and by $\triangle$ the diagonal in $M \times M$. It follows from the construction of the family $\{\varphi_t\}$, that for every t, $\mathrm{Fix}(\varphi_t) = \mathrm{Fix}(h)$. Hence, L_t intersects the diagonal at the same set of points for every t. Moreover, we assumed that all the fixed points of h are non-degenerate, therefore for every t, L_t transversely intersect the diagonal. In order to prove Proposition 2.2, we first need the following lemma.

Lemma 5.1. *Let $x, y \in Fix^\circ(\varphi_t) = Fix^\circ(h)$, i.e., intersection points of the family $\{L_t\}$ and the diagonal in $M \times M$. Take a smooth curve $\alpha : [0, 1] \to M$ with $\alpha(0) = x$ and $\alpha(1) = y$ and let $\Sigma : [0, 1] \times [0, 1] \to M,\ \Sigma(t, s) = \varphi_t(\alpha(s))$ be a 2-disc with $\partial\Sigma([0, t] \times [0, 1]) = \varphi_t\alpha - \varphi_0\alpha = \varphi_t\alpha - h\alpha$. Then the symplectic area of $\Sigma_t = \Sigma([0, t] \times [0, 1])$ vanishes for all t.*

Proof. By a direct computation of the symplectic area of Σ_t, we obtain that

$$\int_{\Sigma_t} \omega = \int_{[0,t]\times[0,1]} \Sigma_t^*\omega = -\int_0^t dt \int_0^1 d\widehat{F}_t\Big(\frac{\partial}{\partial s}\varphi_t\alpha(s)\Big)\, ds = \int_0^t \widehat{F}_t(\varphi_t(x))dt - \int_0^t \widehat{F}_t(\varphi_t(y))dt,$$

where $\widehat{F}$ is the Hamiltonian function generating the flow $\{\varphi_t\}$. A straightforward computation shows that $\widehat{F}(t, x) = F(t, h^{-1}x)$, where F is the Hamiltonian function generating the flow $\{f_t\}$. Recall that by definition, $F(x, t)$ is equal to a constant c_0 outside the ball B. Moreover, since $x,\ y \in Fix^\circ(h)$ and $h(B) \cap B = \emptyset$, then $x, y \notin$ B. Therefore, $\widehat{F}_t(\varphi_t(x)) = \widehat{F}_t(\varphi_t(y)) = c_0$ for every t. Thus, we conclude that for every t, the symplectic area of Σ_t vanishes as required. $\square$

Proof of Proposition 2.2. We shall proceed along the following lines. By the Lagrangian tubular neighborhood theorem (see [W]), there exists a symplectic identification between a small tubular neighborhood U_s of L_s in $M \times M$ and a tubular neighborhood V_s of the zero section in the cotangent bundle T^*L_s. Moreover, it follows from a standard compactness argument that there exists $\delta_s = \delta(s, U_s) > 0$ such that $L_{s'} \subset U_s$ for every s' with

$|s' - s| \leq \delta_s$. Next, denote $I_s = (s - \delta_s, s + \delta_s) \cap [0, t]$, and consider an open cover of the interval $[0, t]$ by the family $\{I_s\}$, that is $[0, t] = \bigcup_{s \in [0,t]} I_s$. By compactness we can choose a finite number of points $S = \{s_1 < \ldots < s_n\}$ such that $[0, t] = \bigcup_{i=1}^{n} I_{s_i}$. Without loss of generality we may assume that $I_{s_j} \cap I_{s_{j+2}} = \emptyset$. Now, for every $s \in$ S, we will construct a Hamiltonian function $\widetilde{H}_s : U_s \to \mathbb{R}$ such that the corresponding Hamiltonian flow sends L_s to $L_{s'}$ for $s' \in I_s$, and leave the diagonal invariant. Next, by smoothly patching together those Hamiltonian flows on the intersections $U_{s_i} \cap U_{s_{i+1}}$, we will achieve the required Hamiltonian isotopy Φ.

We fix $s_0 \in$ S. Let (p, q) be canonical local coordinates on $T^*L_{s_0}$ (where q is the coordinate on L_{s_0} and p is the coordinate on the fiber). Moreover, we fix a Riemannian metric on L_{s_0}, and denote by $\| \cdot \|_{s_0}$ the induced fiber norm on $T^*L_{s_0}$. Consider the aforementioned tubular neighborhood U_{s_0} of L_{s_0} in $M \times M$. For every $x \in L_{s_0} \cap \triangle$ denote by $\sigma_{s_0}(x)$ the component of the intersection of U_{s_0} and $\triangle$ containing the point x. Note that we may choose U_{s_0} small enough such that the sets $\{\sigma_{s_0}(x)\}$, $x \in L_{s_0} \cap \triangle$, are mutually disjoint. In what follows we shall denote the image of $\sigma_{s_0}(x)$ under the above identification between U_{s_0} and V_{s_0}, by $\sigma_{s_0}(x)$ as well.

We first claim that there exists a Hamiltonian symplectomorphism $\widetilde{\varphi} : V_{s_0} \to V_{s_0}$ which for every intersection point $x \in L_{s_0} \cap \triangle$ sends $\sigma_{s_0}(x)$ to the fiber over x and which leaves L_{s_0} invariant. Indeed, since L_{s_0} transversely intersects the diagonal, and since $\sigma_{s_0}(x)$ is a Lagrangian submanifold, $\sigma_{s_0}(x)$ is the graph of a closed 1-form of p-variable i.e, $\sigma_{s_0}(x) = \{(p, \alpha(p))\}$ where $\alpha(p)$ is locally defined near the intersection point x, and $\alpha(0) = 0$. Define a family of local diffeomorphisms by $\varphi_t(p, q) = (p, q - t\alpha(p))$. Since the 1-form $\alpha(p)$ is closed, $\{\varphi_t\}$ is a Hamiltonian flow. Denote by $K(p, q)$ the Hamiltonian function generating $\{\varphi_t\}$. A simple computation shows that $K(p, q) = -\int \alpha(p)dp$. Hence $K(p, q)$ is independent on the q-variable i.e, $K(p, q) = K(p)$. Furthermore, we may assume that $K(0) = 0$. Next, we cut off the Hamiltonian function $K(p)$ outside a neighborhood of the intersection point x. Let $\beta(r)$ be a smooth cut-off function that vanishes for $r \geq 2\varepsilon$ and equal to 1 when $r \leq \varepsilon$, for sufficiently small ε. Define

$$\widetilde{K}(p, q) = \beta(\|p\|) \cdot \beta(\|q\|) \cdot K(p).$$

A straightforward computation shows that, $\frac{\partial \widetilde{K}}{\partial q}(0, \cdot) = \frac{\partial \widetilde{K}}{\partial p}(0, \cdot) = 0$. Hence the time-1-map of the Hamiltonian flow corresponding to $\widetilde{K}(p, q)$ is the required symplectomorphism. Therefore, we now can assume that $\sigma_{s_0}(x)$ coincide with the fiber over the point x.

Next, since Ψ is an exact Lagrangian isotopy, we have that for every $s \in I_{s_0}$, L_s is a graph of an exact 1-form dG_s in the symplectic tubular neighborhood V_{s_0} of L_{s_0}. Hence, in the above local coordinates (p, q) on $T^*L_{s_0}$, L_s takes the form $L_s = (dG_s(q), q)$. Moreover, note that $dG_s(0) = 0$.

Define

$$\widetilde{H}_{s_0}(p, q) = \beta(\|p\|) \cdot G_s(q).$$

Consider the Hamiltonian vector field corresponding to $\widetilde{H}_{s_0}$,

$$\widetilde{\xi} = \begin{cases} \dot{p} = -\frac{\partial \widetilde{H}}{\partial q} = -\beta(\|p\|) \cdot \frac{\partial G_s(q)}{\partial q} \\ \dot{q} = \frac{\partial \widetilde{H}}{\partial p} = \frac{\partial}{\partial p}\{\beta(\|p\|)\} \cdot G_s(q) \end{cases}$$

It follows that for every $s \in I_{s_0}$ such that $L_s \subset \{(p, q) \mid \|p\| < \varepsilon\}$, the Hamiltonian flow is given by

$$(p, q) \to \left(p + \frac{\partial G_s(q)}{\partial q}, q\right)$$

Hence, locally, the Hamiltonian flow sends L_{s_0} to L_s as required. It remains to prove that $\widetilde{\xi}$ vanishes on the diagonal. First, since $dG_s(0) = 0$, it follows that $\dot{p} = 0$. Next, consider x and y, two intersection points of the family $\{L_s\}$ and the diagonal. It follows from Lemma 5.1 that the symplectic area between L_{s_0} and L_s in V_{s_0} vanishes for every $s \in I_{s_0}$. Hence, by the same argument as in Lemma 5.1, for every such s we have

$$0 = \int_{\Sigma_s} \omega = \int_{[0,s]\times[0,1]} \Sigma_s^* \omega = \int_0^s \Big(G_s(x) - G_s(y)\Big) ds$$

Thus, we get that $G_s(x) - G_s(y) = 0$. Note that by changing the functions $\{G_s\}$ by a summand depending only on s, we can assume that for every s, G_s vanishes on $L_s \cap \triangle$. Hence, we obtain that $\widetilde{\xi}_{|\triangle} = 0$. Therefore, we have that the diagonal is invariant under the Hamiltonian flow. Finally, by smoothly patching together all the Hamiltonian flows corresponding to the Hamiltonian functions $\widetilde{H}_{s_i}$, for $i = 1, \ldots, n$, we conclude that there exists a Hamiltonian isotopy Φ such that $\Phi_s(L_0) = L_s$ and $\Phi_s(\triangle) = \triangle$. This completes the proof of the proposition. $\square$

References

[B] Banyaga, A. (1978). *Sur la structure du groupe des difféomorphisms qui préservent une forme symplectique.* Comm. Math. Helv. 53, 174-227.

[BP] Bialy, M. and Polterovich, L. *Geodesics of Hofer's metric on the group of Hamiltonian diffeomorphisms*, Duke Math. J., 76 (1994), 273–292.

[C] Chekanov, Yu. *Invariant Finsler metrics on the space of Lagrangian embeddings*, Math. Z., 234 (2000), 605–619.

[H] Hofer, H. *On the topological properties of symplectic maps.* Proceedings of the Royal Society of Edinburgh, 115 (1990), 25-38.

[HZ] Hofer, H. and Zehnder, E. *Symplectic invariants and Hamiltonian dynamics*, Birkhauser Advanced Texts, Birkhauser Verlag, 1994.

[LM1] Lalonde, F. and McDuff, D. *The geometry of symplectic energy*, Ann. of Math 141 (1995), 349-371.

[LM2] Lalonde, F. and McDuff, D. *Hofer's L^∞ - geometry: energy and stability of Hamiltonian flows*, parts 1 and 2, Invent. Math. 122 (1995), 1-33 and 35-69.

[M] Milinković , D. *Geodesics on the space of Lagrangian submanifolds in cotangent Bundles*, Proc. Amer. Math. Soc. 129 (2001), no 6, 1843–1851.

[MS] McDuff, D. and Salamon, D. *Introduction to Symplectic Topology*, 2nd edition, Oxford University Press, Oxford, England (1998).

[P1] Polterovich, L. (1993). *Symplectic displacement energy for Lagrangian submanifolds.* Ergodic Theory and Dynamical Systems, 13, 357-67.

[P2] Polterovich, L. *The Geometry of the group of Symplectic Diffeomorphisms*, Lectures in Math, ETH, Birkhauser (2001).

[P3] Polterovich, L. *Growth of maps, distortion in groups and symplectic geometry*, Preprint Math.DS/0111050.

[S] Schwarz, M. *On the action spectrum for closed symplectically aspherical manifolds*, *Pac. Journ. Math* **193** (2000), 419–461.

[W] Weinstein, A. (1971). *Symplectic manifolds and their Lagrangian submanifolds.* Advances in Mathematics, 6, 329-46.